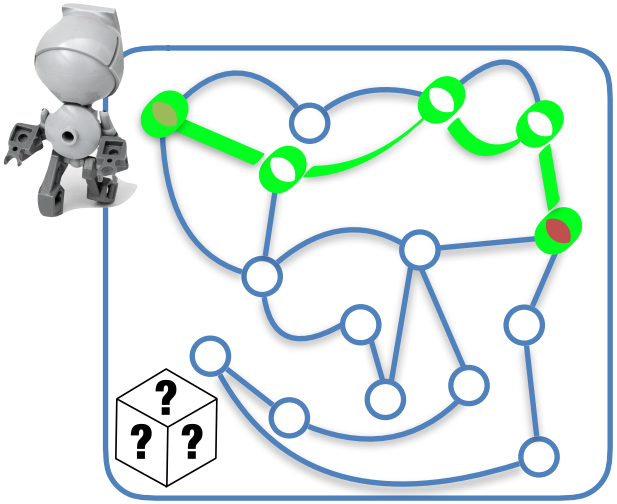
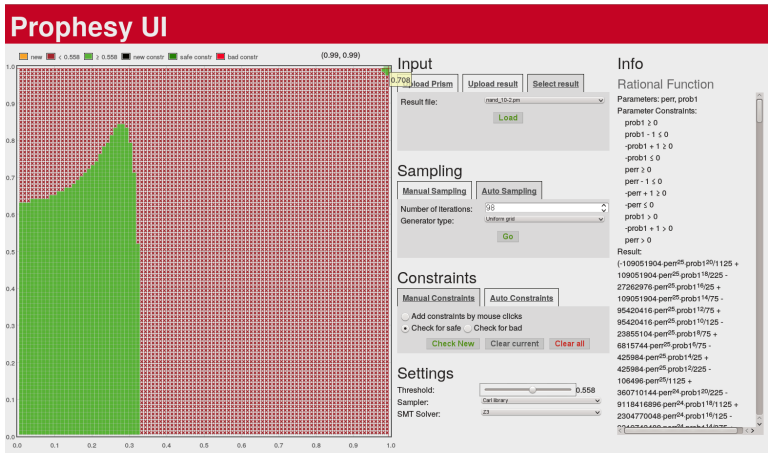
Convex Optimization meets Parameter Synthesis for MDPs



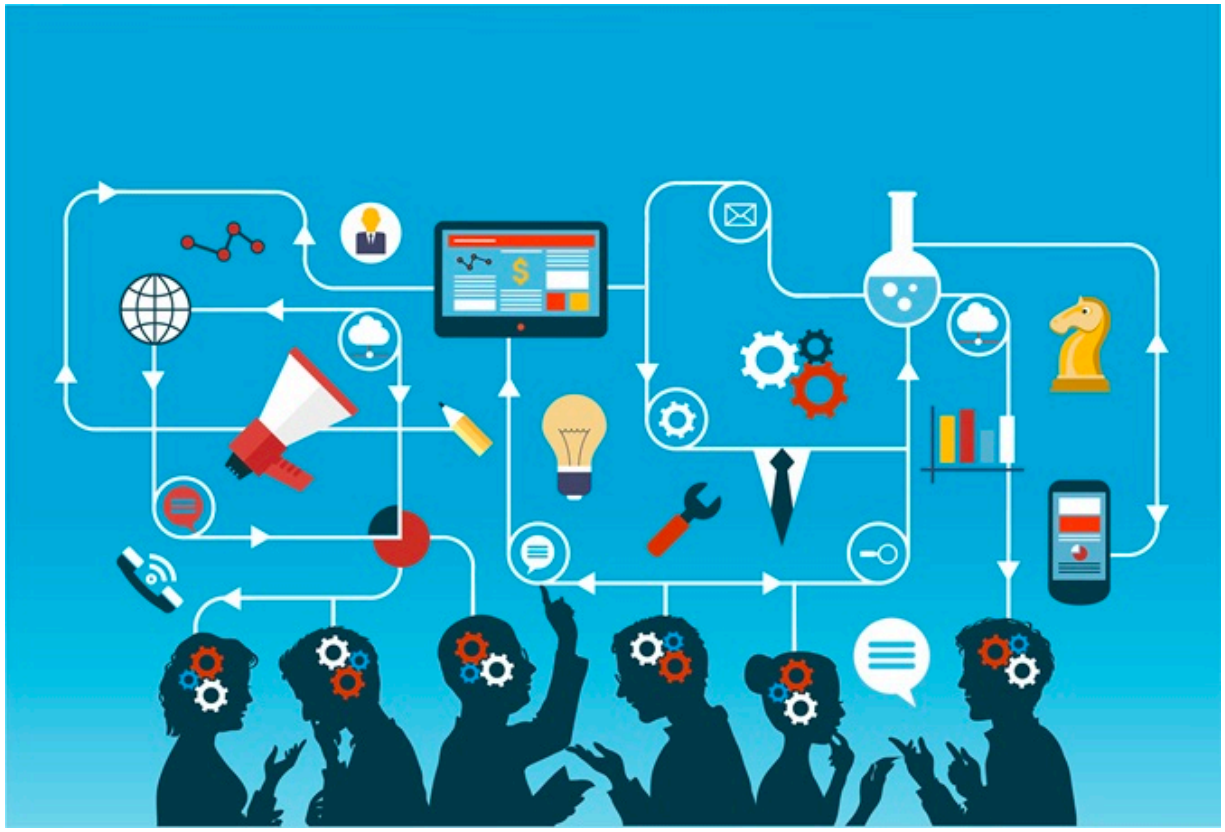


Nils Jansen

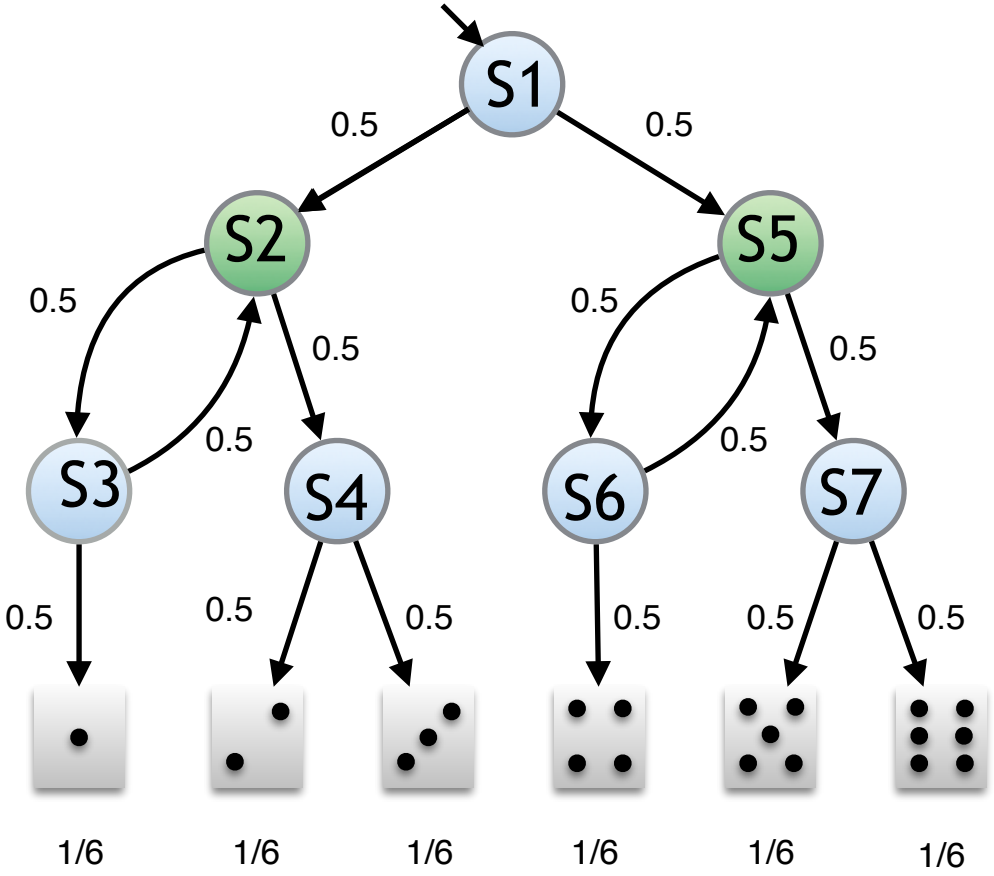
SynCoP 2019

joint work with:

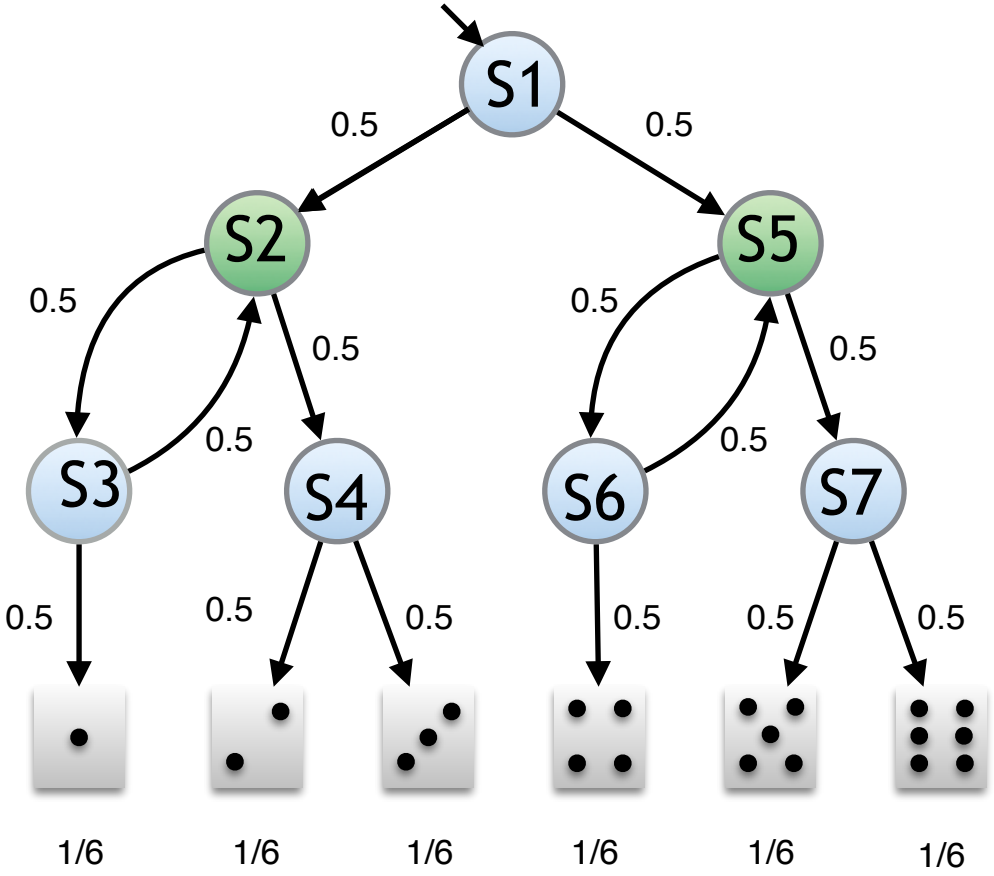
Murat Cubuktepe, Sebastian Junges, Joost-Pieter Katoen, Ivan Papusha, Hasan Poonawala, Ufuk Topcu



Knuth-Yao's Die

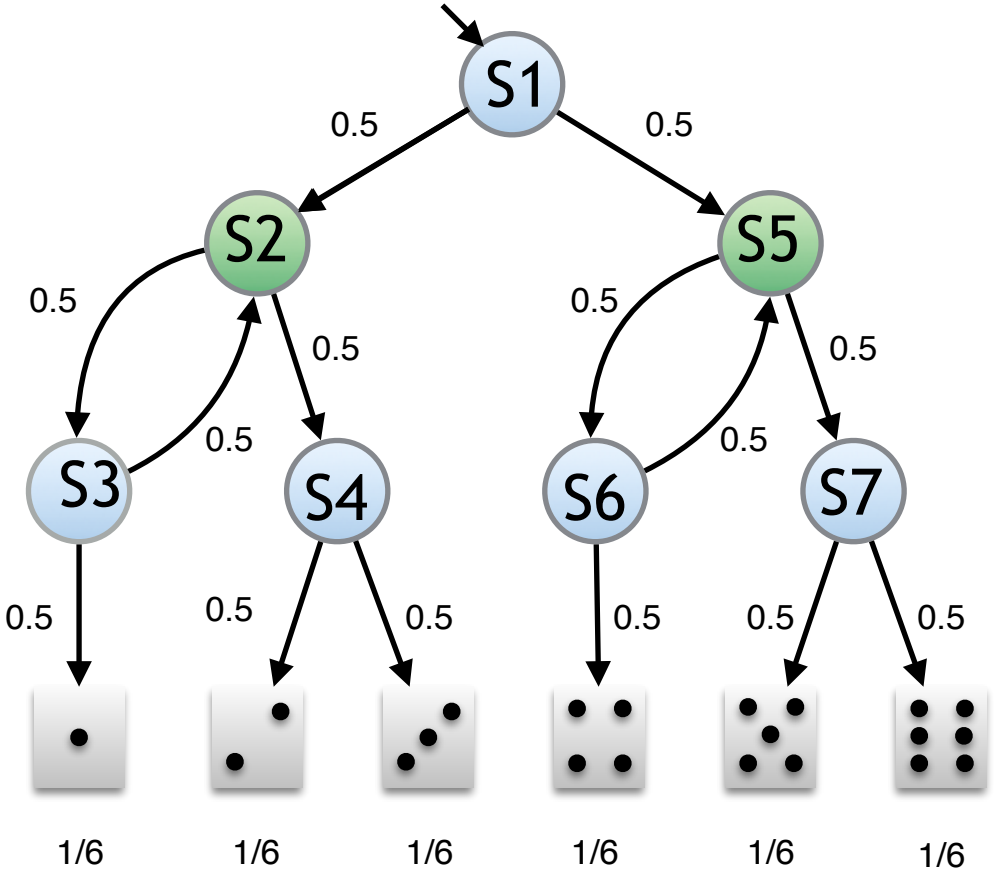


Knuth-Yao's Die



Simulation of a fair die
- fair coins
- uniform distribution

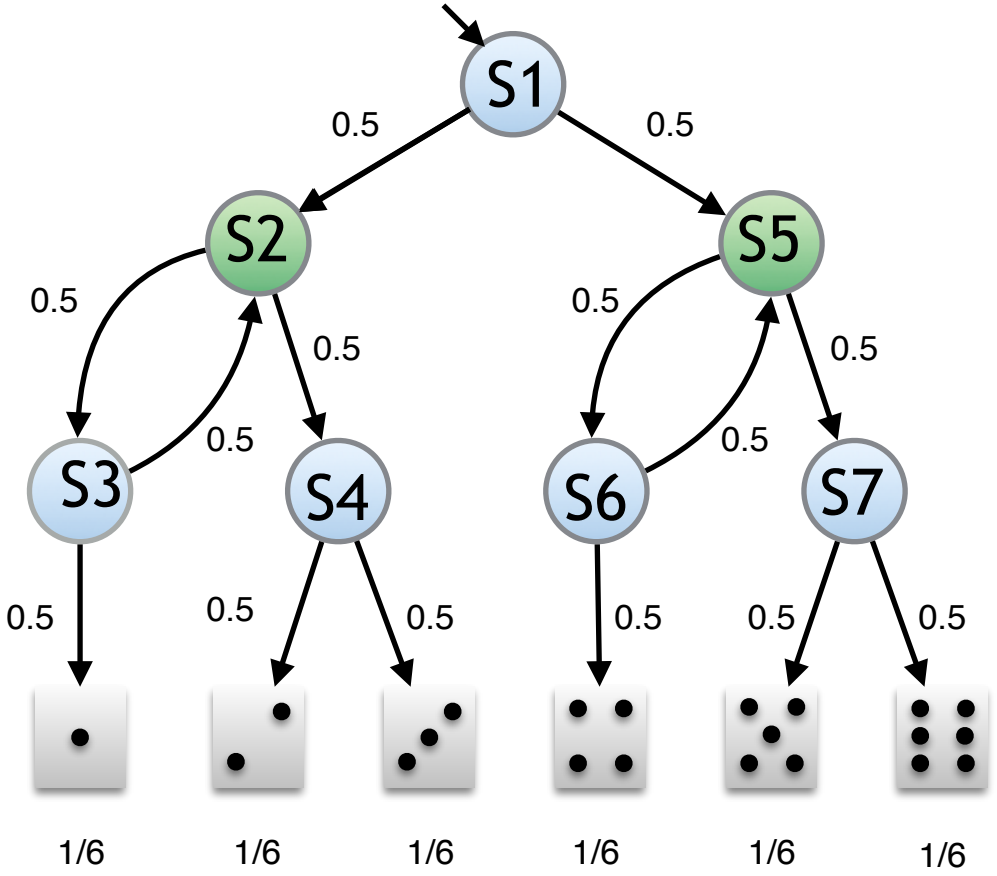
Knuth-Yao's Die



Simulation of a fair die
- fair coins
- uniform distribution

Does this model simulate a fair die?

Knuth-Yao's Die

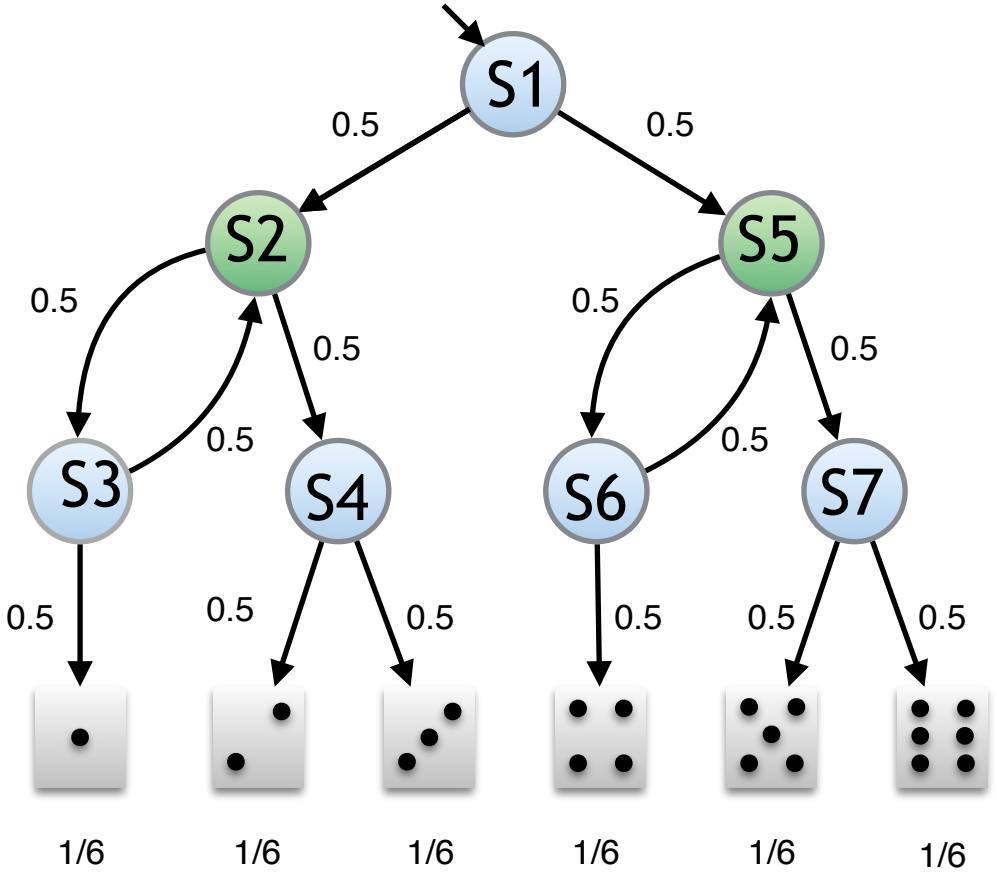


Simulation of a fair die
- fair coins
- uniform distribution

Does this model simulate a fair die?

Probability of reaching  needs to be 1/6.

Knuth-Yao's Die



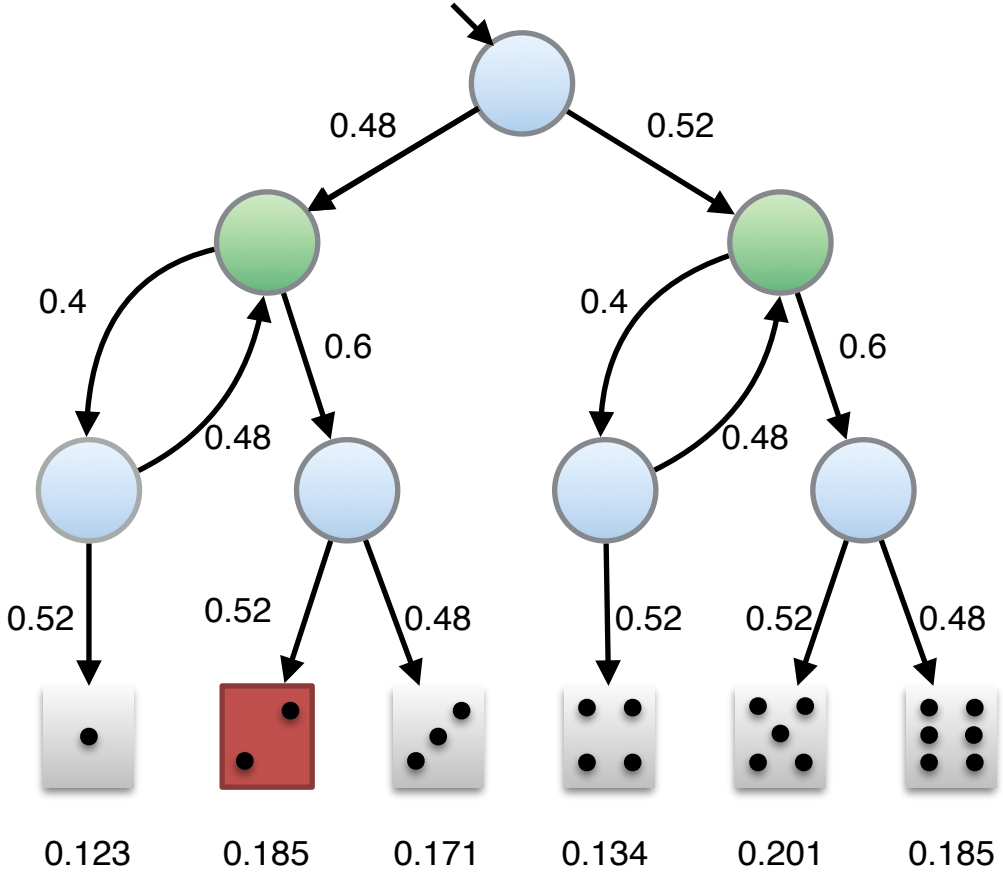
Simulation of a fair die
- fair coins
- uniform distribution

Does this model simulate a fair die?

Probability of reaching  needs to be 1/6.

Computation:
Linear Equation System.

Parameter Synthesis

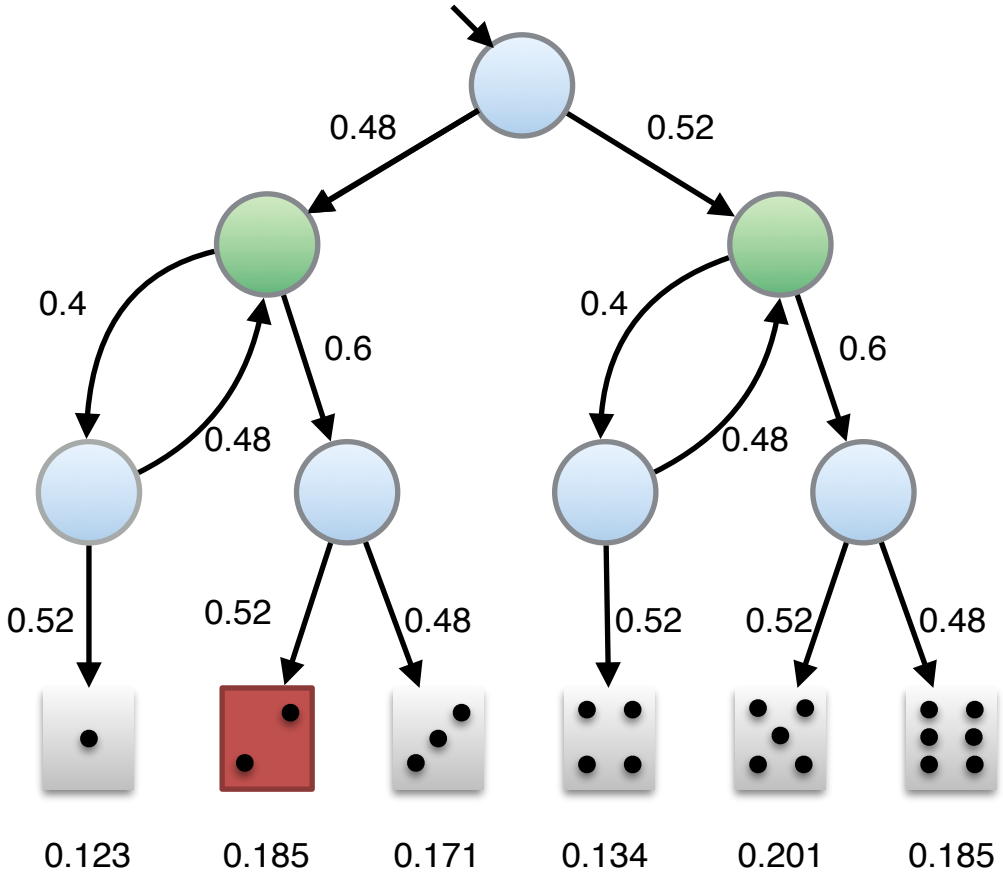


Requirement:

- increase probability of ‘two’
- introduce unfair coins

Bartocci, E., Grosu, R., Katsaros, P., Ramakrishnan, C., Smolka, S.: Model repair for probabilistic systems. TACAS 2011.

Parameter Synthesis



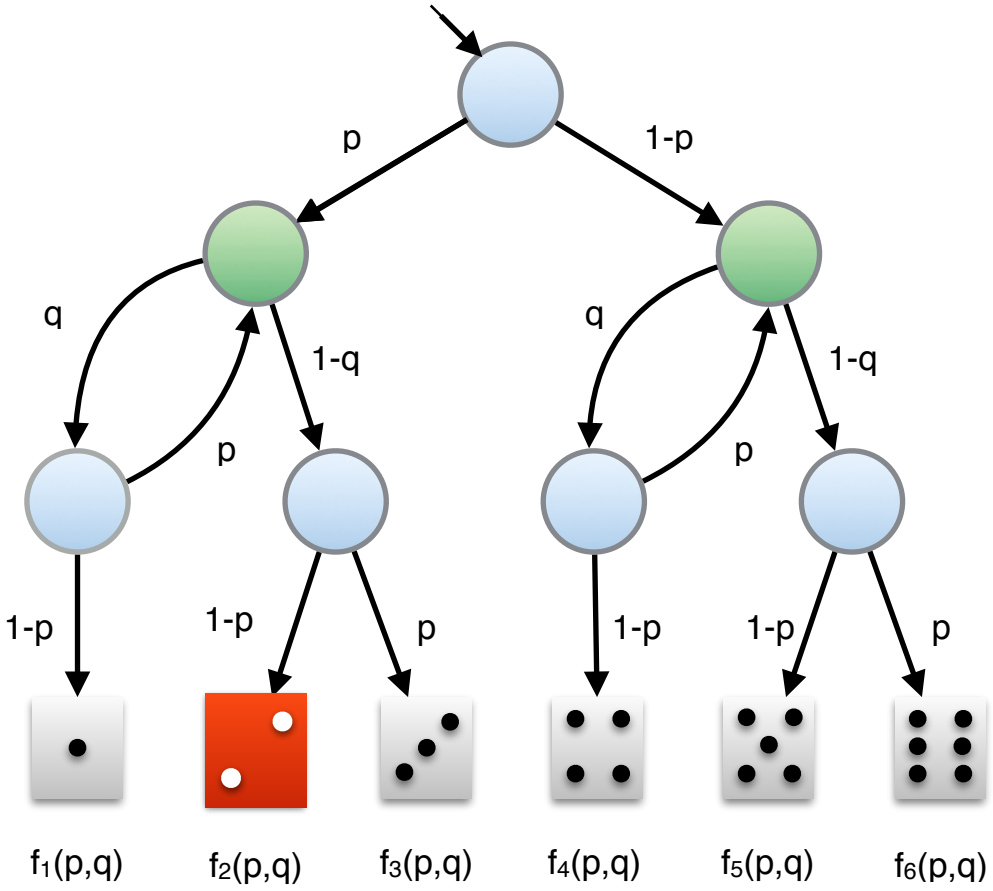
Requirement:

- increase probability of ‘two’
- introduce unfair coins

Model repair: (Minimally) change model such that specifications are met

Bartocci, E., Grosu, R., Katsaros, P., Ramakrishnan, C., Smolka, S.: Model repair for probabilistic systems. TACAS 2011.

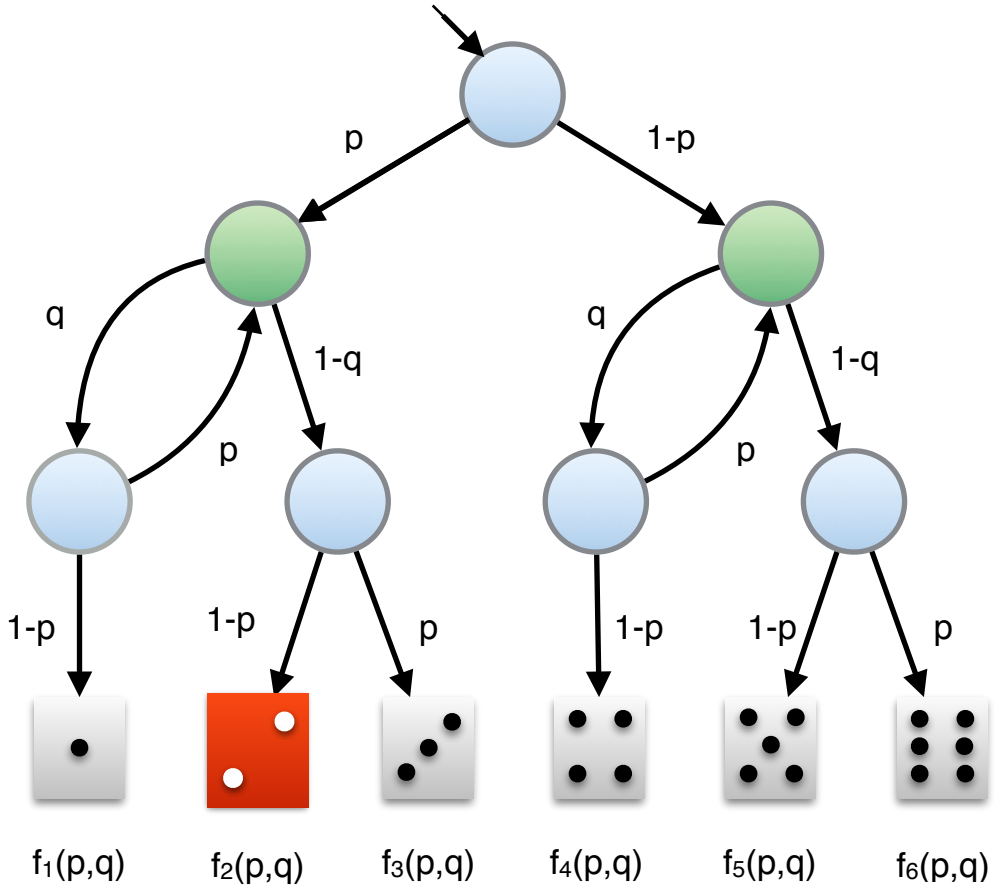
Parameter Synthesis



More general:

- for which coins are requirements met?
- introduce parameters

Parameter Synthesis

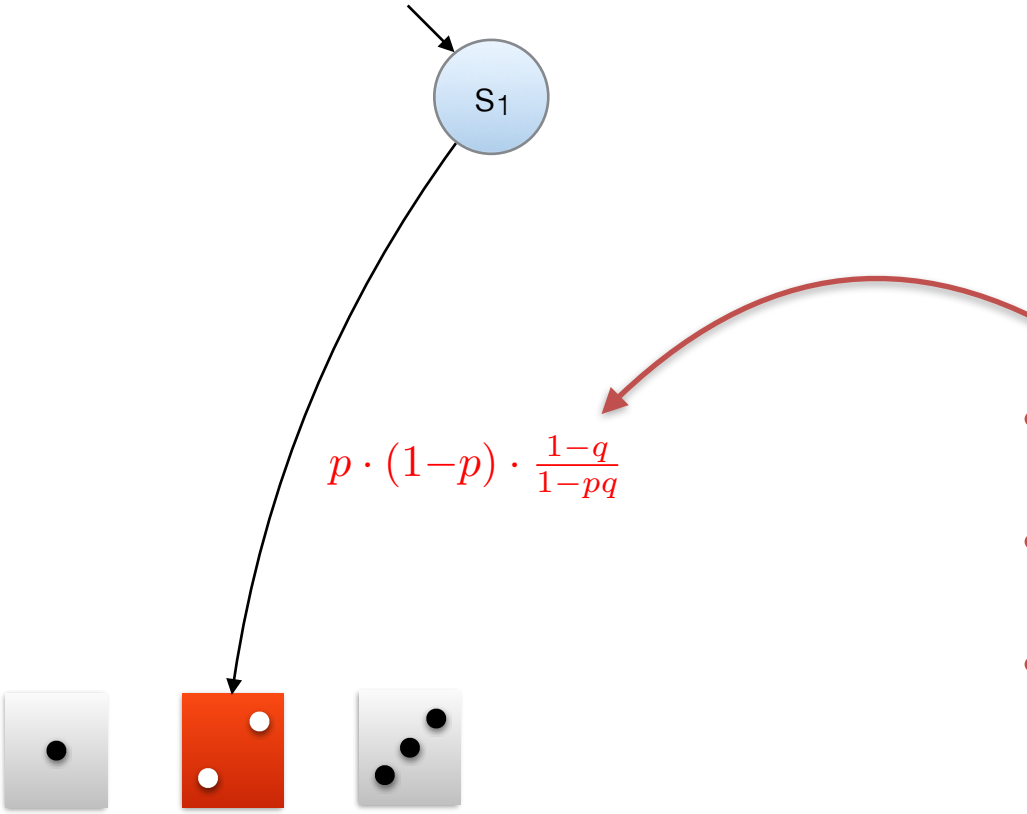


More general:

- for which coins are requirements met?
- introduce parameters

Parameter synthesis: Find parameter values where specifications are satisfied or violated?

Parameter Synthesis



More general:

- for which coins are requirements met?
- introduce parameters

- rational function over parameters
- obtained via variants of state elimination
- probability of reaching 

Parameter synthesis: Find parameter values where specifications are satisfied or violated?

Hahn, E.M., Hermanns, H., Wachter, B., Zhang, L.: PARAM: A model checker for parametric Markov models. CAV 2010

The Need for Parameter Synthesis

The Need for Parameter Synthesis

Probabilistic Model Checking

- applicable for a wide range of benchmarks (network protocols, systems biology, randomized algorithms, security, planning and synthesis, power management)
- tools like PRISM, Storm, iscasMC, MRMC, ...

The Need for Parameter Synthesis

Probabilistic Model Checking

- applicable for a wide range of benchmarks (network protocols, systems biology, randomized algorithms, security, planning and synthesis, power management)
- tools like PRISM, Storm, iscasMC, MRMC, ...

Limitation

- concrete probabilities need to be known a priori
- is this a valid assumption?
- what about results when probabilities fluctuate?

The Need for Parameter Synthesis

Probabilistic Model Checking

- applicable for a wide range of benchmarks (network protocols, systems biology, randomized algorithms, security, planning and synthesis, power management)
- tools like PRISM, Storm, iscasMC, MRMC, ...

Limitation

- concrete probabilities need to be known a priori
- is this a valid assumption?
- what about results when probabilities fluctuate?

Goal

- treat parametric models
- synthesize “good" or "robust" parameter values

Outline

1. Parameter Synthesis for Markov Chains
2. Parameter Synthesis for MDPs
3. Sequential Convex Optimization
4. The Convex-concave Procedure
5. Numerical Examples

Parameter Synthesis

Input

- parametric discrete-time Markov chain (PDTMC)
- property (reachability, expected cost)
- threshold

} specification

Parameter Synthesis

Input

- parametric discrete-time Markov chain (PDTMC)
- property (reachability, expected cost)
- threshold

} specification

Method

- compute rational function for property
- evaluate function against threshold and well-defined probability distributions

Parameter Synthesis

Input

- parametric discrete-time Markov chain (PDTMC)
- property (reachability, expected cost)
- threshold

} specification

Method

- compute rational function for property
- evaluate function against threshold and well-defined probability distributions

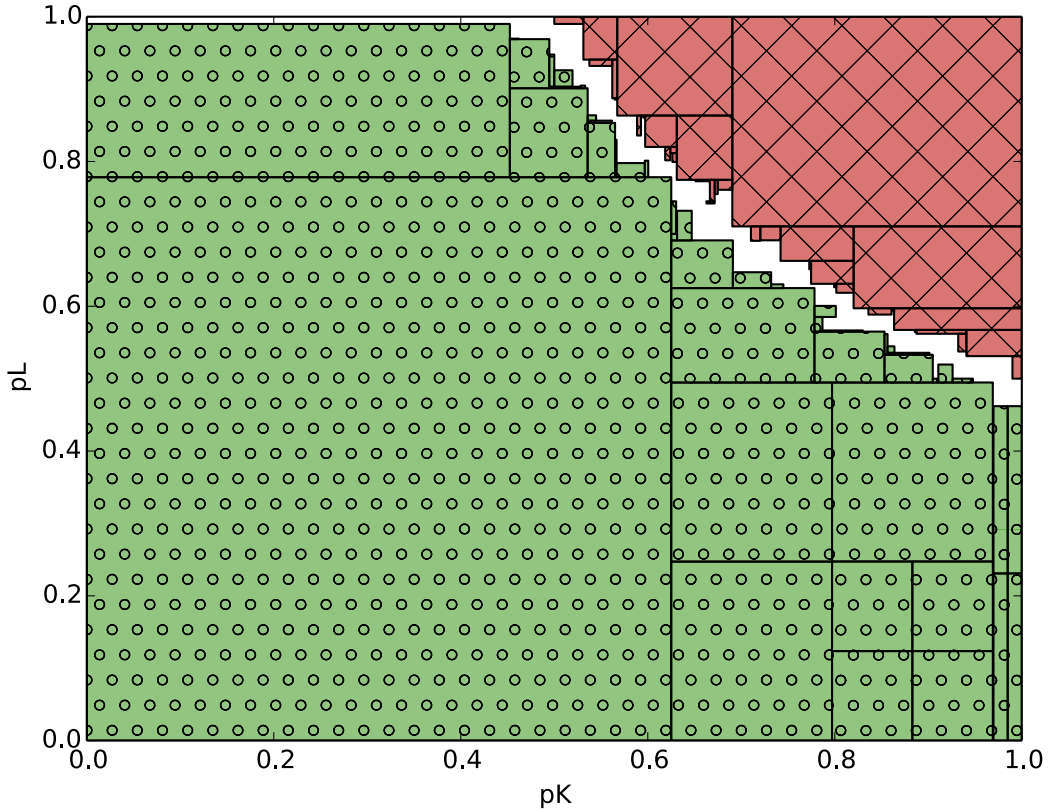
Output

- for which parameter values does PDTMC satisfy the property with the given threshold?

Parameter Space Partitioning

Partitioning into
safe and unsafe regions

Parameter Space Partitioning



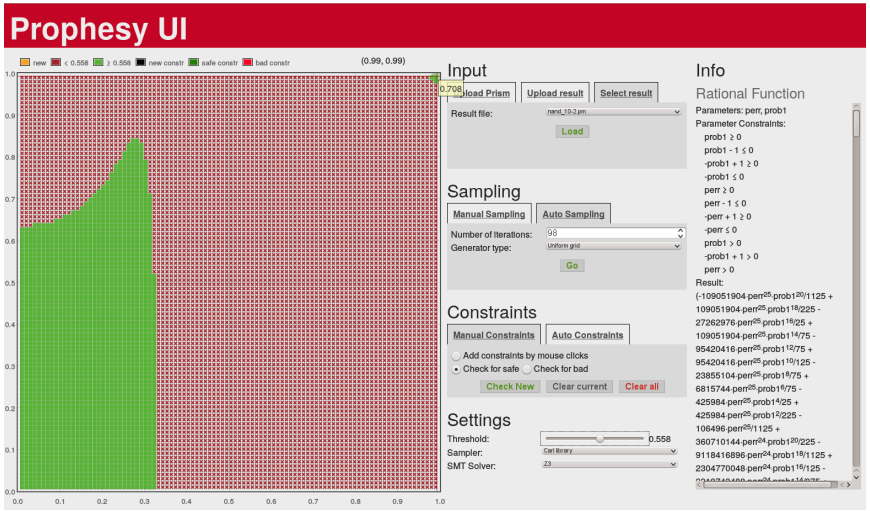
Partitioning into
safe and unsafe regions

PROPhESY - A PRObabilistic ParamEter SYnthesis Tool

CAV 2015

ATVA 2016

Journal under submission



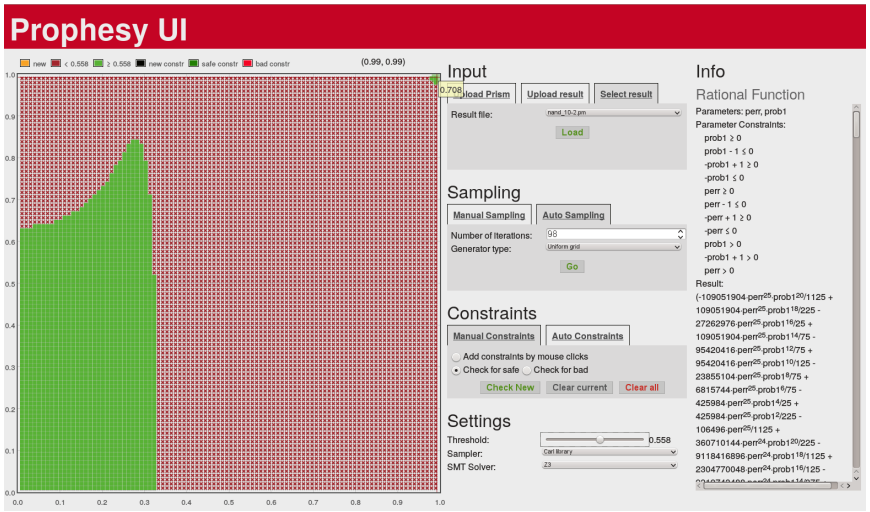
Evaluation of rational function using SMT solving or approximation

CAV 2015

ATVA 2016

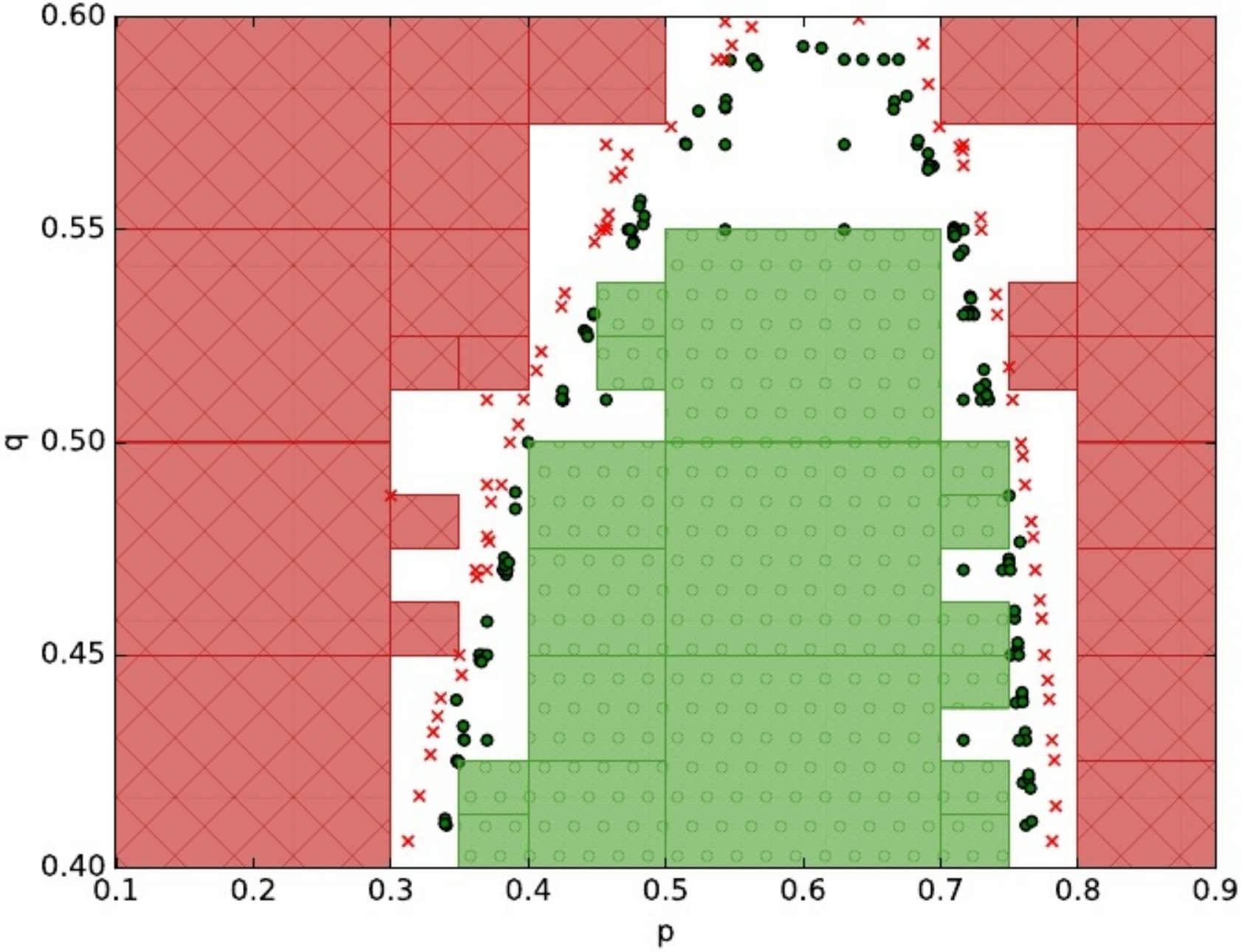
Journal under submission

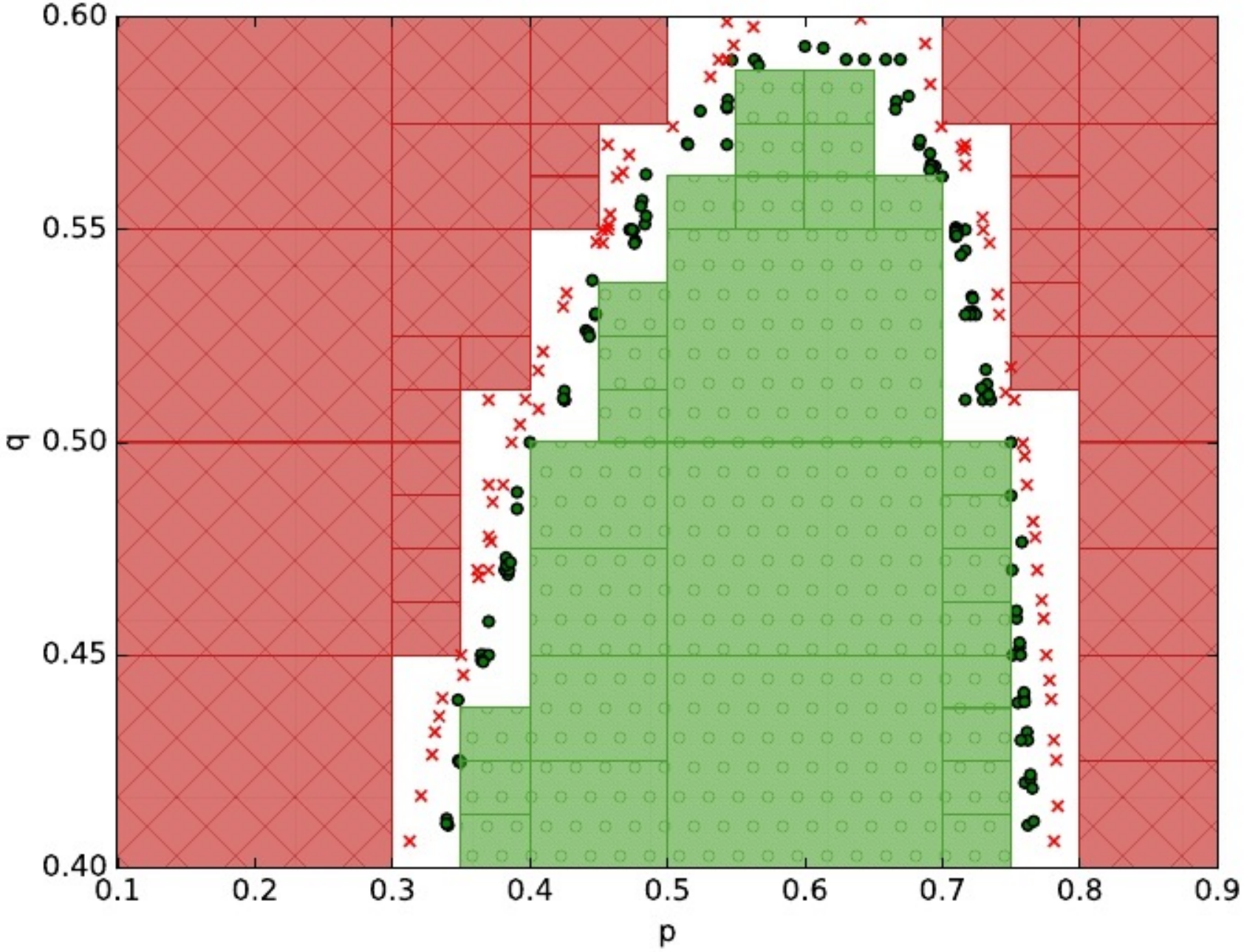
PROPhESY - A PRObabilistic ParamEter SYnthesis Tool

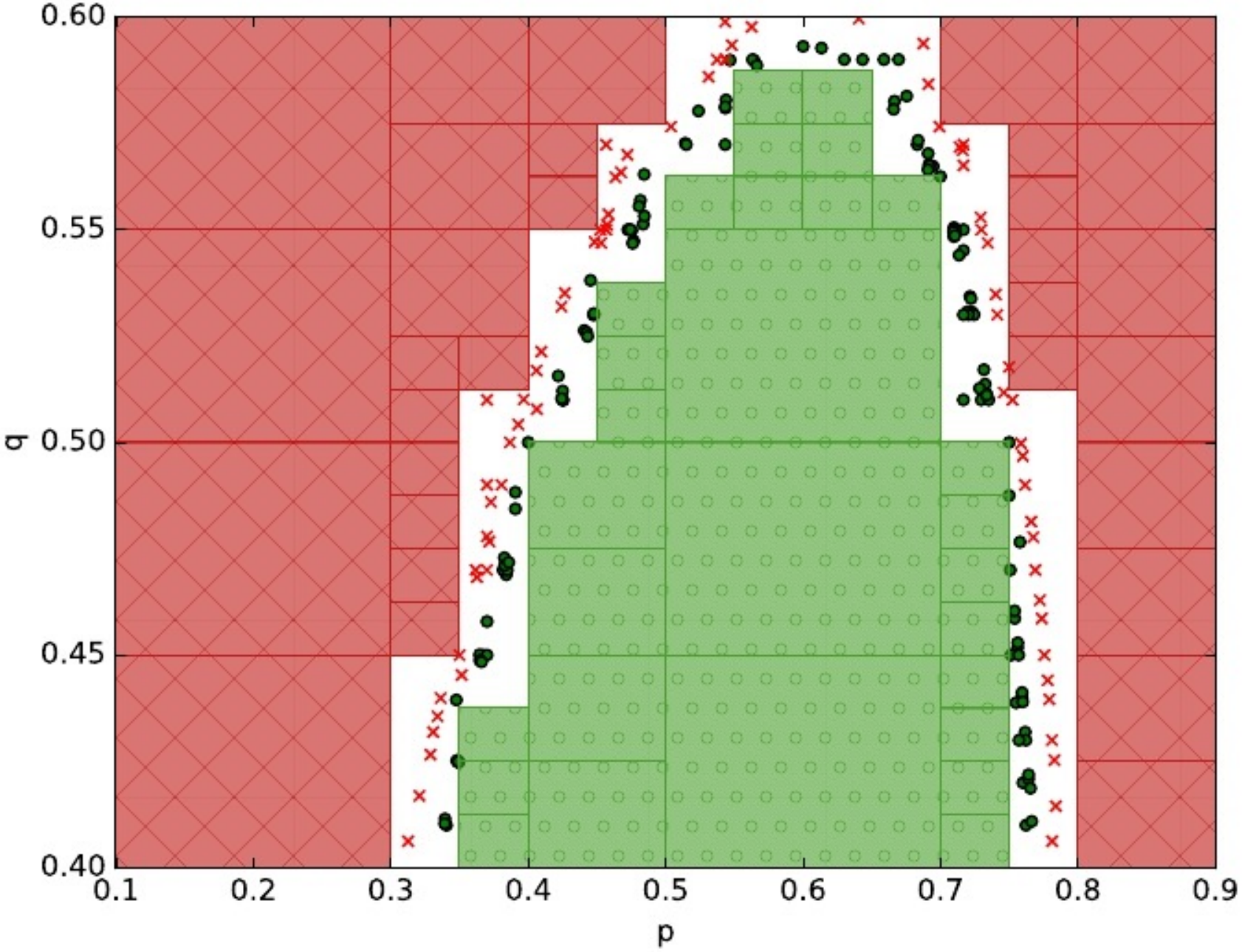


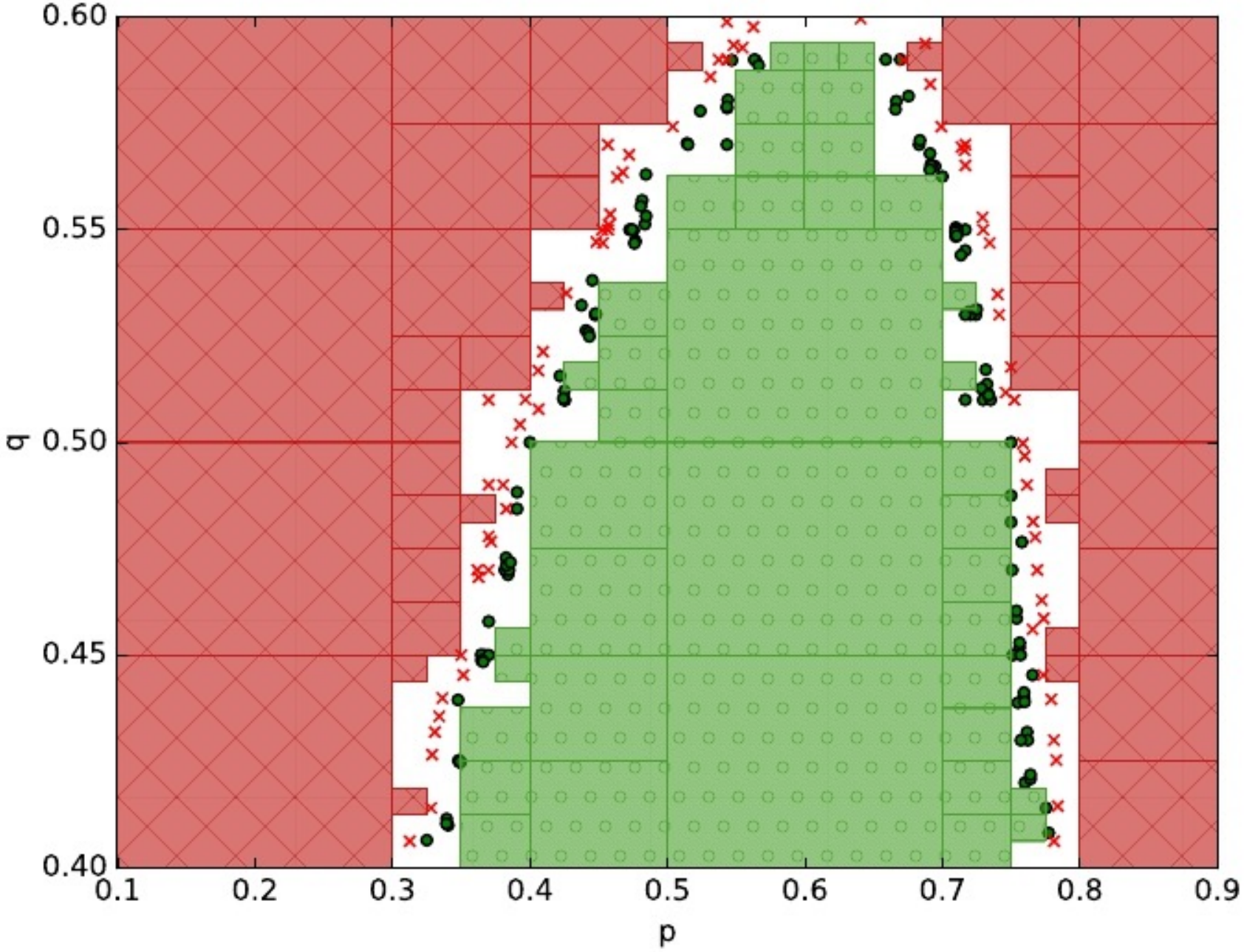
Evaluation of rational function using SMT solving or approximation

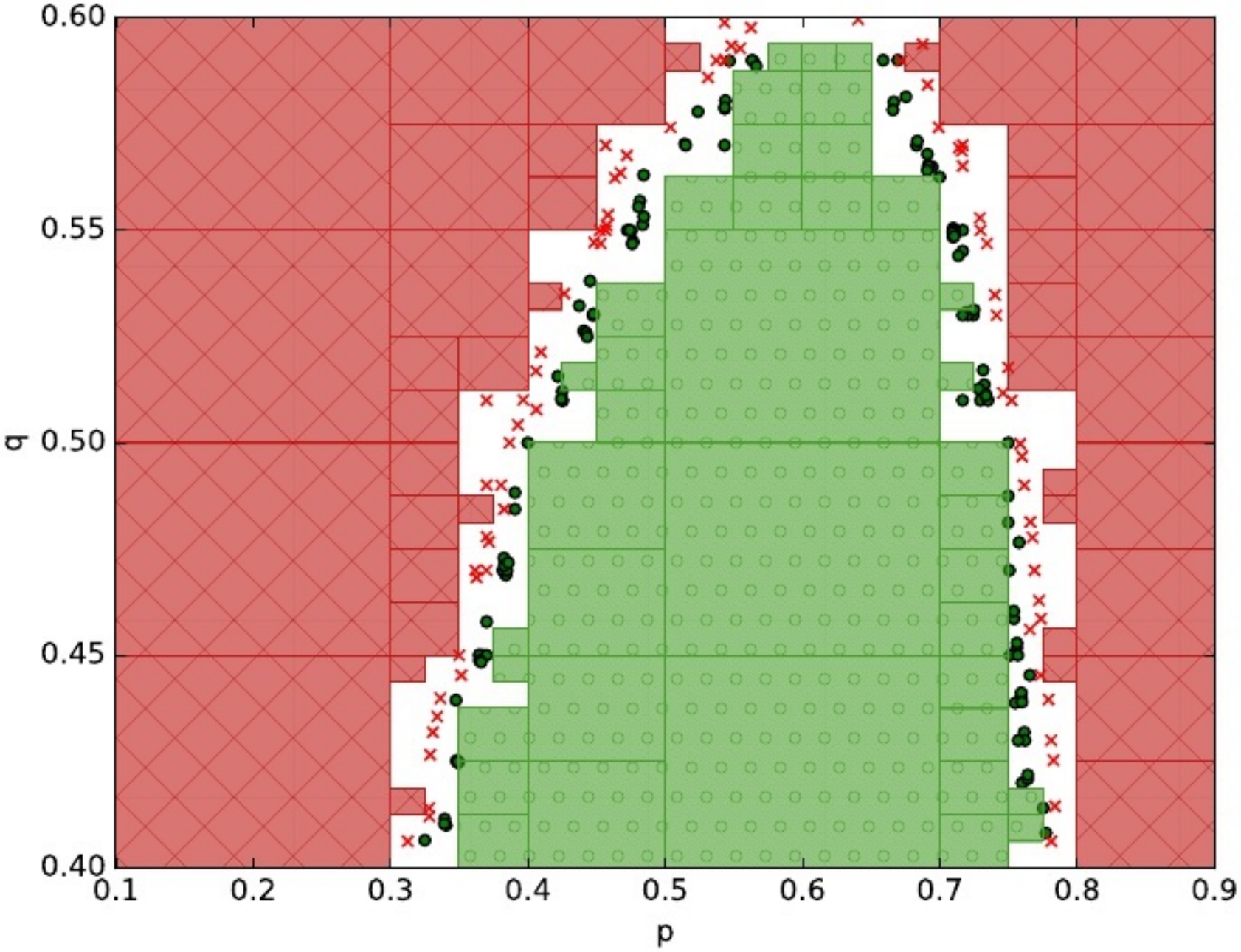
- applicable to models with millions of states
- restricted to few parameters
- SMT performance often unpredictable

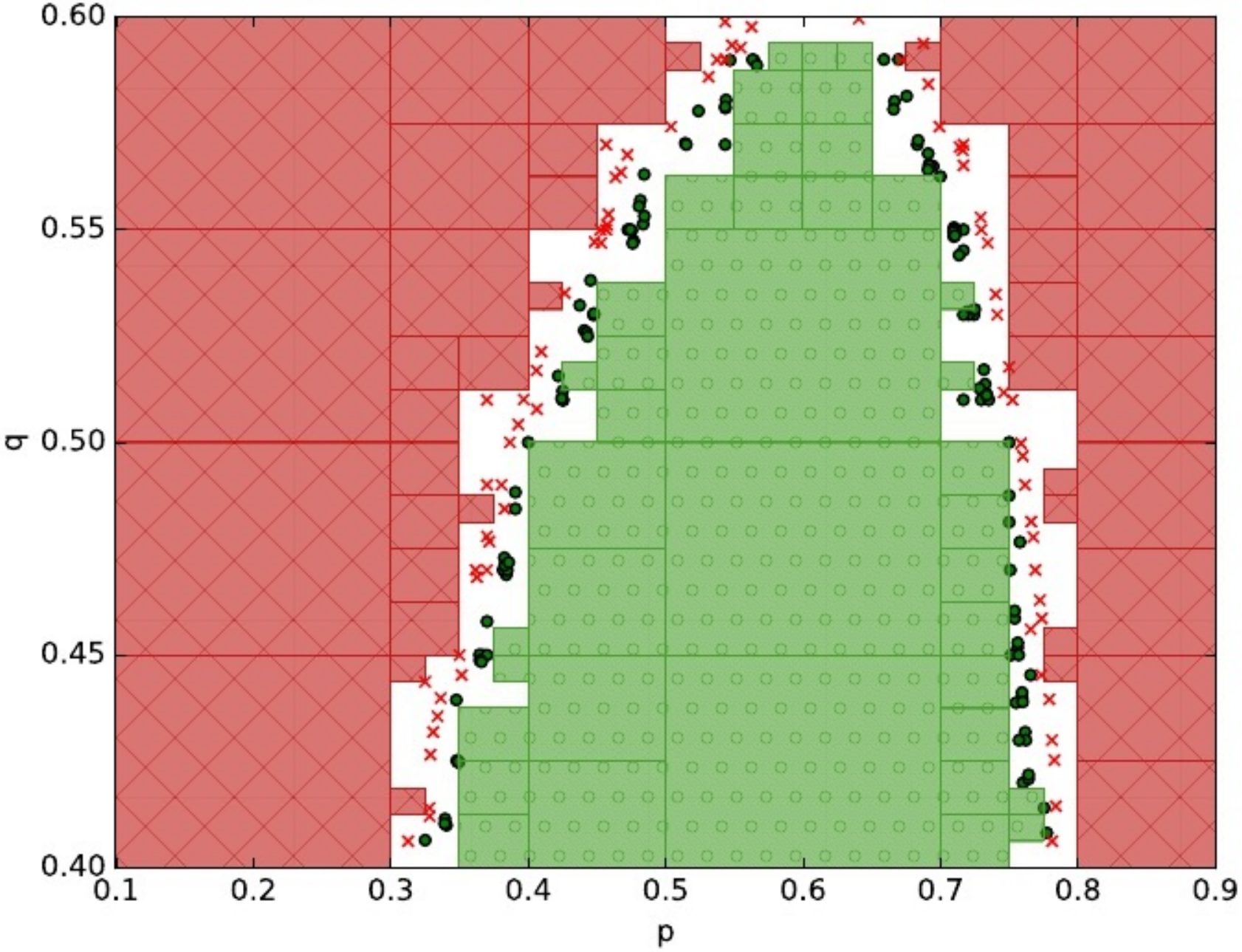

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

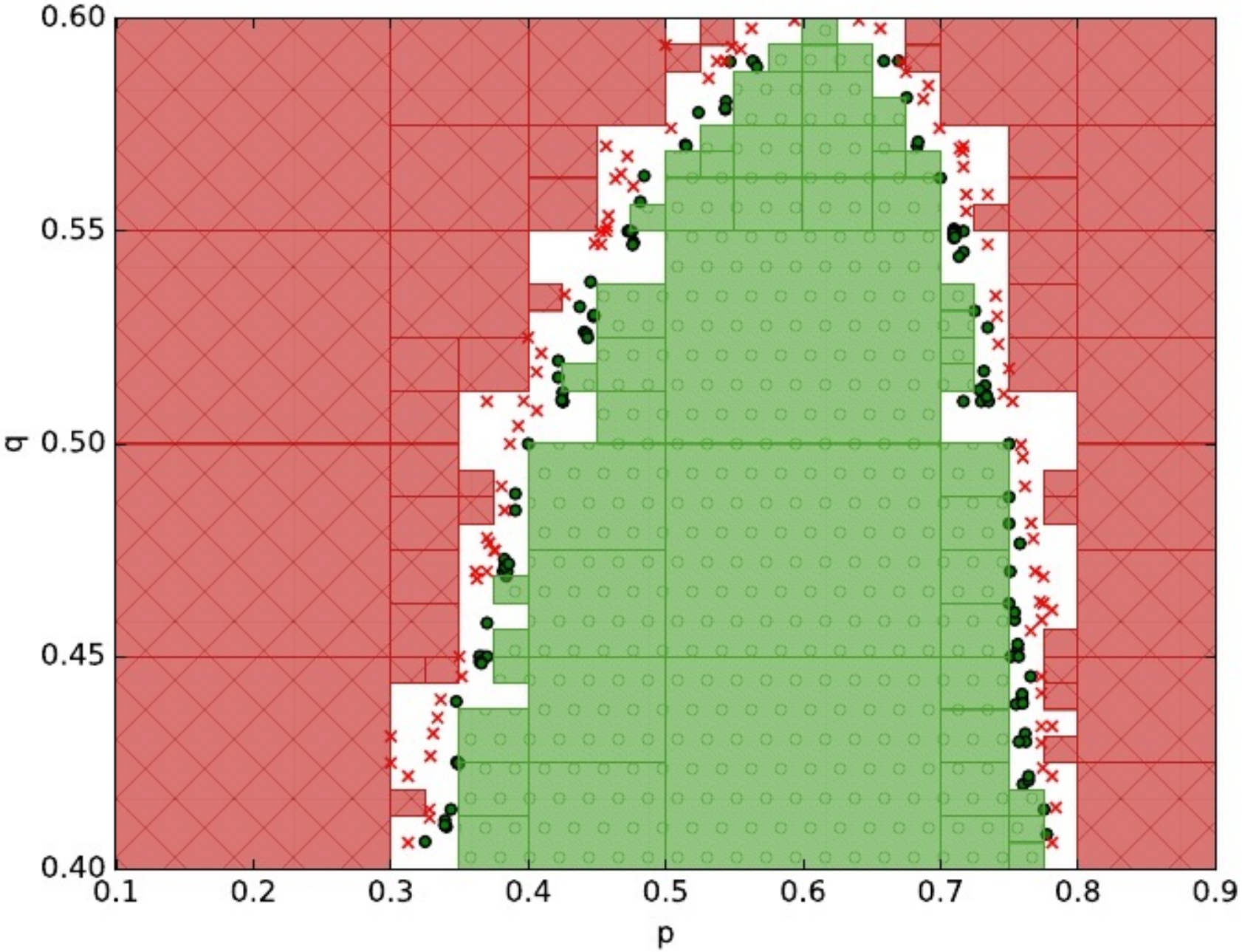

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

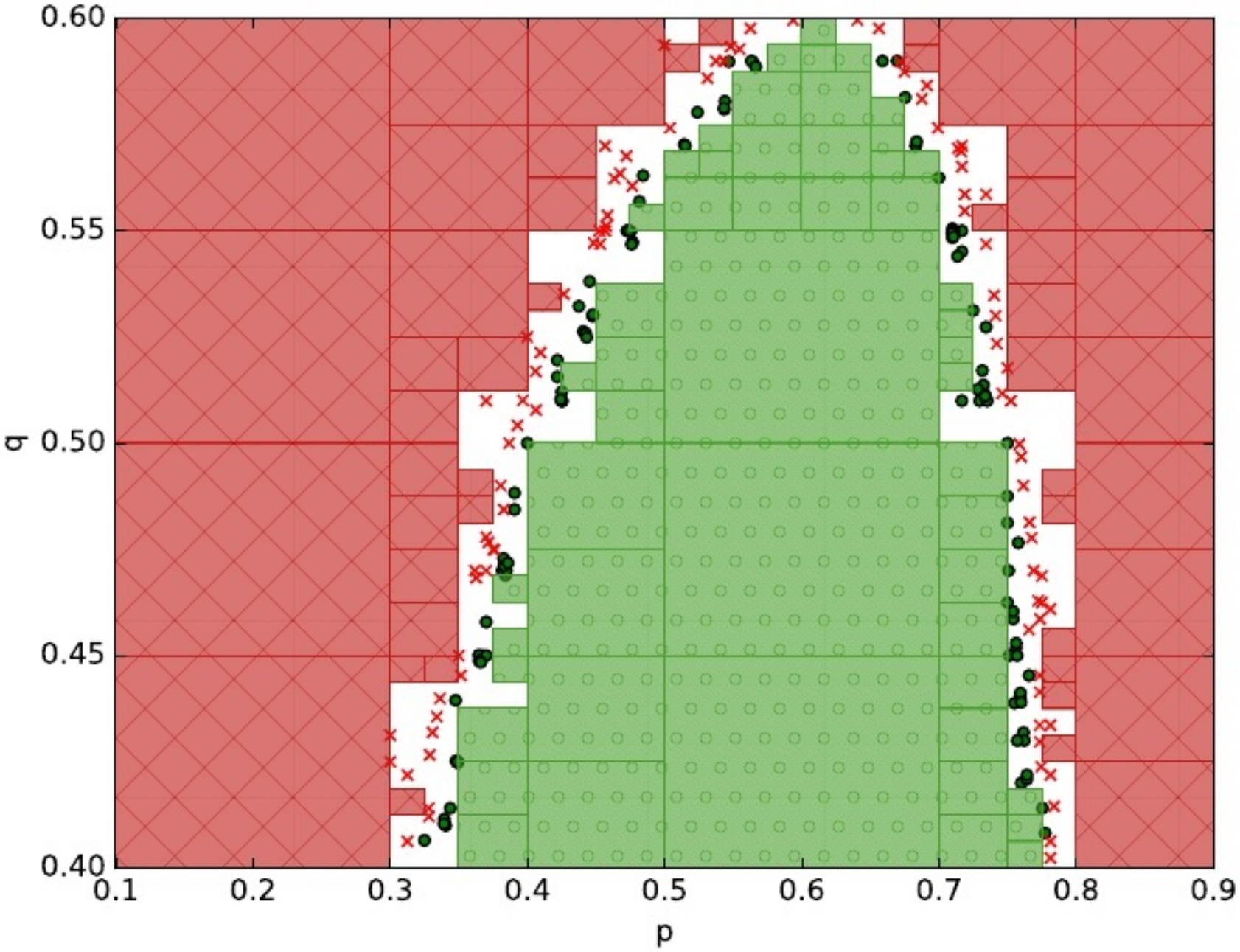

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

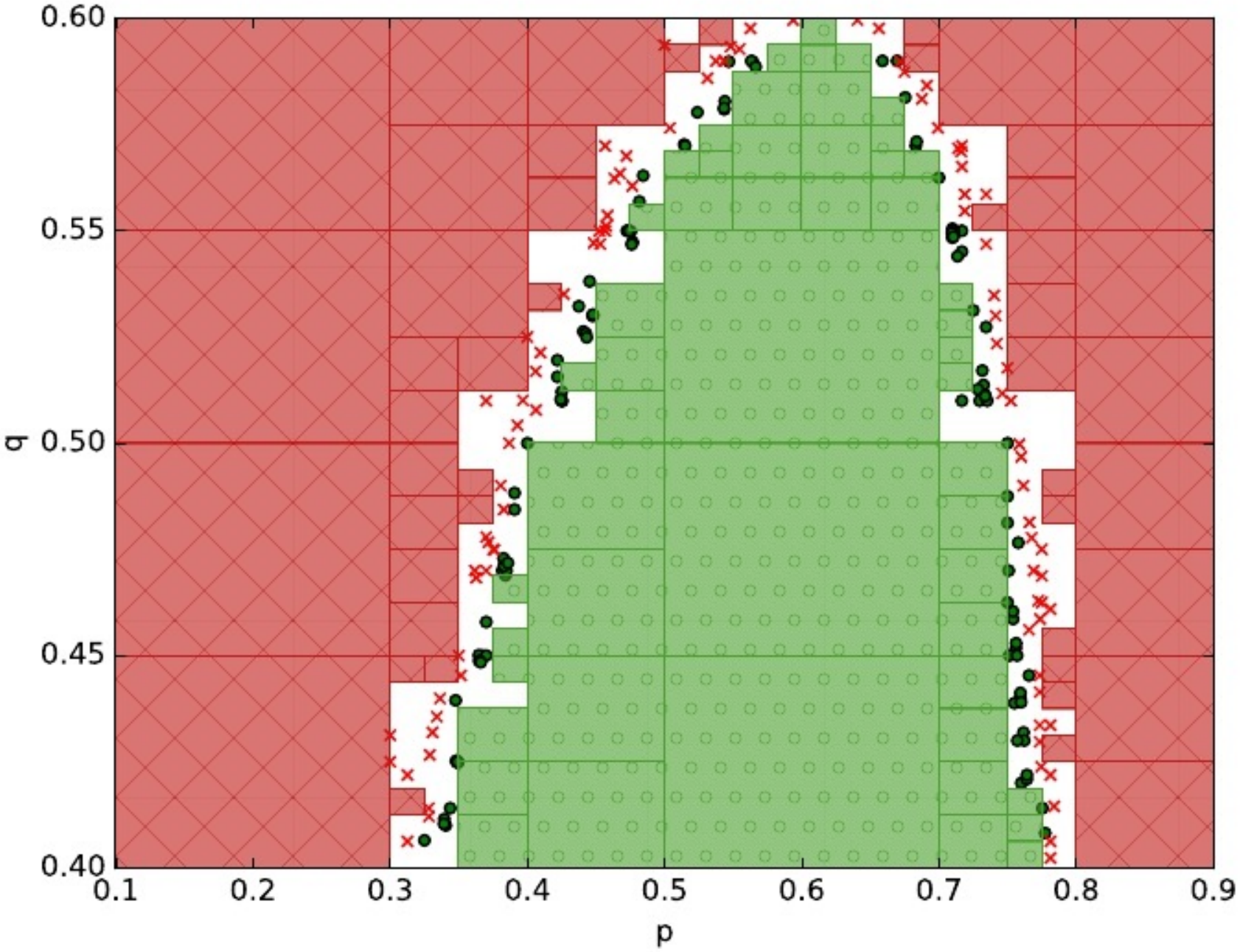

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

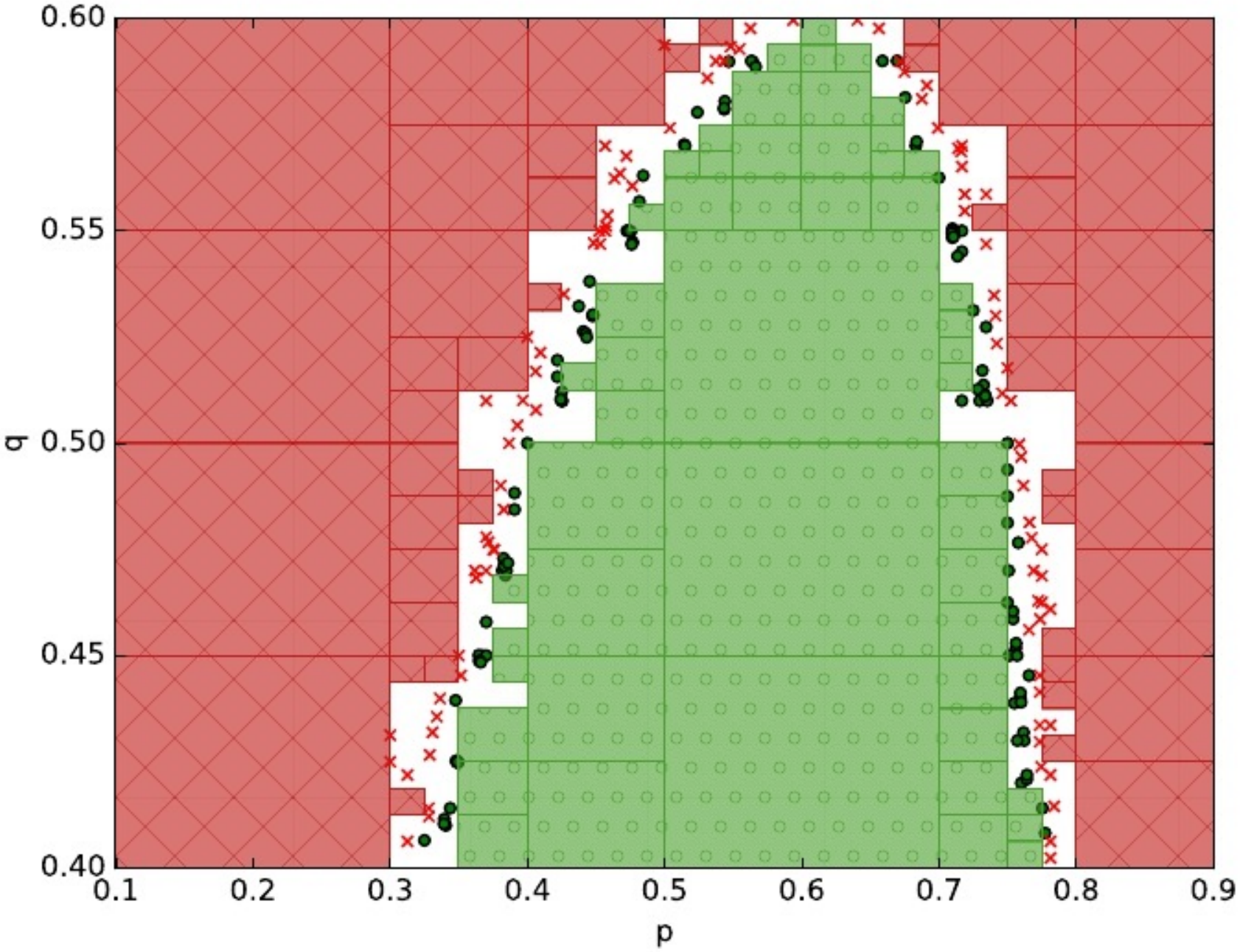

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

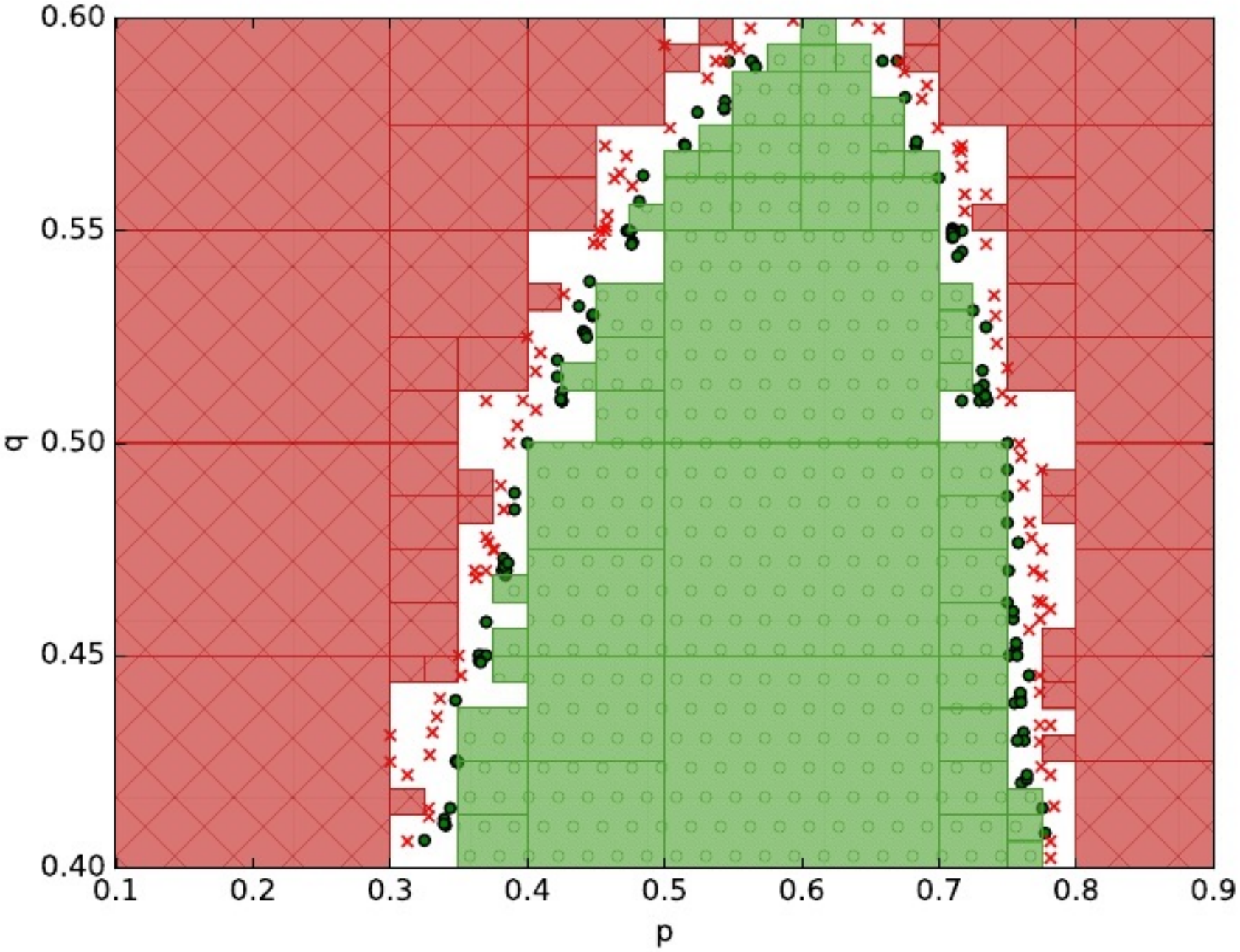

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

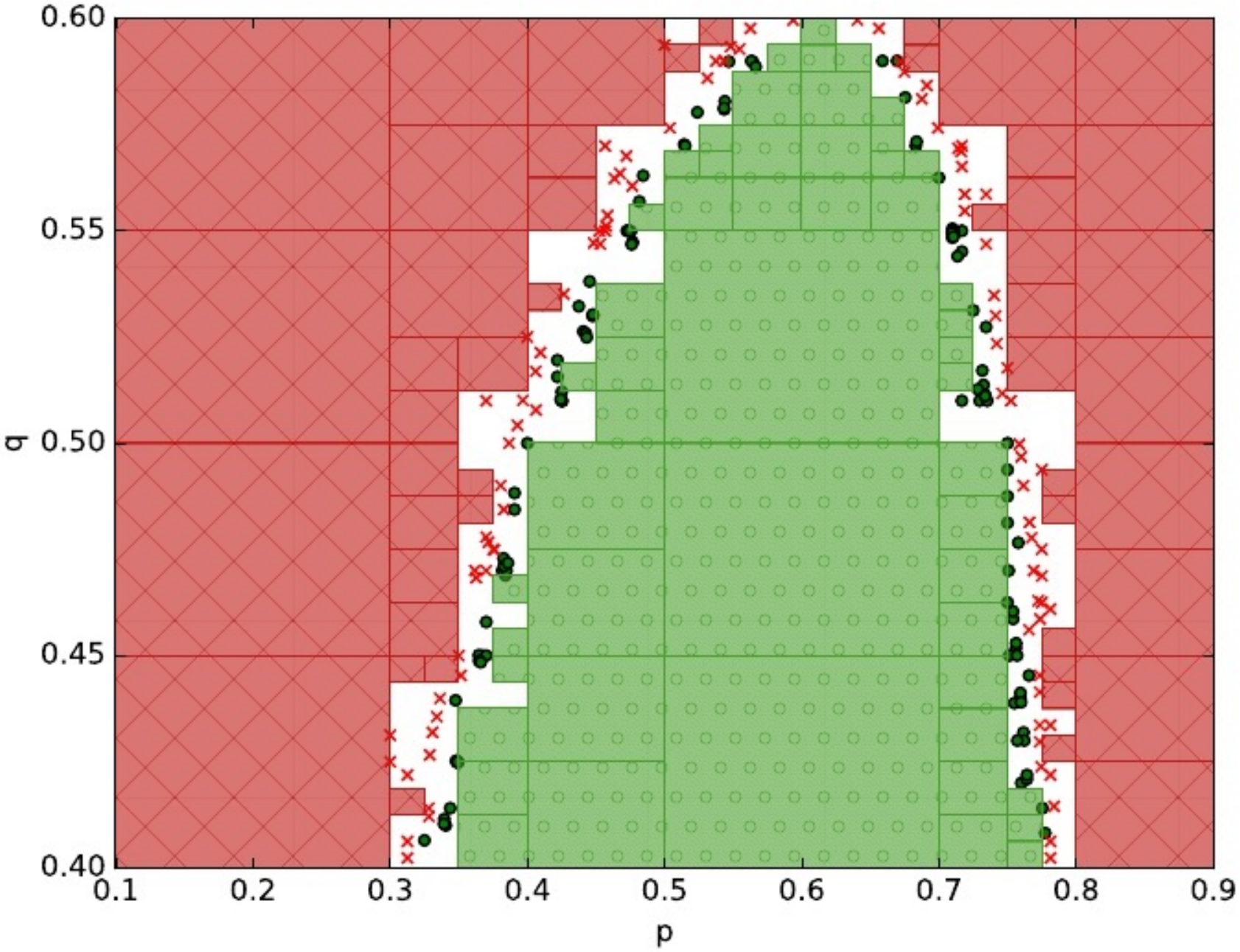

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

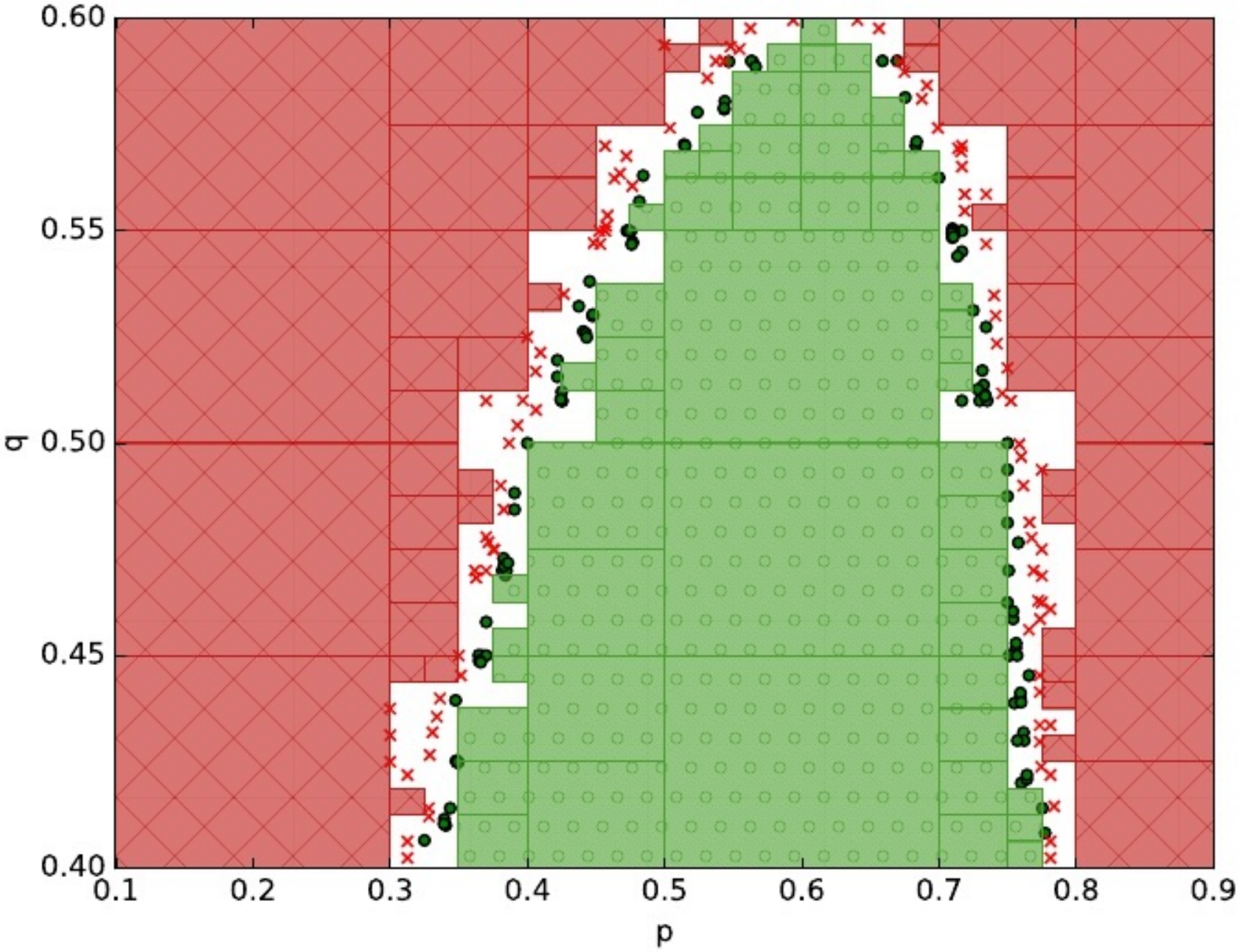

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

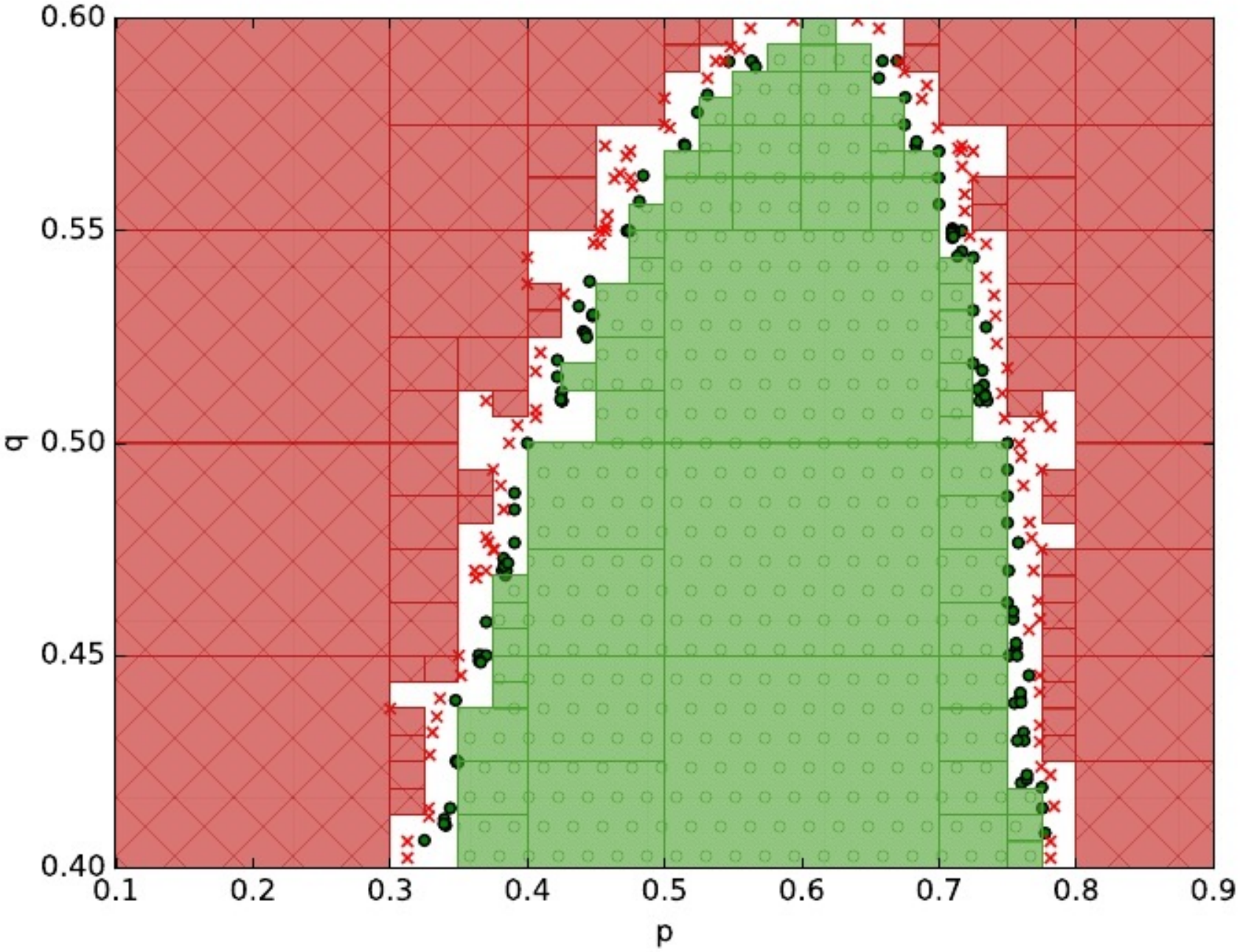

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

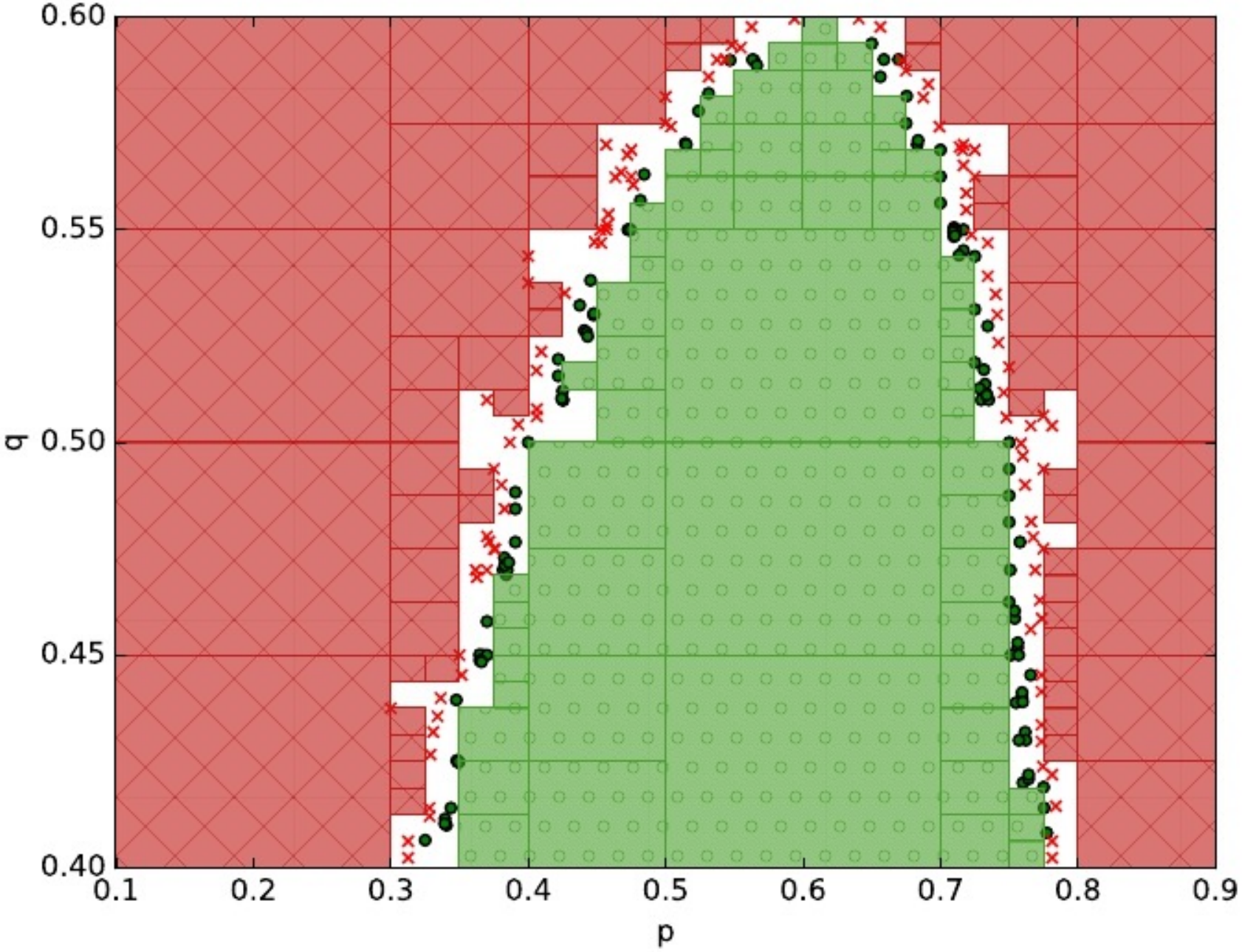

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

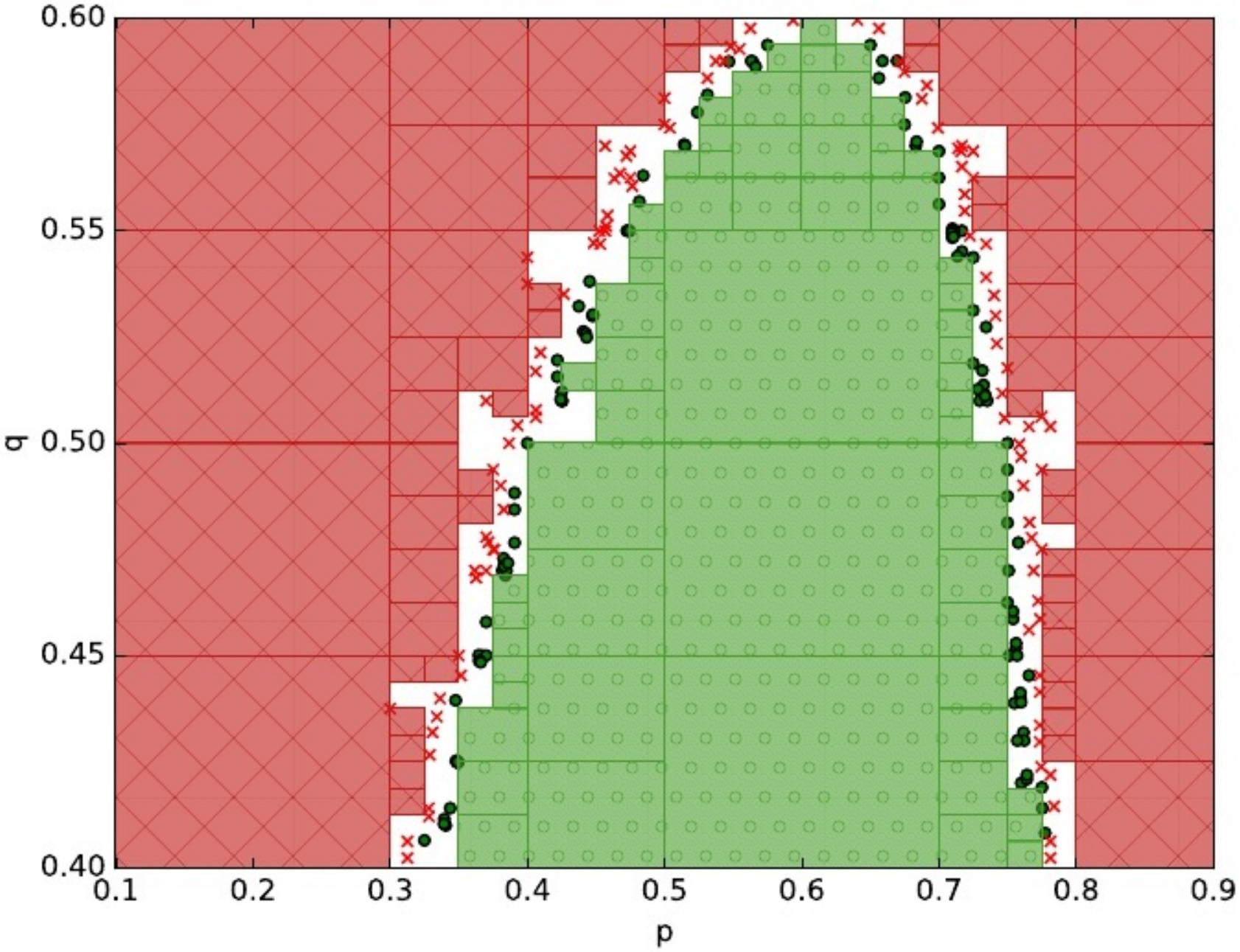

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

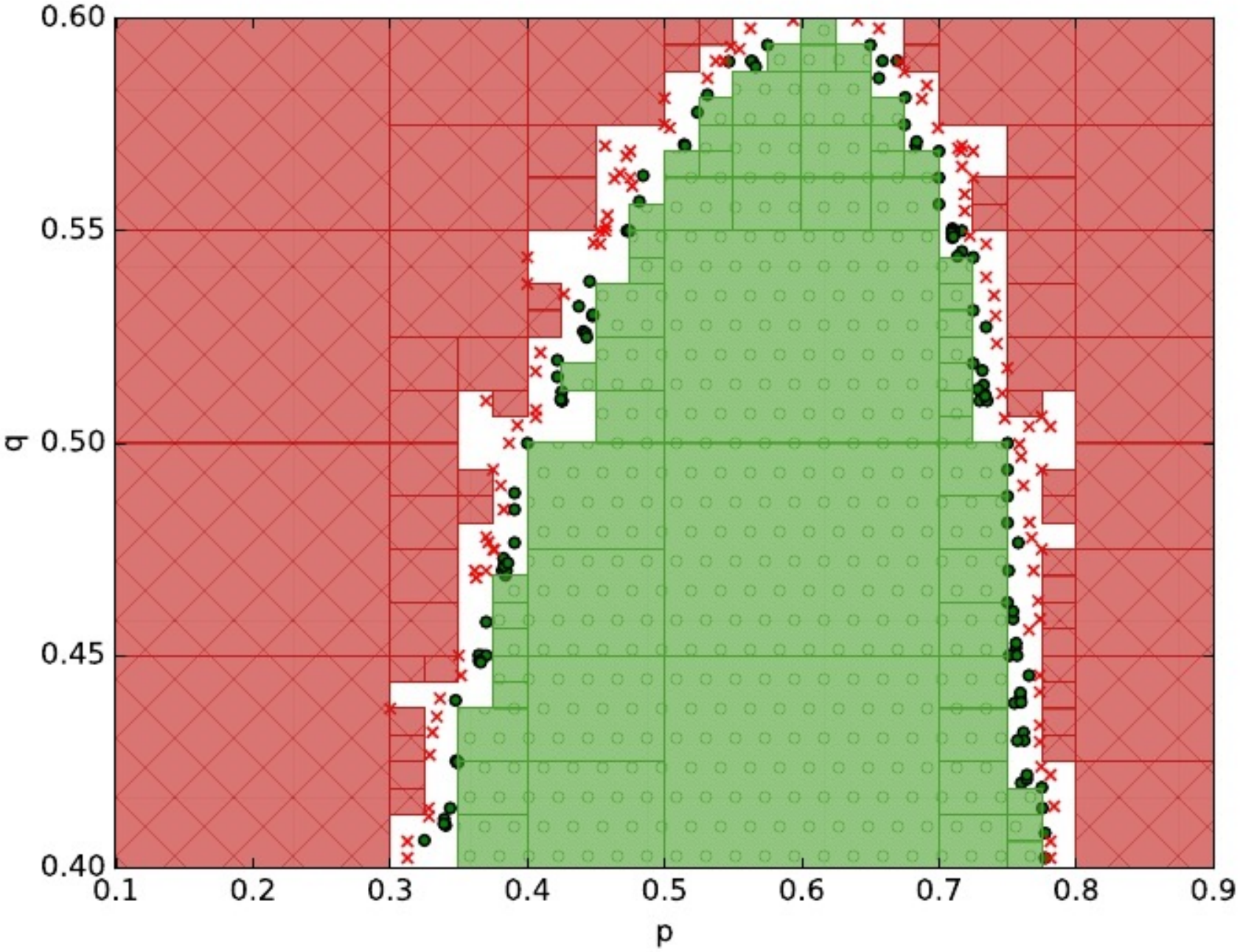

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

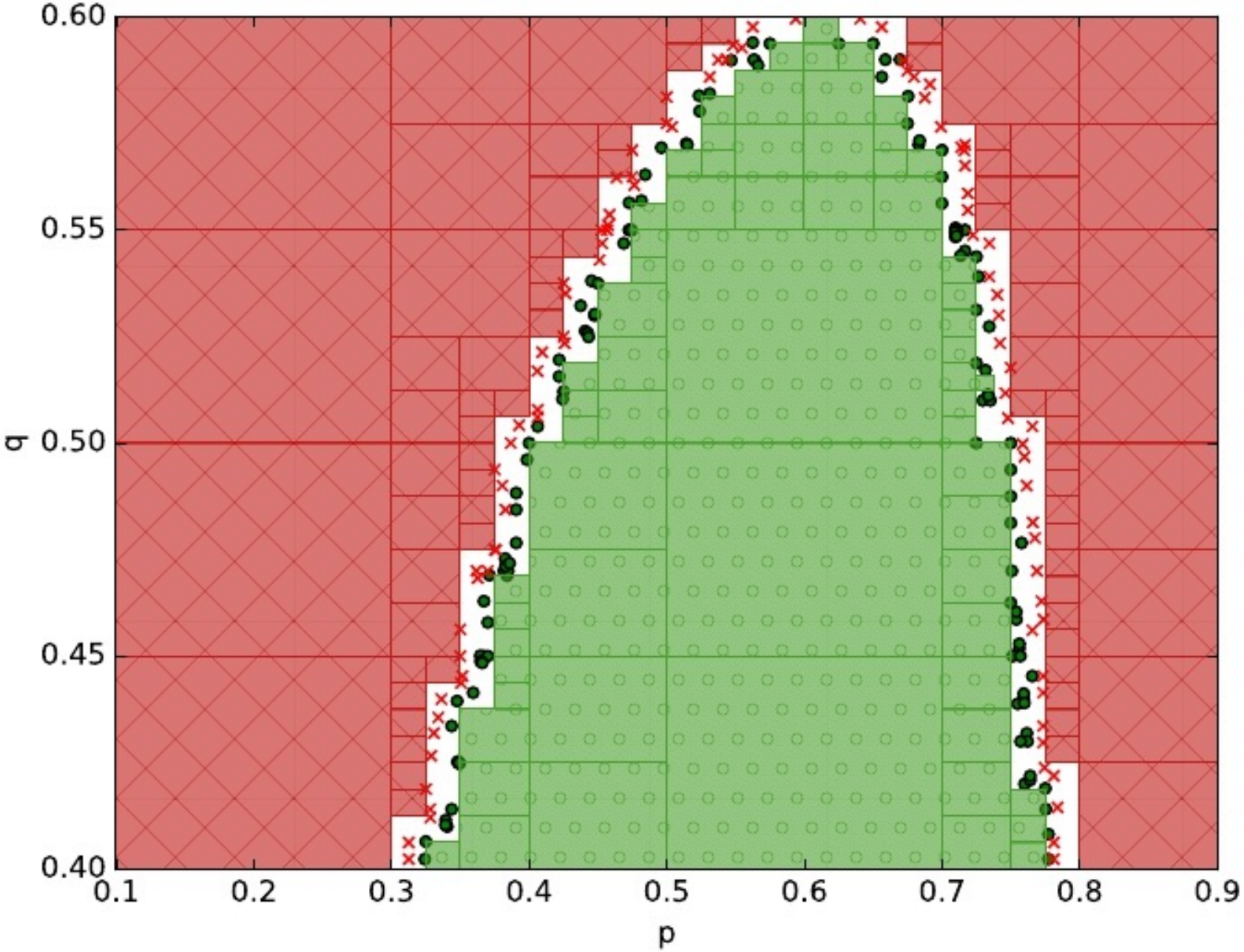

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

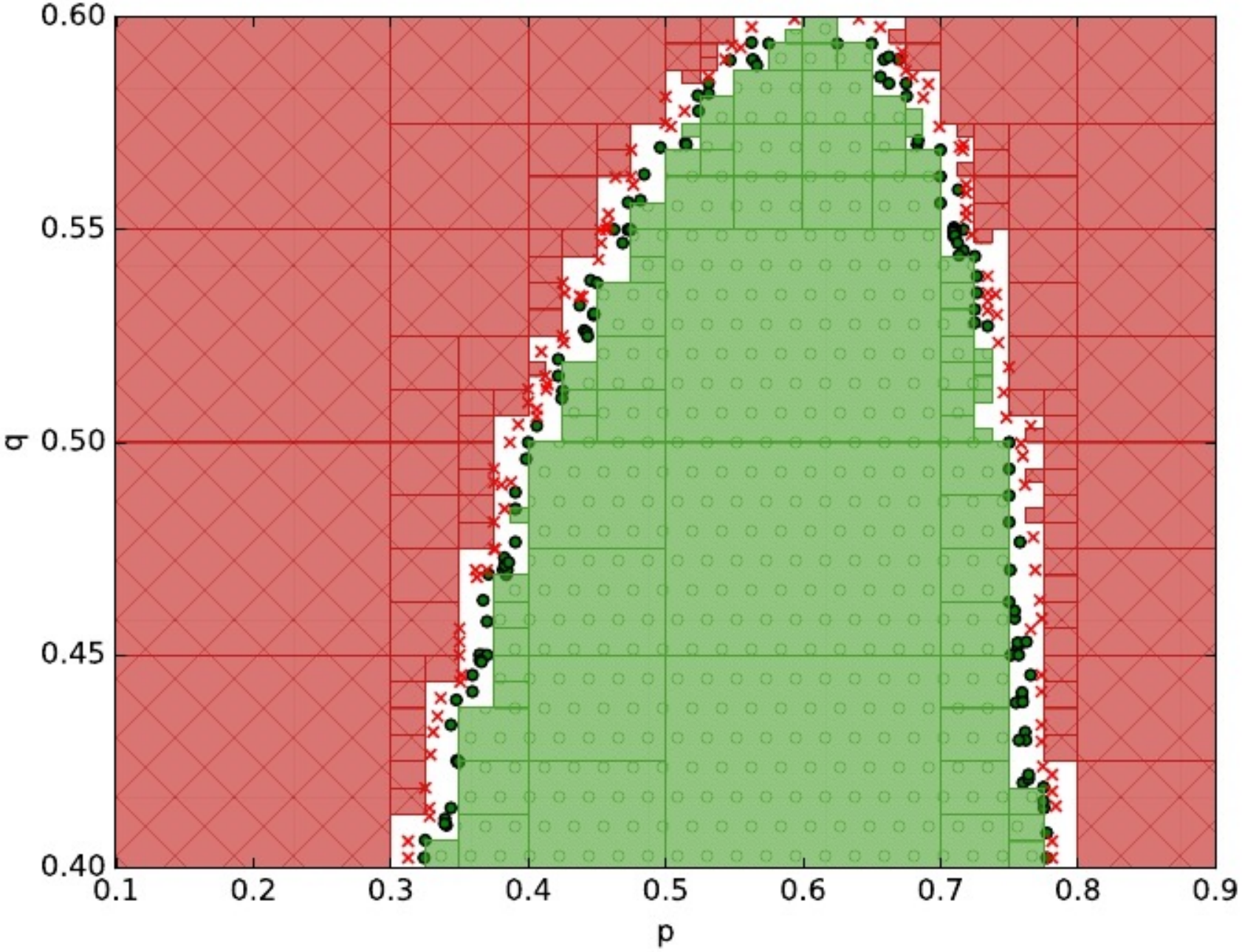

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

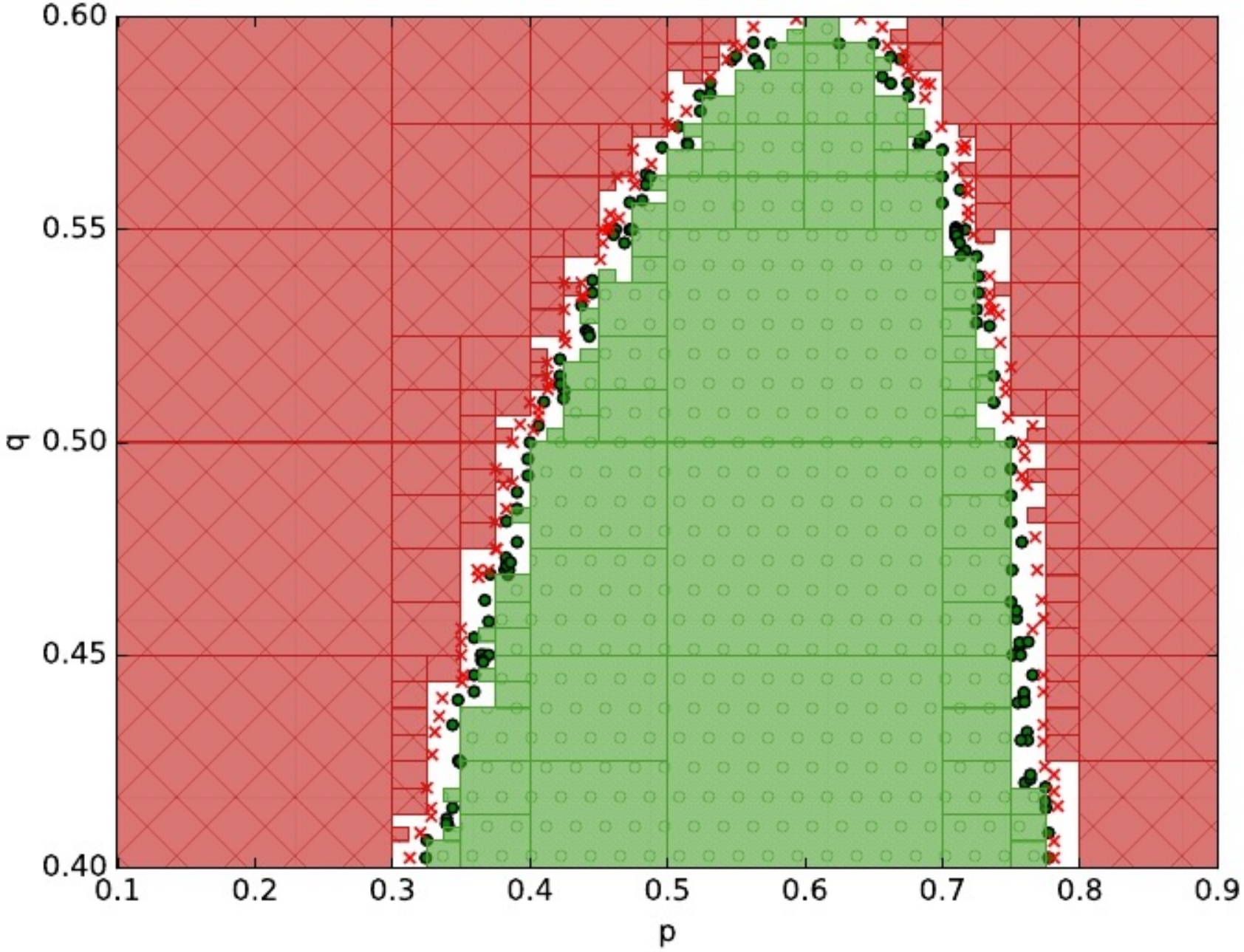

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

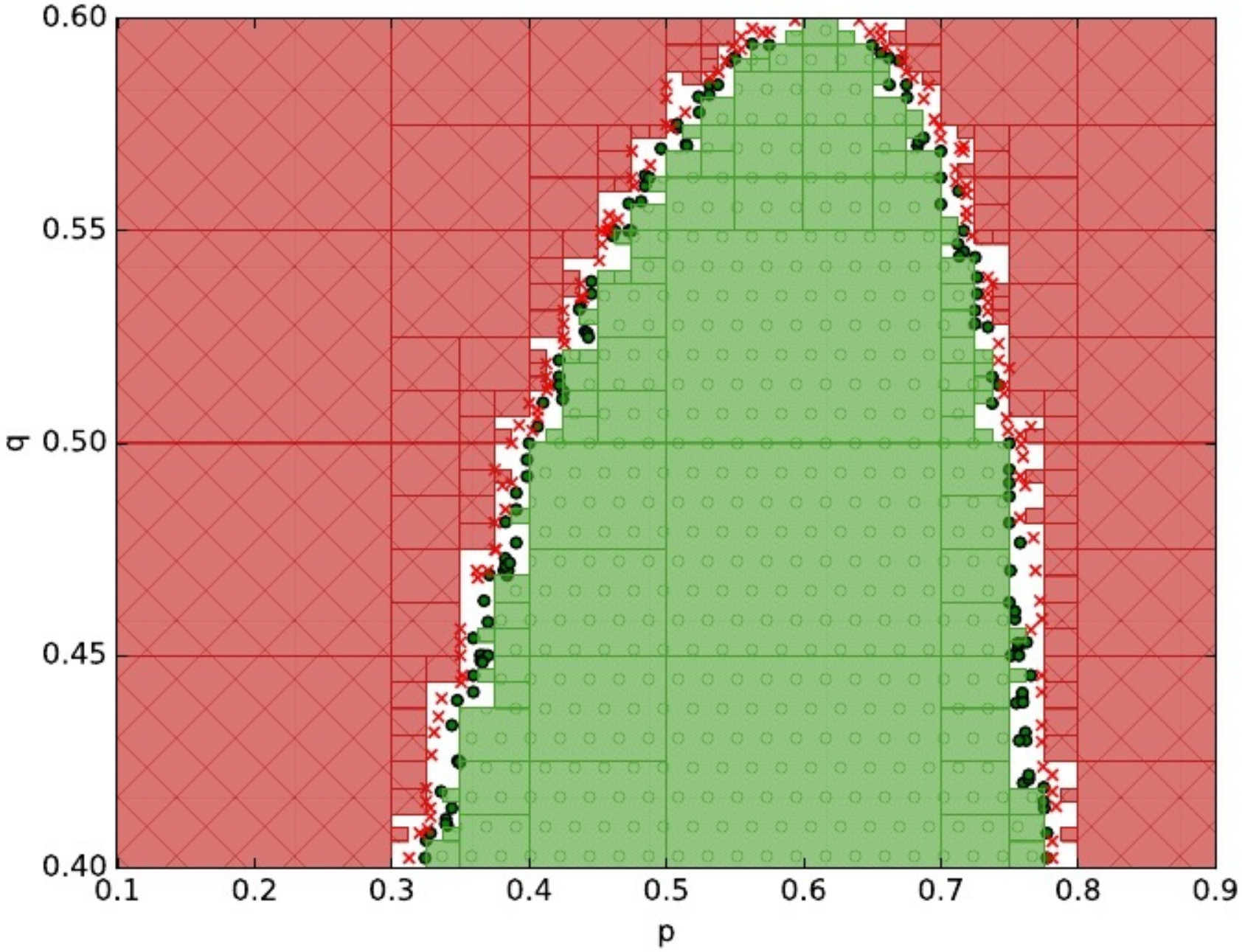

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

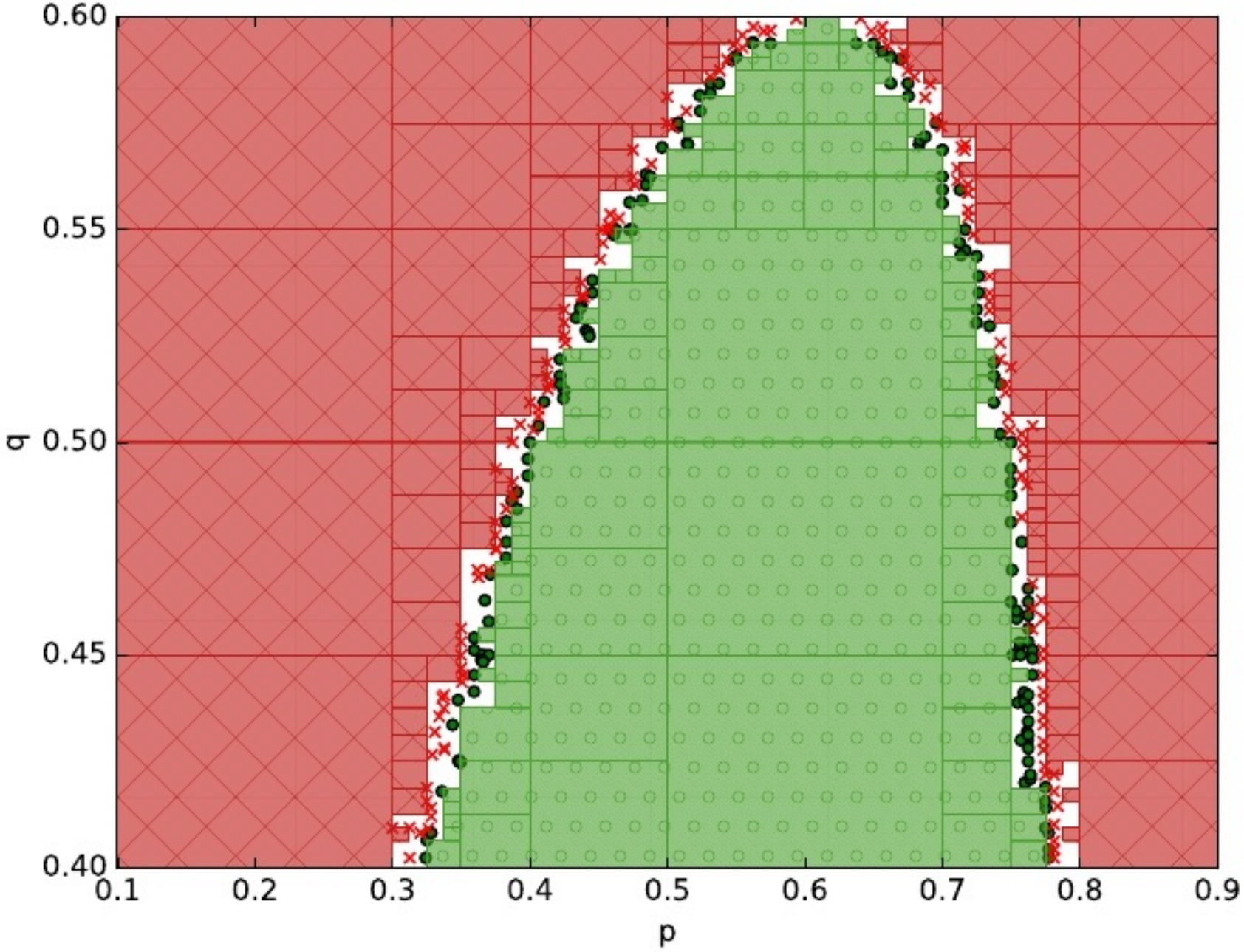

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

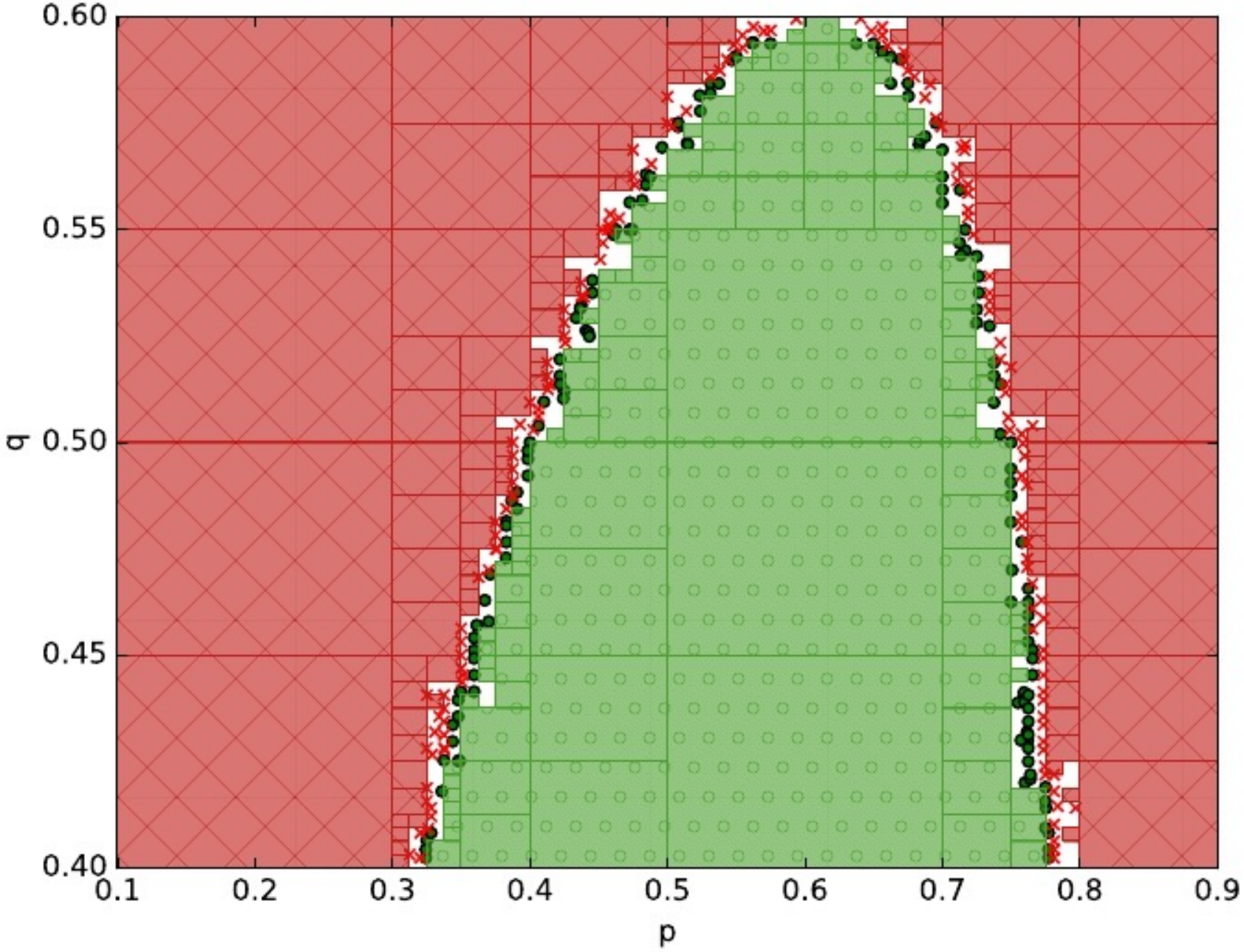

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

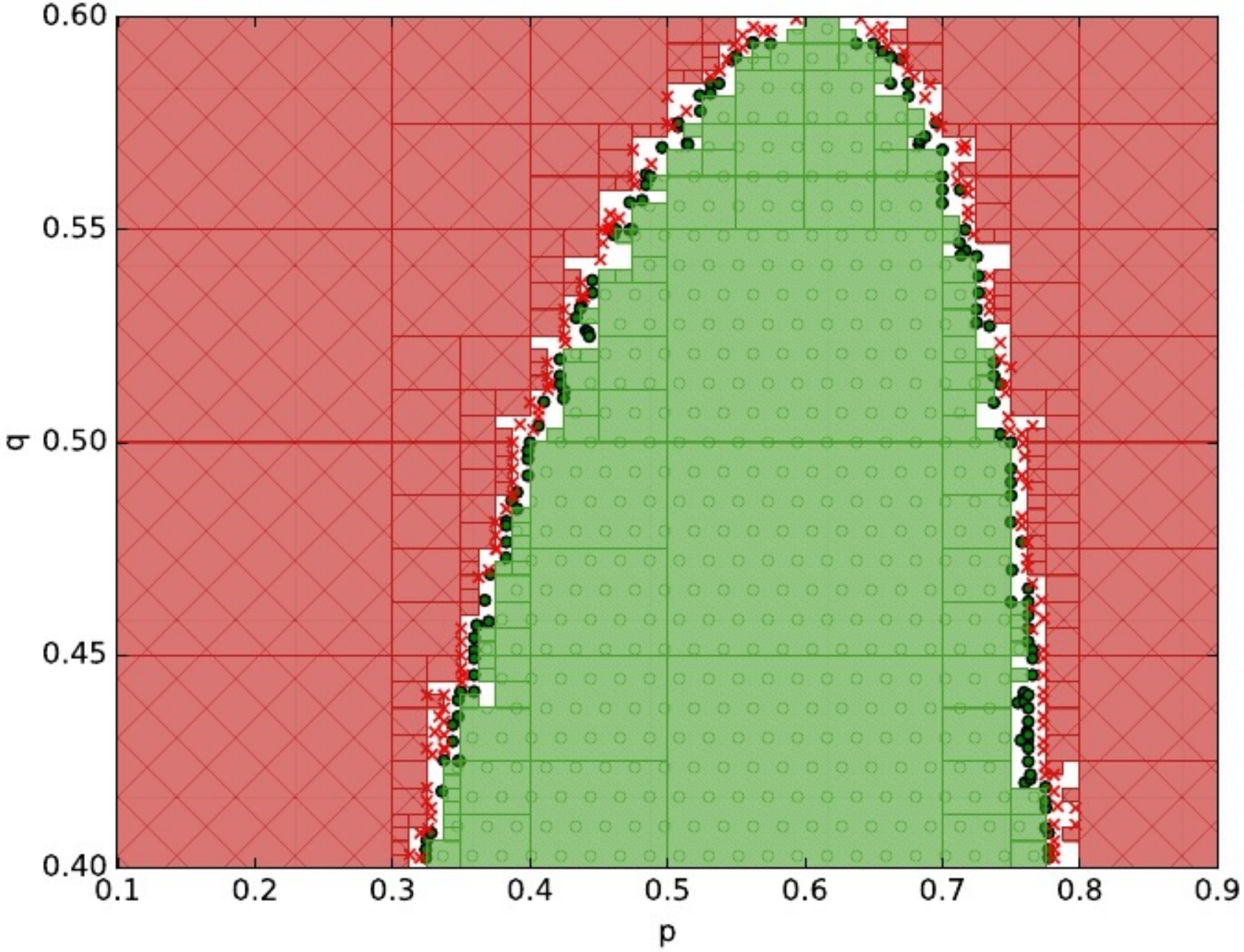

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

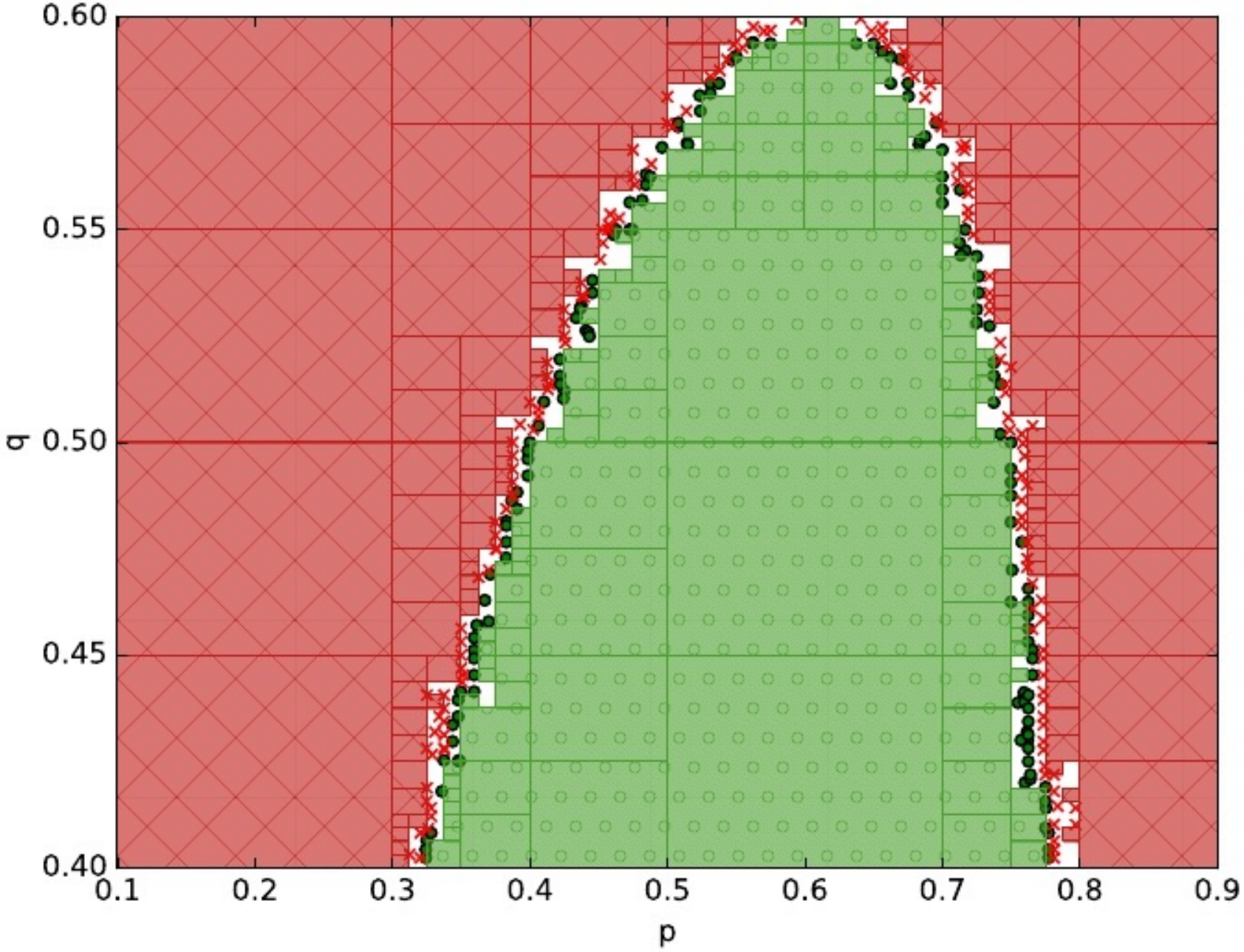

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

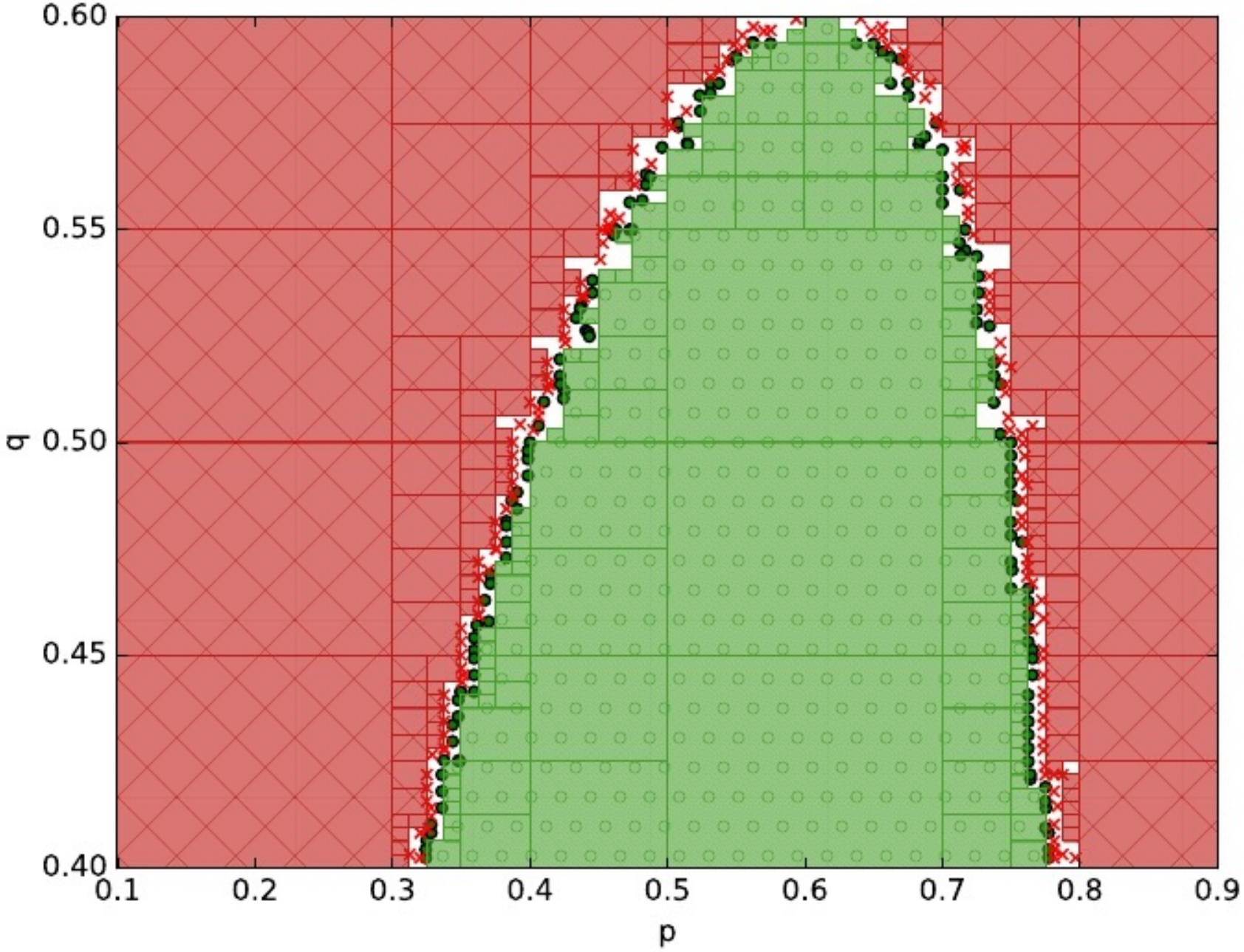

0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p


0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p


0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p


0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p


0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p


0.60
0.55
0.50
0.45
0.40
q
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
p

Rational Function

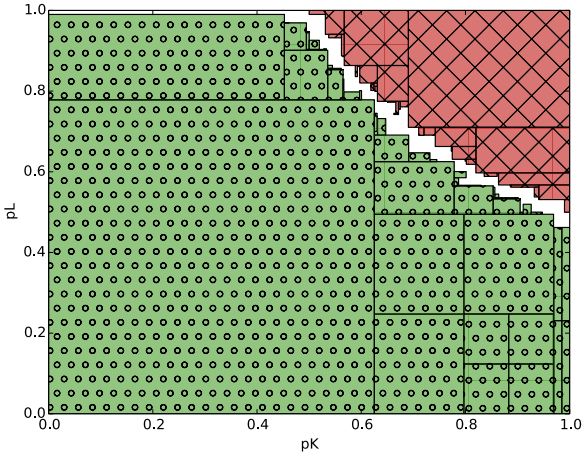
Rational Function

Nils Jansen

Radboud University

Rational Function

Output of Parameter Synthesis



Parameter Space Partitioning

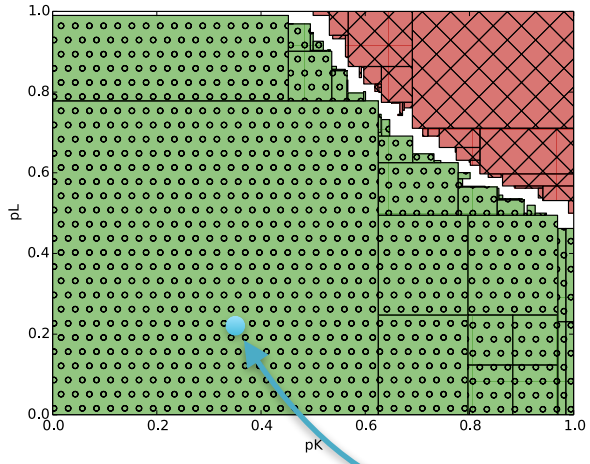
- concise description of parameter values that yield (un)satisfactory results

$$p \cdot (1-p) \cdot \frac{1-q}{1-pq}$$

Rational Function

- generalization of non-parametric model checking

Output of Parameter Synthesis



Parameter Space Partitioning

- concise description of parameter values that yield (un)satisfactory results

Feasible Solution

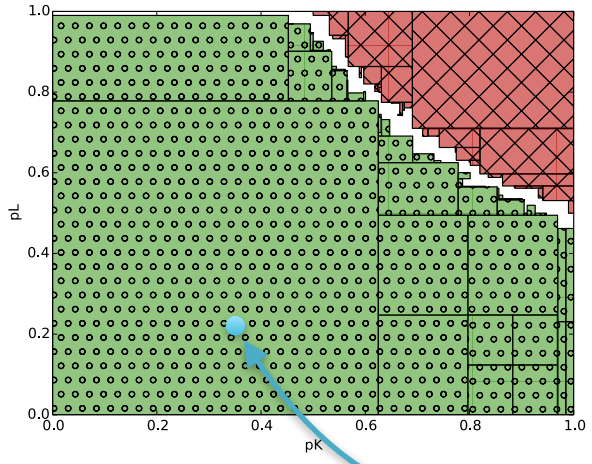
- one parameter valuation that is satisfactory

$$p \cdot (1-p) \cdot \frac{1-q}{1-pq}$$

Rational Function

- generalization of non-parametric model checking

Output of Parameter Synthesis



Parameter Space Partitioning

- concise description of parameter values that yield (un)satisfactory results

Feasible Solution

- one parameter valuation that is satisfactory

$$p \cdot (1-p) \cdot \frac{1-q}{1-pq}$$

Rational Function

- generalization of non-parametric model checking

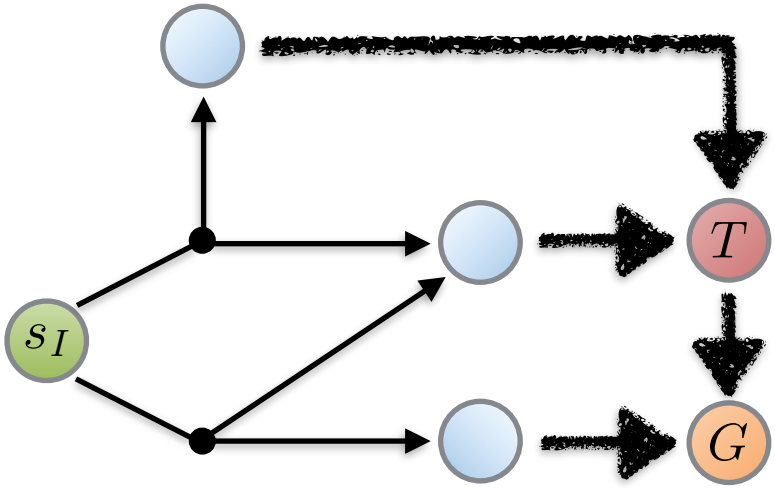
Already the feasibility problem is ETR-hard! (existential theory of the reals)

Winkler T., Junges S., Pérez, G. A., Katoen, J.-P.: On the Complexity of Reachability in Parametric Markov Decision Processes

Outline

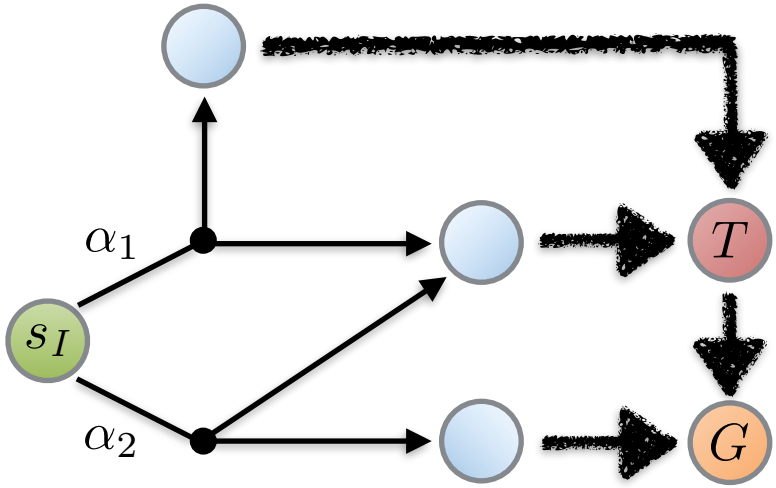
1. Parameter Synthesis for Markov Chains
2. Parameter Synthesis for MDPs
3. Sequential Convex Optimization
4. The Convex-concave Procedure
5. Numerical Examples

Formal Setting — Parametric MDP



States S

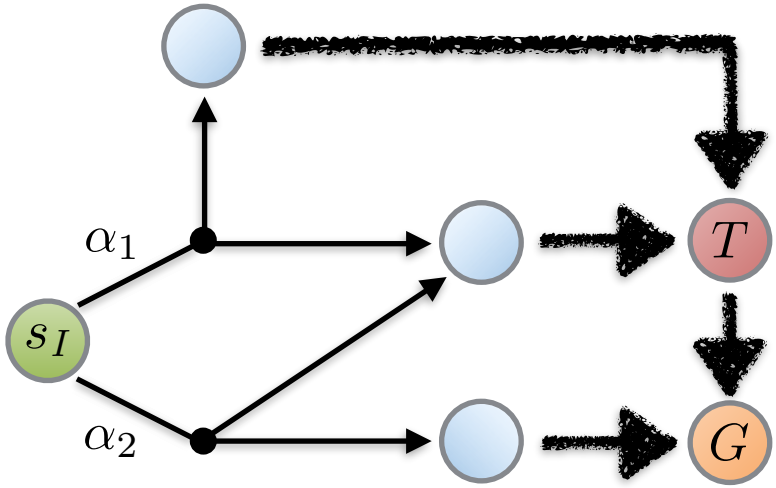
Formal Setting — Parametric MDP



States S

Actions $\alpha_1, \alpha_2 \in A$

Formal Setting — Parametric MDP

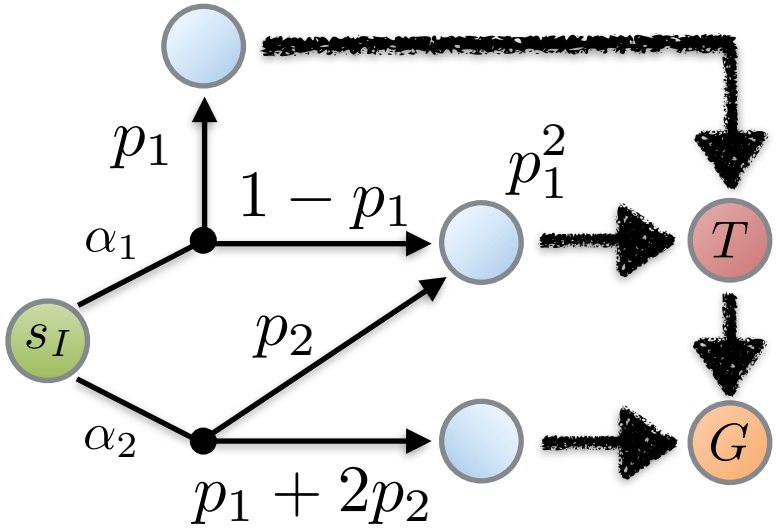


States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Formal Setting — Parametric MDP



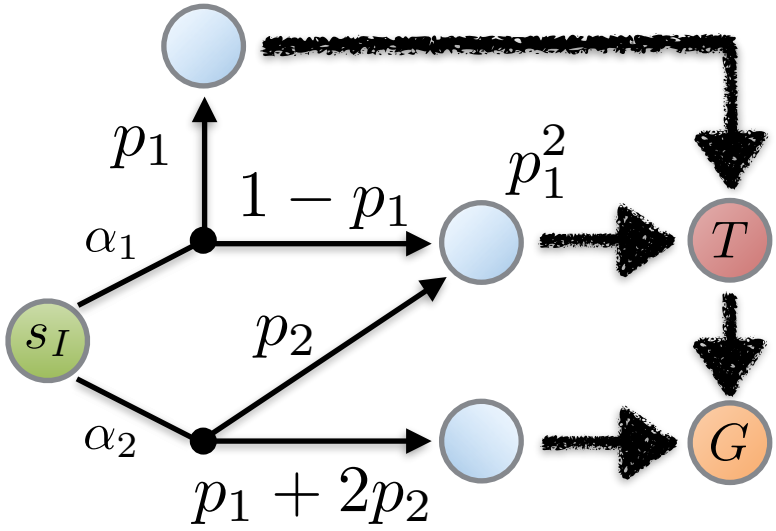
States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Formal Setting — Parametric MDP



States S

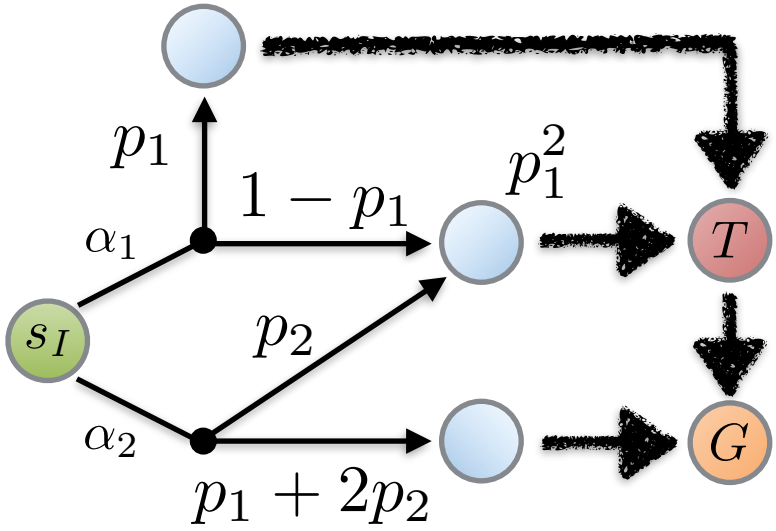
Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

State elimination not applicable.

Formal Setting — Parametric MDP



States S

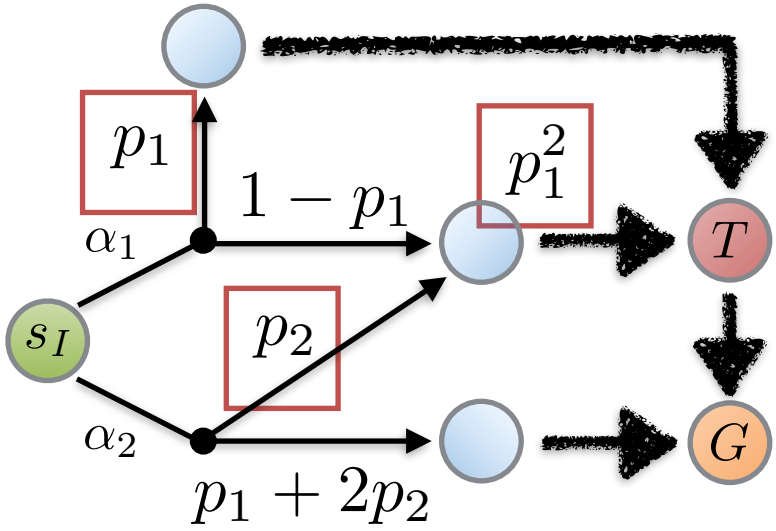
Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Types of functions

Formal Setting — Parametric MDP



States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

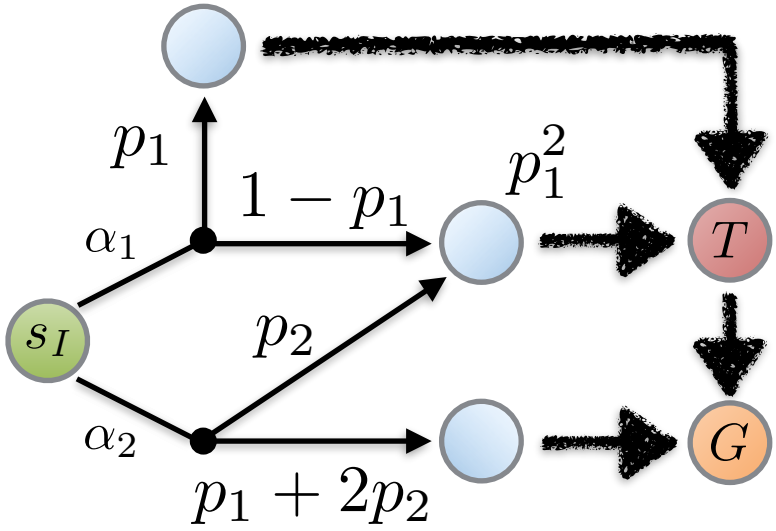
Probability and cost functions over parameters

Types of functions

$g = c \cdot p_1^{a_1} \cdots p_n^{a_n}$ with

$a_i \in \mathbb{R}, c, p_i \in \mathbb{R}_{>0}$: Monomial

Formal Setting — Parametric MDP



States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

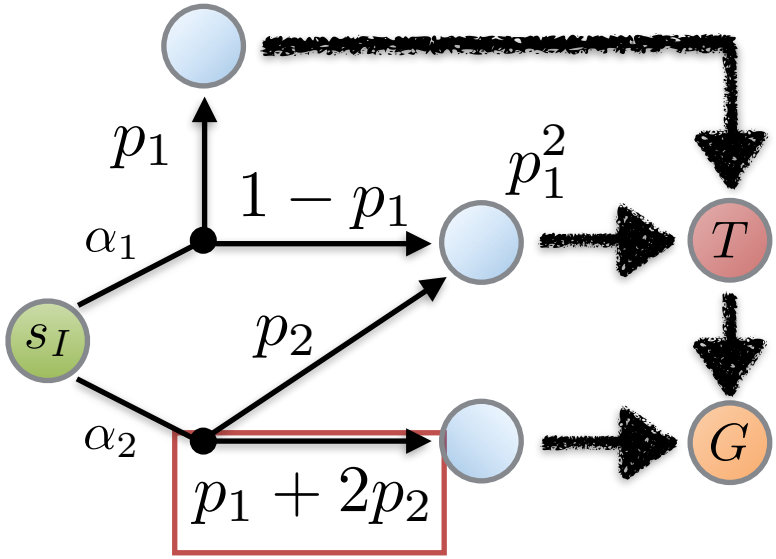
Types of functions

$g = c \cdot p_1^{a_1} \cdots p_n^{a_n}$ with

$a_i \in \mathbb{R}, c, p_i \in \mathbb{R}_{>0}$: Monomial

$$f = \sum_{k=1}^{K} c_k \cdot p_1^{a_{1k}} \cdots p_n^{a_{nk}}$$

Formal Setting — Parametric MDP



States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Types of functions

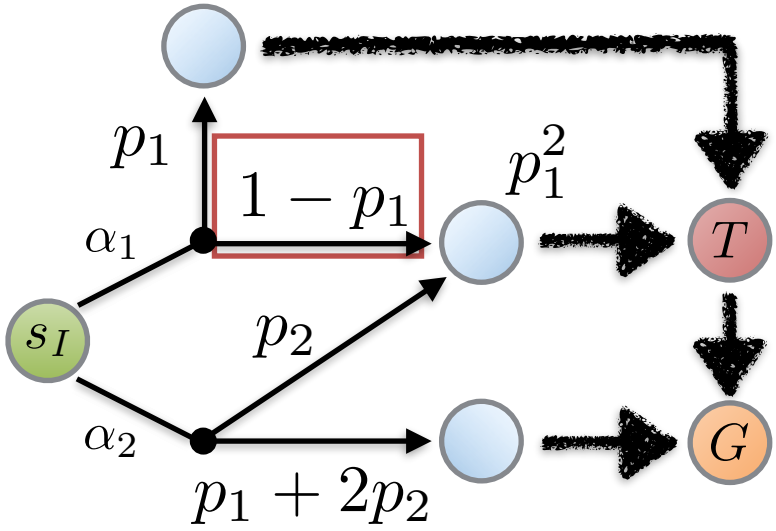
$g = c \cdot p_1^{a_1} \cdots p_n^{a_n}$ with

$a_i \in \mathbb{R}, c, p_i \in \mathbb{R}_{>0}$: Monomial

$$f = \sum_{k=1}^{K} c_k \cdot p_1^{a_{1k}} \cdots p_n^{a_{nk}}$$

$c_k \in \mathbb{R}_{>0}$: Posynomial

Formal Setting — Parametric MDP



States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Types of functions

$g = c \cdot p_1^{a_1} \cdots p_n^{a_n}$ with

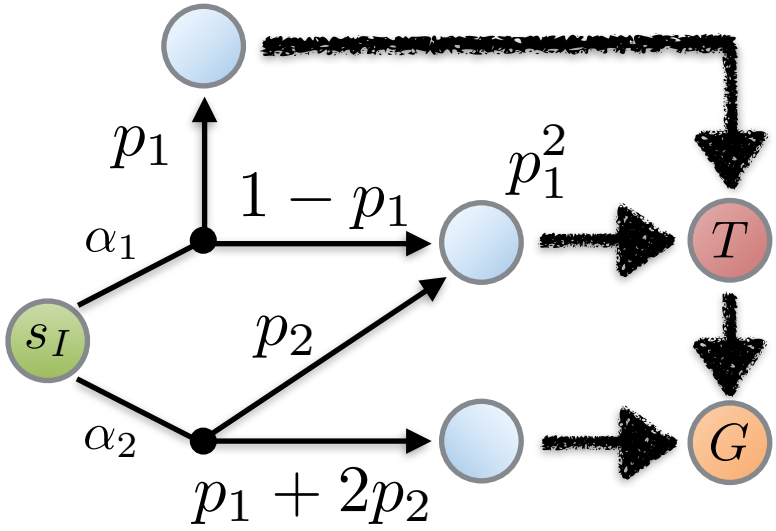
$a_i \in \mathbb{R}, c, p_i \in \mathbb{R}_{>0}$: Monomial

$$f = \sum_{k=1}^{K} c_k \cdot p_1^{a_{1k}} \cdots p_n^{a_{nk}}$$

$c_k \in \mathbb{R}_{>0}$: Posynomial

$c_k \in \mathbb{R}$: Signomial

Formal Problem — Parameter Synthesis



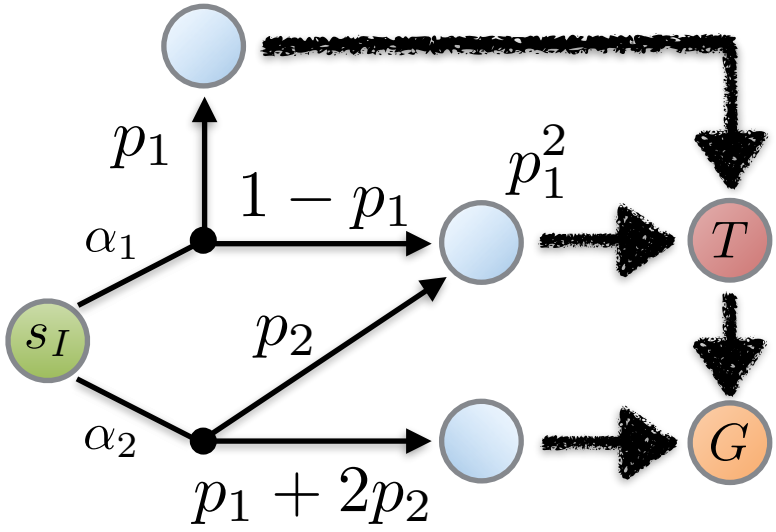
Safety specification

$$\varphi = \mathbb{P}_{\leq\lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

$$\psi = \mathbb{E}_{\leq\kappa}(\Diamond G),\ G \subseteq S$$

Formal Problem — Parameter Synthesis



Safety specification

$$\varphi = \mathbb{P}_{\leq\lambda}(\Diamond T),\ T \subseteq S$$

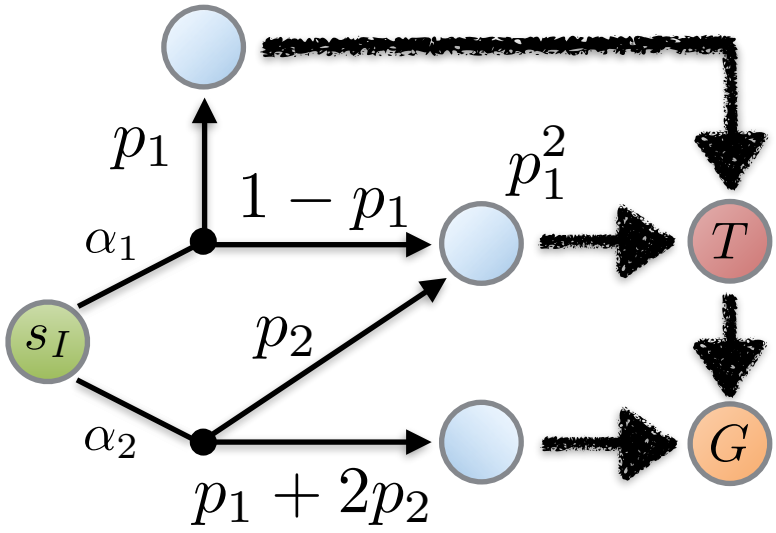
Performance specification

$$\psi = \mathbb{E}_{\leq\kappa}(\Diamond G),\ G \subseteq S$$

Given pMDP $\mathcal{M}$, find a well-defined valuation of parameters and a scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

$$\mathcal{M}^{\sigma} \models \varphi \wedge \psi$$

Formal Problem — Parameter Synthesis



Safety specification

$$\varphi = \mathbb{P}_{\leq\lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

$$\psi = \mathbb{E}_{\leq\kappa}(\Diamond G),\ G \subseteq S$$

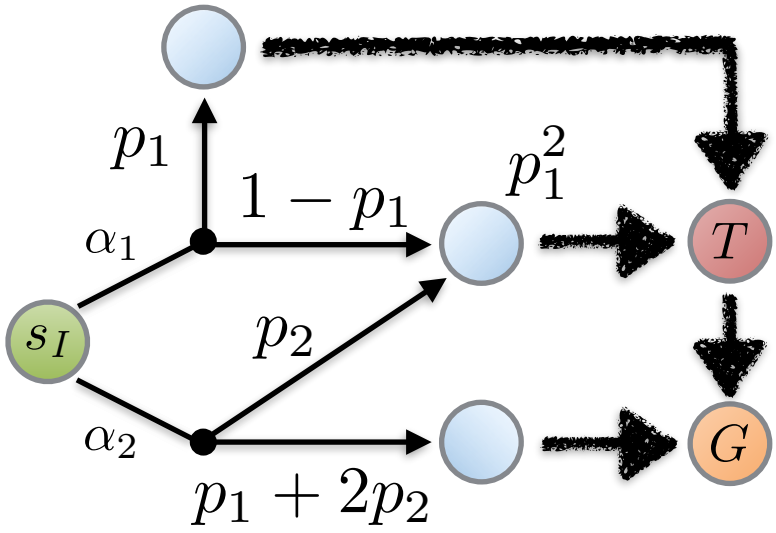
Objective function $f : V \to \mathbb{R}$

Given pMDP $\mathcal{M}$, find a well-defined valuation of parameters and a scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

$$\mathcal{M}^{\sigma} \models \varphi \wedge \psi$$

and value for objective function $f : V \to \mathbb{R}$ is minimal.

Formal Problem — Parameter Synthesis



Safety specification

$$\varphi = \mathbb{P}_{\leq\lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

$$\psi = \mathbb{E}_{\leq\kappa}(\Diamond G),\ G \subseteq S$$

Objective function $f : V \to \mathbb{R}$

Given pMDP $\mathcal{M}$, find a well-defined valuation of parameters and a scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

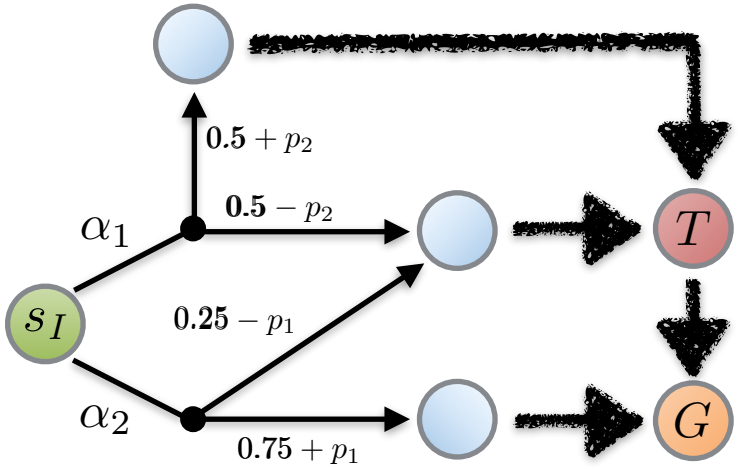
$$\mathcal{M}^\sigma \models \varphi \wedge \psi$$

feasibility

optimality

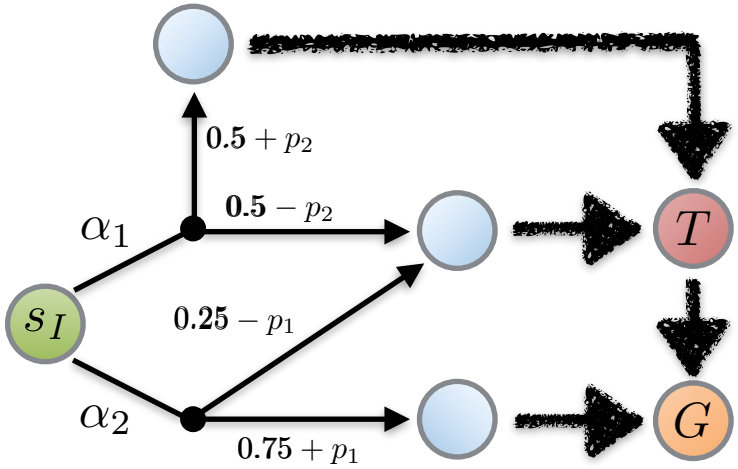
and value for objective function $f : V \to \mathbb{R}$ is minimal.

Direct Application — Model Repair



Concrete MDP, minimal probability to reach T is 0.25

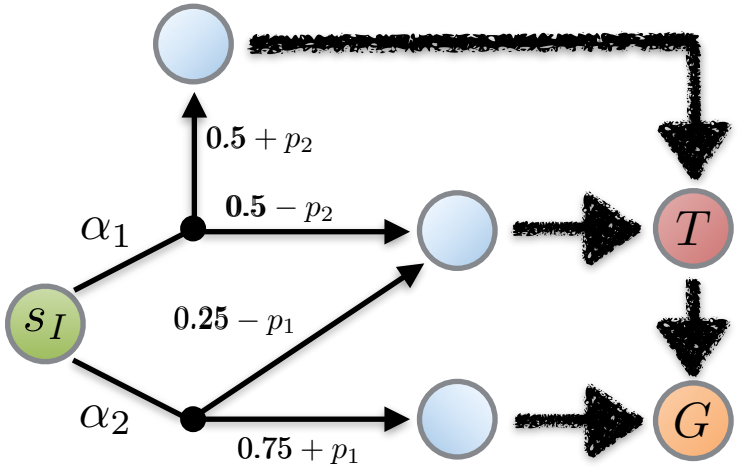
Direct Application — Model Repair



Concrete MDP, minimal probability to reach T is 0.25

Repair the MDP such that the minimal probability is 0.2

Direct Application — Model Repair

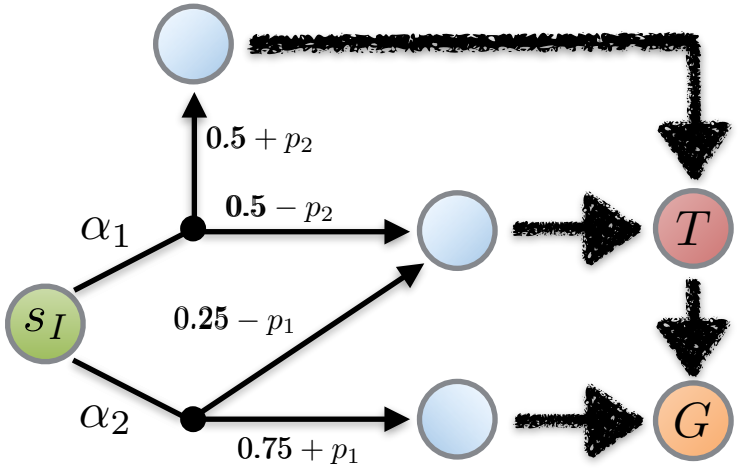


Concrete MDP, minimal probability to reach T is 0.25

Repair the MDP such that the minimal probability is 0.2

Introduce parameters to adjust transition probabilities

Direct Application — Model Repair



Cost function:

$$\min p_1^2 + p_2^2$$

Concrete MDP, minimal probability to reach T is 0.25

Repair the MDP such that the minimal probability is 0.2

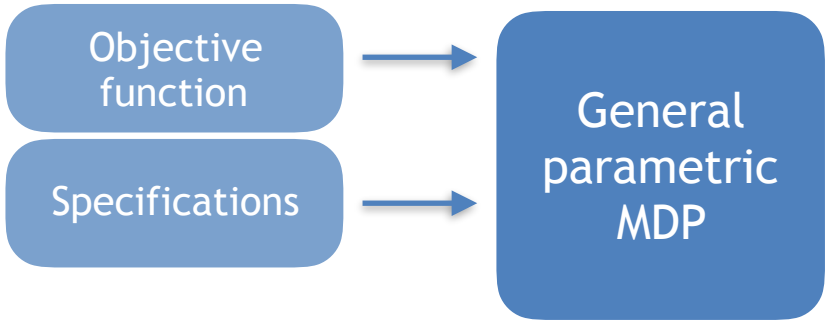
Introduce parameters to adjust transition probabilities

Find parameter valuation such that cost is minimal

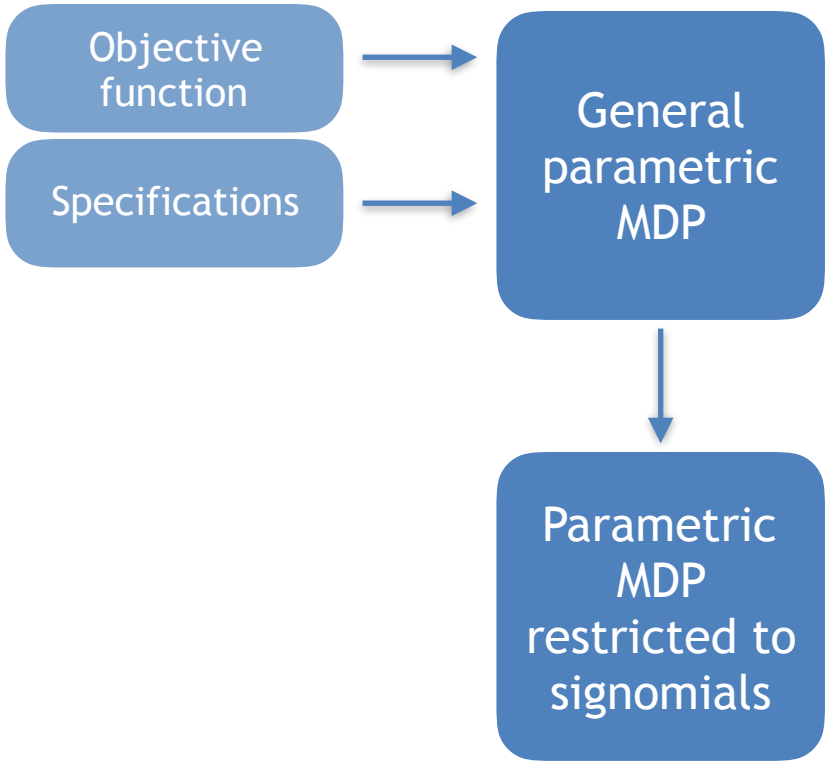
Outline

1. Parameter Synthesis for Markov Chains
2. Parameter Synthesis for MDPs
3. Sequential Convex Optimization
4. The Convex-concave Procedure
5. Numerical Examples

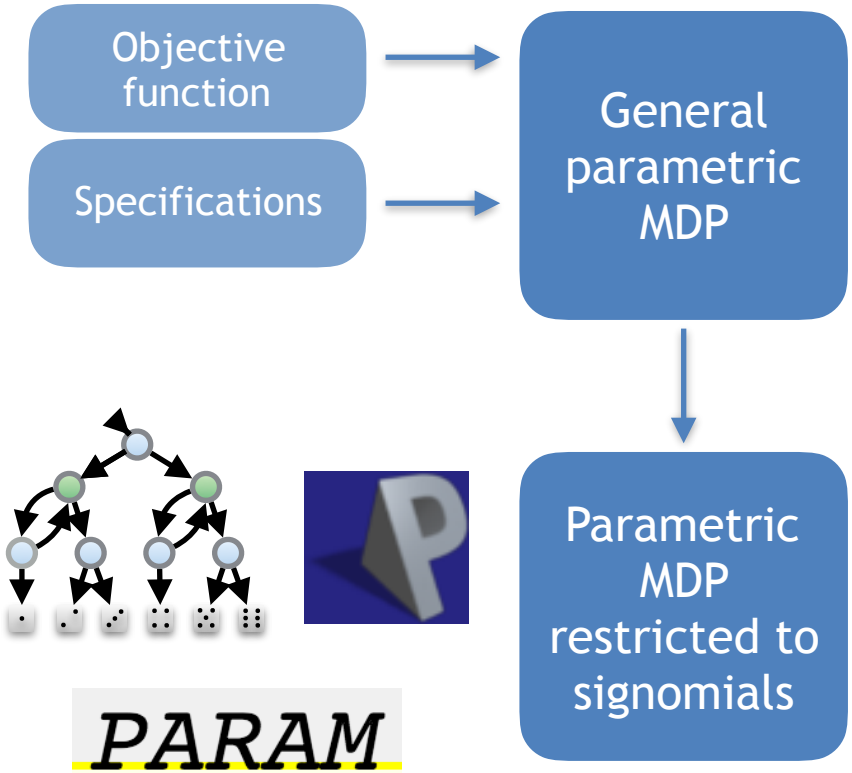
Story - Sequential Convex Optimization



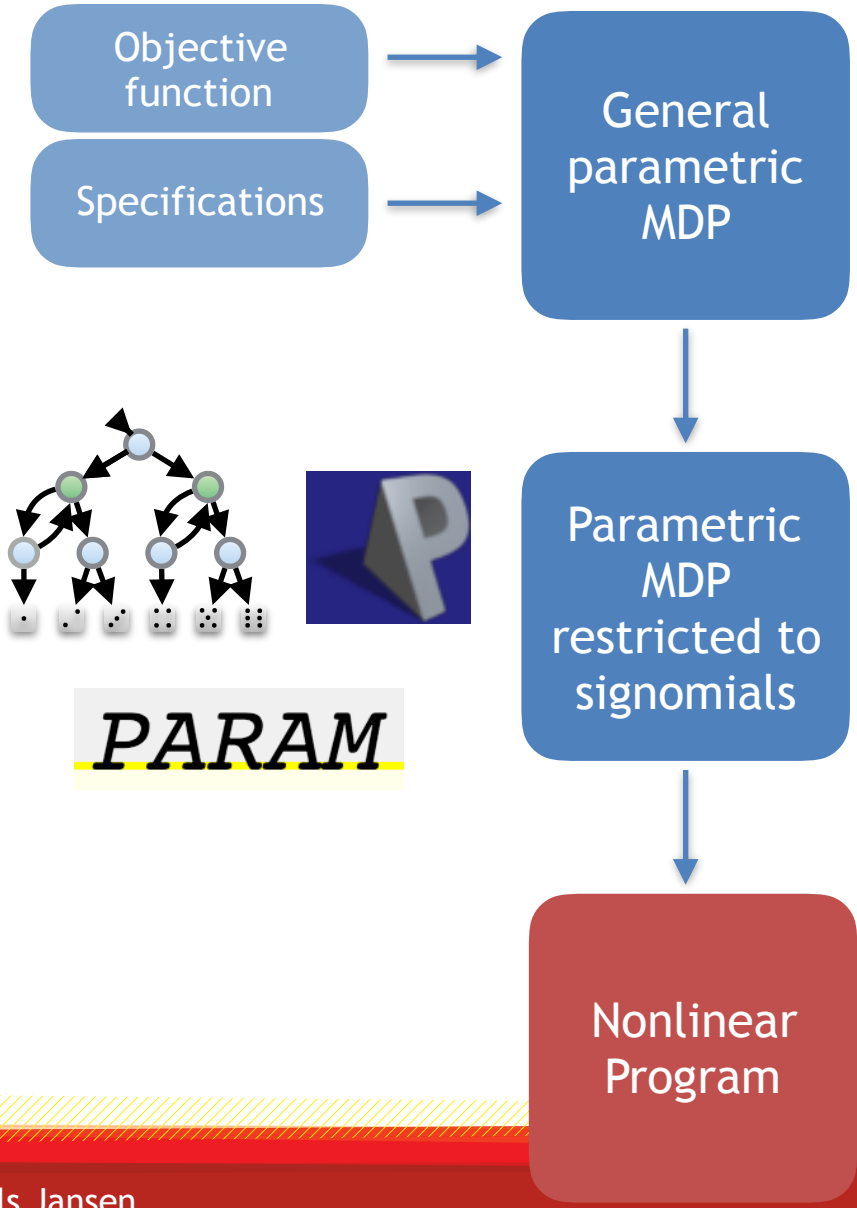
Story - Sequential Convex Optimization



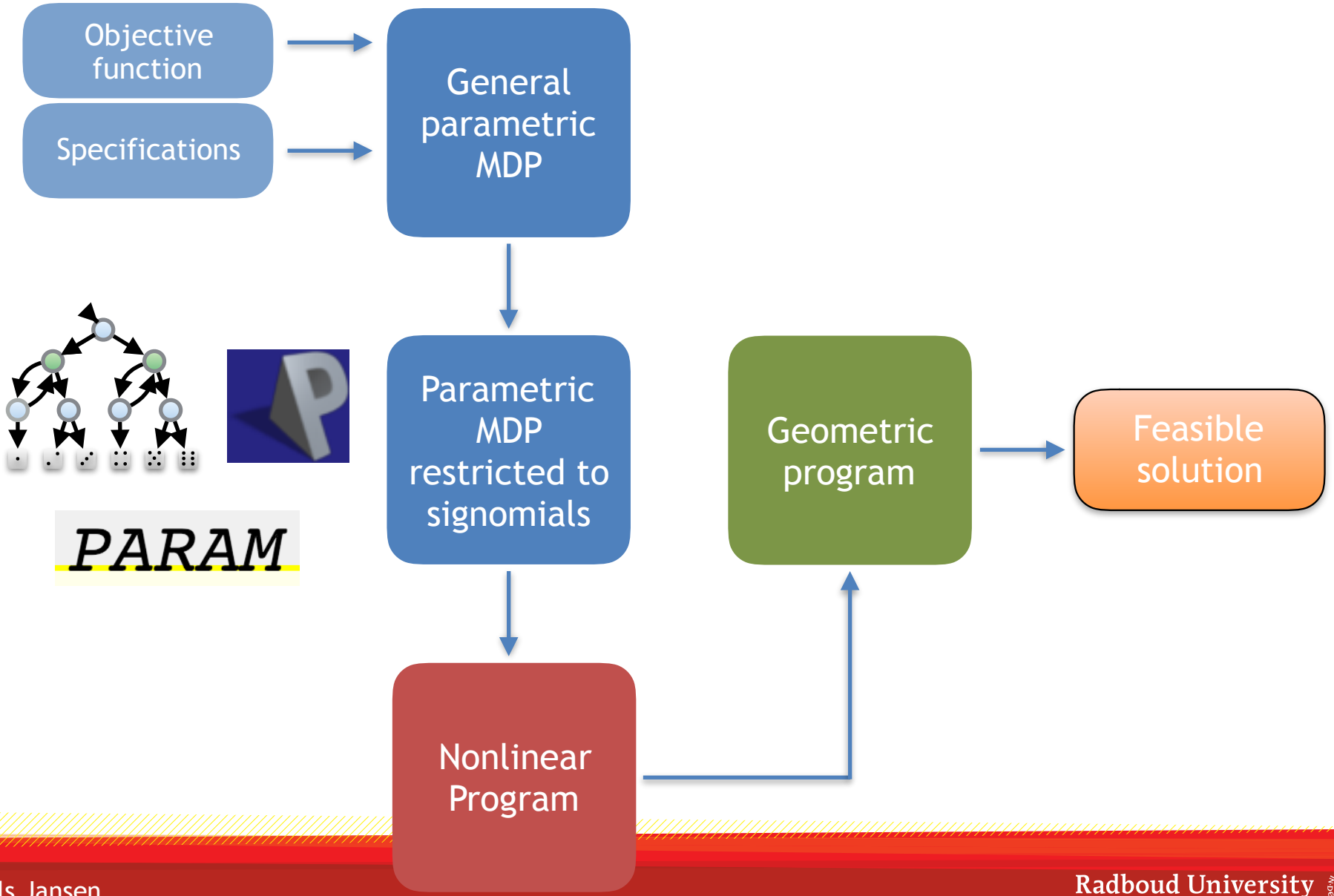
Story - Sequential Convex Optimization



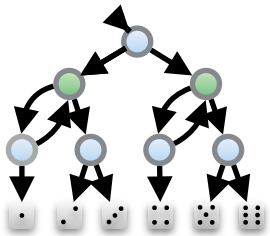
Story - Sequential Convex Optimization



Story - Sequential Convex Optimization



Story - Sequential Convex Optimization



PARAM

Objective function

Specifications

General parametric MDP

Parametric MDP restricted to signomials

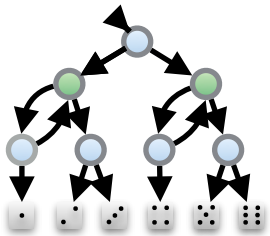
Nonlinear Program

Geometric program

Feasible solution

convex, interior point methods can be used

Story - Sequential Convex Optimization



PARAM

Objective function

Specifications

General parametric MDP

Parametric MDP restricted to signomials

Nonlinear Program

Geometric program

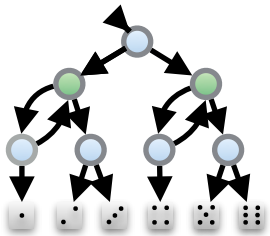
Sequential convex programming

Local optimum

Feasible solution

convex, interior point methods can be used

Story - Sequential Convex Optimization



PARAM

Objective function

Specifications

General parametric MDP

Parametric MDP restricted to signomials

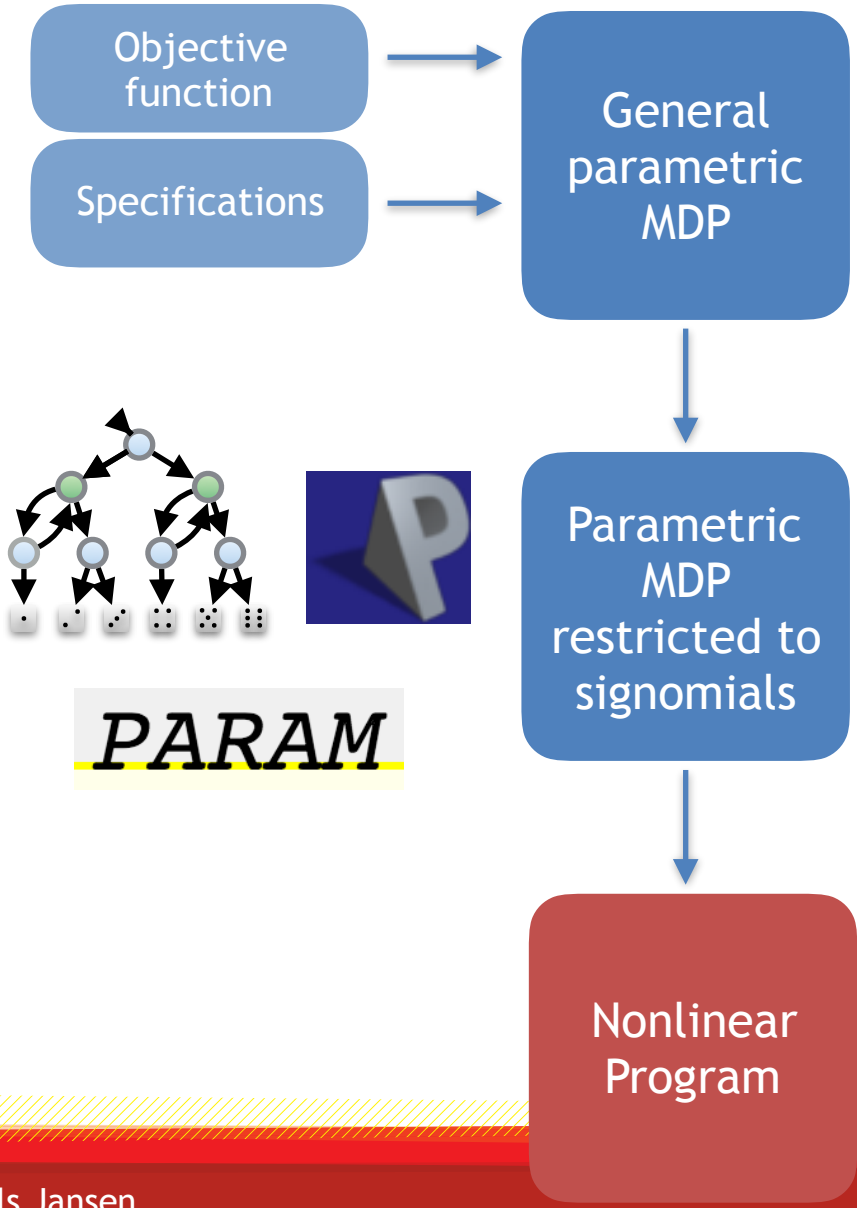
Nonlinear Program

Geometric program

Feasible solution

convex, interior point methods can be used

Story - Sequential Convex Optimization



Nonlinear Programming

$$
\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & \\
\forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\
\forall j. 1 \leq i \leq p \quad & h_j = 1
\end{aligned}
$$

Nonlinear Programming

$$
\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & \\
\forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\
\forall j. 1 \leq i \leq p \quad & h_j = 1
\end{aligned}
$$

f, g_i, h_j arbitrary functions: general nonlinear program

- finding feasible or optimal solution is NP hard
- only applicable for small problem instances

Nonlinear Programming

$$
\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & \\
\forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\
\forall j. 1 \leq i \leq p \quad & h_j = 1
\end{aligned}
$$

f, g_i, h_j arbitrary functions: general nonlinear program

- finding feasible or optimal solution is NP hard
- only applicable for small problem instances

f, g_i, h_j signomials: signomial program

- if feasible solution is found, local optima can be computed efficiently

Nonlinear Programming

$$
\begin{aligned}
&\text{minimize} && f \\
&\text{subject to} && \\
&\forall i. 1 \leq i \leq m && g_i \leq 1 \\
&\forall j. 1 \leq i \leq p && h_j = 1
\end{aligned}
$$

f, g_i, h_j arbitrary functions: general nonlinear program

- finding feasible or optimal solution is NP hard
- only applicable for small problem instances

$1 - p_1$

f, g_i, h_j signomials: signomial program

- if feasible solution is found, local optima can be computed efficiently

Nonlinear Programming

$$
\begin{aligned}
&\text{minimize} && f \\
&\text{subject to} && \\
&\forall i. 1 \leq i \leq m && g_i \leq 1 \\
&\forall j. 1 \leq i \leq p && h_j = 1
\end{aligned}
$$

f, g_i, h_j arbitrary functions: general nonlinear program

- finding feasible or optimal solution is NP hard
- only applicable for small problem instances

$1 - p_1$

f, g_i, h_j signomials: signomial program

- if feasible solution is found, local optima can be computed efficiently

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Nonlinear Programming

$$\begin{aligned} &\text{minimize} && f \\ &\text{subject to} && \\ &\forall i. 1 \leq i \leq m && g_i \leq 1 \\ &\forall j. 1 \leq i \leq p && h_j = 1 \end{aligned}$$

f, g_i, h_j arbitrary functions: general nonlinear program

- finding feasible or optimal solution is NP hard
- only applicable for small problem instances

$1 - p_1$

f, g_i, h_j signomials: signomial program

- if feasible solution is found, local optima can be computed efficiently

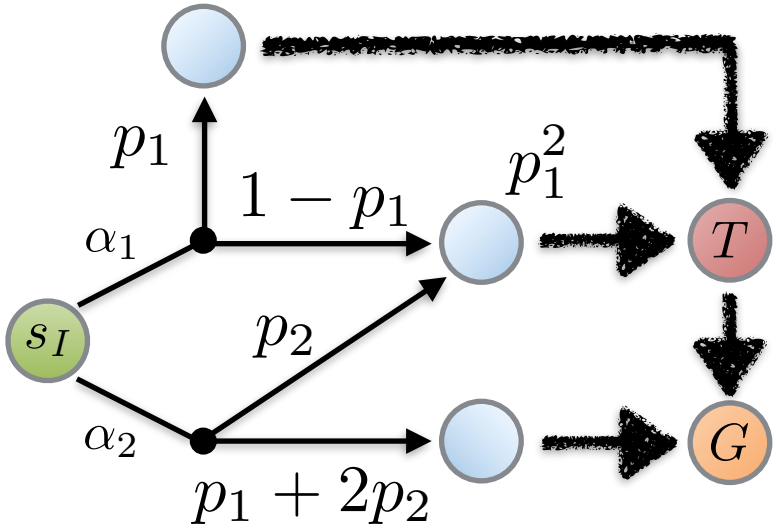
$p_1 + 2p_2$

$p_1 \;\; p_2 \;\; p_1^2$

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Formal Problem — Parameter Synthesis



Safety specification

$$\varphi = \mathbb{P}_{\leq \lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

$$\psi = \mathbb{E}_{\leq \kappa}(\Diamond G),\ G \subseteq S$$

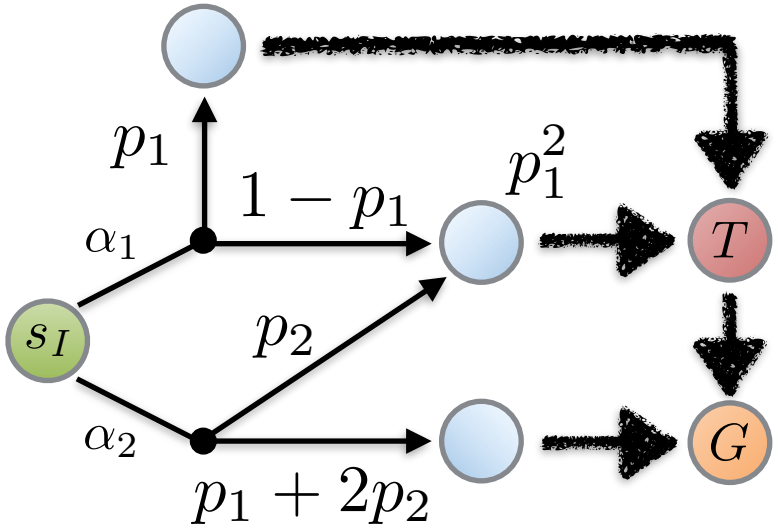
Objective function $f : V \to \mathbb{R}$

Given pMDP $\mathcal{M}$, find a valuation of parameters and a scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

$$\mathcal{M}^{\sigma} \models \varphi \wedge \psi$$

and value for objective function $f : V \to \mathbb{R}$ is minimal.

Formal Problem — Parameter Synthesis



Safety specification

$$\varphi = \mathbb{P}_{\leq \lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

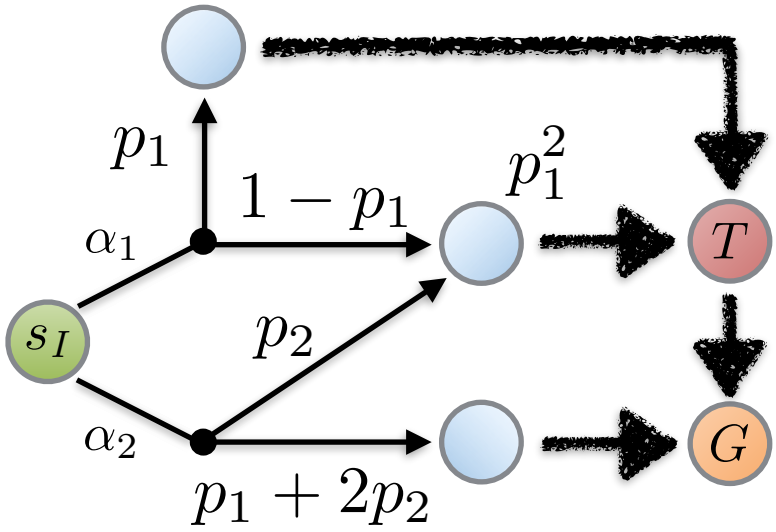
$$\psi = \mathbb{E}_{\leq \kappa}(\Diamond G),\ G \subseteq S$$

Objective function $f : V \to \mathbb{R}$

Given pMDP $\mathcal{M}$, find a valuation of parameters and a scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

$$\mathcal{M}^{\sigma} \models \varphi \wedge \psi$$

Problem Encoding — Variables



Safety specification

$$\varphi = \mathbb{P}_{\leq\lambda}(\Diamond T),\ T \subseteq S$$

Performance specification

$$\psi = \mathbb{E}_{\leq\kappa}(\Diamond G),\ G \subseteq S$$

Objective function $f : V \to \mathbb{R}$

$\{\sigma^{s,\alpha} \mid s \in S, \alpha \in Act(s)\}$ Randomized scheduler

$\{p_s \mid s \in S\}$ Probability of reaching T

$\{c_s \mid s \in S\}$ Expected cost of reaching G

Feasibility Problem

Feasibility Problem

$$p_{s_I} \leq \lambda$$
$$c_{s_I} \leq \kappa$$

safety and performance specifications

Feasibility Problem

$$p_{s_I} \leq \lambda$$
$$c_{s_I} \leq \kappa$$

safety and performance specifications

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} = 1$$

$$\forall s \in S \ \ \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1$$

well-defined schedulers and parameter instantiations

Feasibility Problem

$$p_{s_I} \leq \lambda$$
$$c_{s_I} \leq \kappa$$

safety and performance specifications

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} = 1$$

$$\forall s \in S \ \ \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1$$

well-defined schedulers and parameter instantiations

$$\forall s \in T. \quad p_s = 1$$

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

probability computation

Feasibility Problem

$$p_{s_I} \leq \lambda$$
$$c_{s_I} \leq \kappa$$

safety and performance specifications

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} = 1$$

$$\forall s \in S \ \ \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1$$

well-defined schedulers and parameter instantiations

$$\forall s \in T. \quad p_s = 1$$

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

probability computation

$$\forall s \in G. \quad c_s = 0$$

$$\forall s \in S \setminus G. \quad c_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)$$

expected cost computation

Nonlinear Program

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

Nonlinear Program

real-valued variable

signomials

real-valued variable

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

Nonlinear Program

real-valued variable

signomials

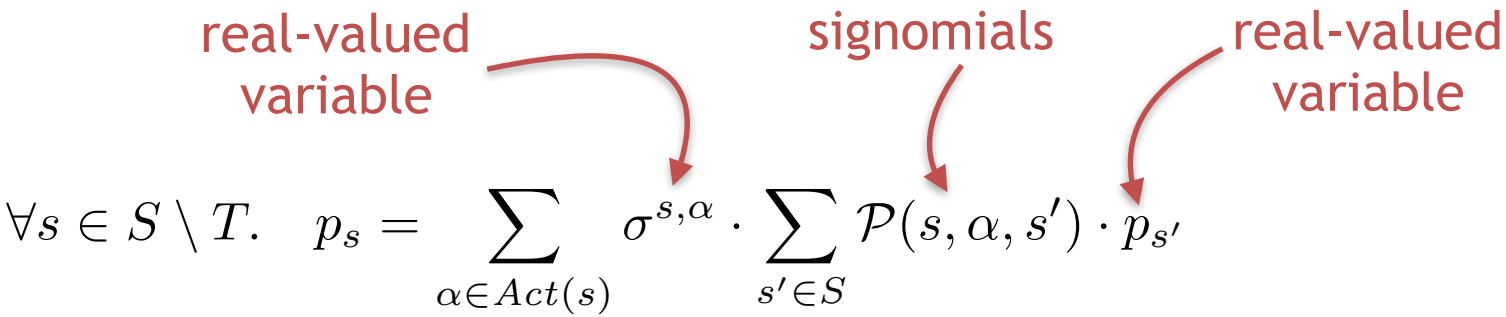
real-valued variable

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

SMT solving over nonlinear arithmetic

- exponential in the number of variables, constraints and the degree of polynomials
- exact solutions

Nonlinear Program



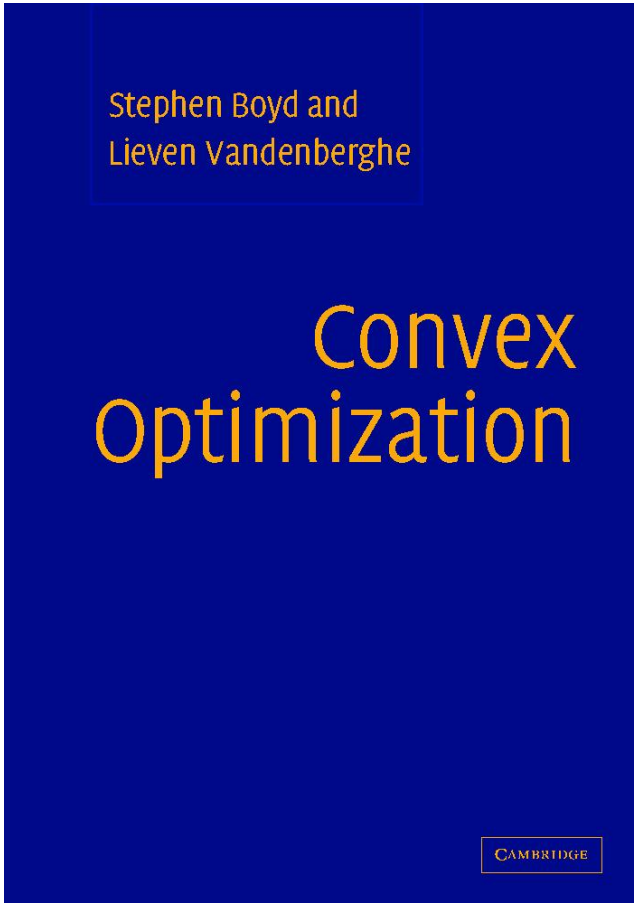
SMT solving over nonlinear arithmetic

- exponential in the number of variables, constraints and the degree of polynomials
- exact solutions

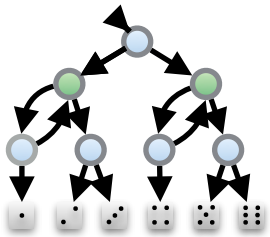
Signomial programming

- If feasible solution is known, local optimum efficiently computable

Convexify it!



Story - Geometric Program



PARAM

Objective function

Specifications

General parametric MDP

Parametric MDP restricted to signomials

Nonlinear Program

Geometric program

Feasible solution

Geometric Programming

$$
\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & \\
\forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\
\forall j. 1 \leq i \leq p \quad & h_j = 1
\end{aligned}
$$

Geometric Programming

$$\begin{aligned} &\text{minimize} && f \\ &\text{subject to} && \\ &\forall i. 1 \leq i \leq m && g_i \leq 1 \\ &\forall j. 1 \leq i \leq p && h_j = 1 \end{aligned}$$

$p_1 + 2p_2$

$p_1 \;\; p_2 \;\; p_1^2$

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Geometric Programming

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & \\ \forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\ \forall j. 1 \leq i \leq p \quad & h_j = 1 \end{aligned}$$

$p_1 + 2p_2$

$p_1 \;\; p_2 \;\; p_1^2$

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Division transformation: $f \leq h \iff \frac{f}{h} \leq 1$

Geometric Programming

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & \\ \forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\ \forall j. 1 \leq i \leq p \quad & h_j = 1 \end{aligned}$$

$p_1 + 2p_2$

$p_1 \;\; p_2 \;\; p_1^2$

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Division transformation: $f \leq h \iff \frac{f}{h} \leq 1$

strictly positive

Geometric Programming

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & \\ \forall i. 1 \leq i \leq m \quad & g_i \leq 1 \\ \forall j. 1 \leq i \leq p \quad & h_j = 1 \end{aligned}$$

$p_1 + 2p_2$

$p_1 \;\; p_2 \;\; p_1^2$

f, g_i posynomials, h_j monomials: geometric program

- transformation to convex optimization problem
- global optima can be computed efficiently

Division transformation: $f \leq h \iff \frac{f}{h} \leq 1$

strictly positive

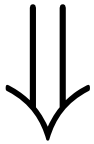
Relaxation: $f = h \;\Rightarrow\; f \leq h \;\Leftrightarrow\; \frac{f}{h} \leq 1$

Convexification

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

Convexification

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$



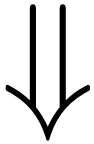
relaxation

$$\forall s \in S \setminus T. \quad p_s \geq \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}$$

Convexification

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

upper bound on actual probability



relaxation

$$\forall s \in S \setminus T. \quad p_s \geq \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

Convexification

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

upper bound on actual probability

⇓ relaxation

$$\forall s \in S \setminus T. \quad p_s \geq \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

⇓ division transformation

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}}{p_s} \leq 1$$

Convexification

$$\forall s \in S \setminus T. \quad p_s = \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

upper bound on actual probability

⇓ relaxation

$$\forall s \in S \setminus T. \quad p_s \geq \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}$$

⇓ division transformation

still signomials (not convex)

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s,\alpha,s') \cdot p_{s'}}{p_s} \leq 1$$

Lifting

$$\mathcal{P}(s, \alpha, \bar{s}) = 1 - \sum_{s' \in S \setminus \{\bar{s}\}} \mathcal{P}(s, \alpha, s')$$

Lifting

$$\mathcal{P}(s, \alpha, \bar{s}) = 1 - \sum_{s' \in S \setminus \{\bar{s}\}} \mathcal{P}(s, \alpha, s')$$

signomial

posynomial

Lifting

$$\mathcal{P}(s,\alpha,\bar{s}) = 1 - \sum_{s' \in S \setminus \{\bar{s}\}} \mathcal{P}(s,\alpha,s') \implies \bar{\mathcal{P}}(s,\alpha,\bar{s}) = \bar{p}_{s,\alpha,\bar{s}} \in L$$

signomial

posynomial

Lifting

$$\mathcal{P}(s,\alpha,\bar{s}) = 1 - \sum_{s' \in S \setminus \{\bar{s}\}} \mathcal{P}(s,\alpha,s') \implies \bar{\mathcal{P}}(s,\alpha,\bar{s}) = \bar{p}_{s,\alpha,\bar{s}} \in L$$

signomial

posynomial

lifting variable

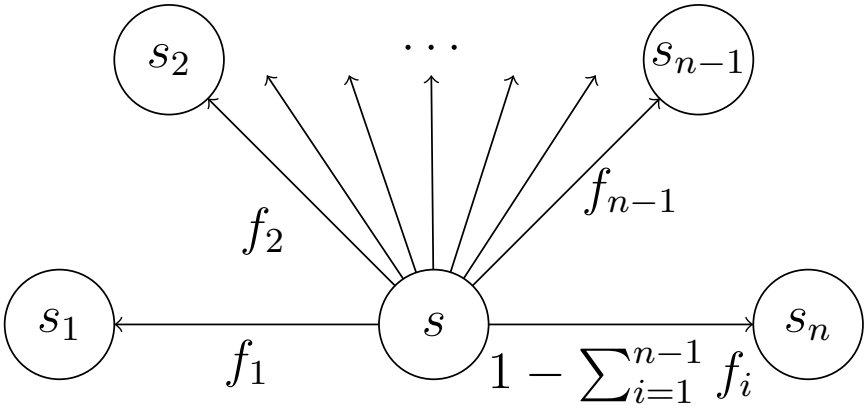
Lifting

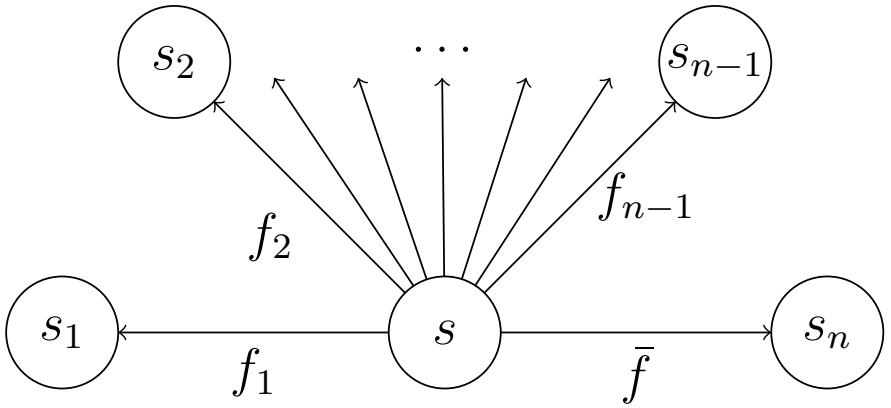
$$\mathcal{P}(s,\alpha,\bar{s}) = 1 - \sum_{s' \in S \setminus \{\bar{s}\}} \mathcal{P}(s,\alpha,s') \Longrightarrow \bar{\mathcal{P}}(s,\alpha,\bar{s}) = \bar{p}_{s,\alpha,\bar{s}} \in L$$

signomial

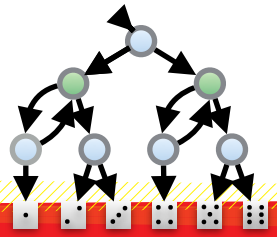
posynomial

lifting variable





PARAM



Geometric Program - Feasibility

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Geometric Program - Feasibility

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

upper bound on actual probability/cost

Geometric Program - Feasibility

specification still met

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

upper bound on actual probability/cost

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Geometric Program - Feasibility

specification still met

substochastic probability distributions

upper bound on actual probability/cost

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Tightening of Equality Constraints

$$\text{minimize} \quad \sum_{p \in V} \frac{1}{p} + \sum_{\bar{p} \in L} \frac{1}{\bar{p}} + \sum_{s \in S, \alpha \in Act(s)} \frac{1}{\sigma_{s,\alpha}}$$

subject to

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Tightening of Equality Constraints

$$\text{minimize} \quad \sum_{p \in V} \frac{1}{p} + \sum_{\bar{p} \in L} \frac{1}{\bar{p}} + \sum_{s \in S, \alpha \in Act(s)} \frac{1}{\sigma_{s,\alpha}}$$

monotonic regularization function

$$\text{subject to}$$

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Tightening of Equality Constraints

$$\text{minimize} \quad \sum_{p \in V} \frac{1}{p} + \sum_{\bar{p} \in L} \frac{1}{\bar{p}} + \sum_{s \in S, \alpha \in Act(s)} \frac{1}{\sigma_{s,\alpha}}$$

monotonic regularization function

$$\text{subject to}$$

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

solution is feasible for the original problem if the constraints are satisfied with equality

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

Tightening of Equality Constraints

no further objective

monotonic regularization function

$$\text{minimize} \quad \sum_{p \in V} \frac{1}{p} + \sum_{\bar{p} \in L} \frac{1}{\bar{p}} + \sum_{s \in S, \alpha \in Act(s)} \frac{1}{\sigma_{s,\alpha}}$$

$$\text{subject to}$$

$$\frac{p_{s_I}}{\lambda} \leq 1$$

$$\frac{c_{s_I}}{\kappa} \leq 1$$

solution is feasible for the original problem if the constraints are satisfied with equality

$$\forall s \in S. \quad \sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \leq 1$$

$$\forall s \in S \, \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \leq 1$$

$$\forall s \in S \setminus T. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}}{p_s} \leq 1$$

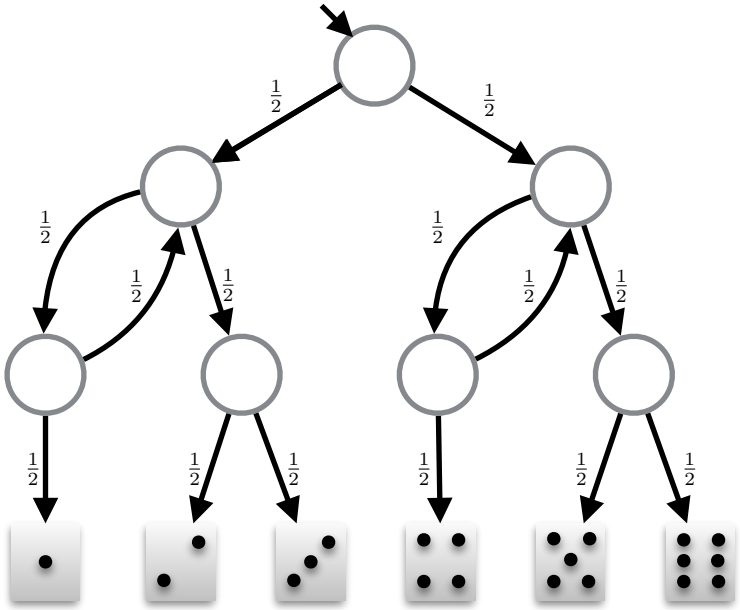
$$\forall s \in S \setminus G. \quad \frac{\sum_{\alpha \in Act(s)} \sigma^{s,\alpha} \cdot \left(c(s, \alpha) + \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot c_{s'} \right)}{c_s} \leq 1$$

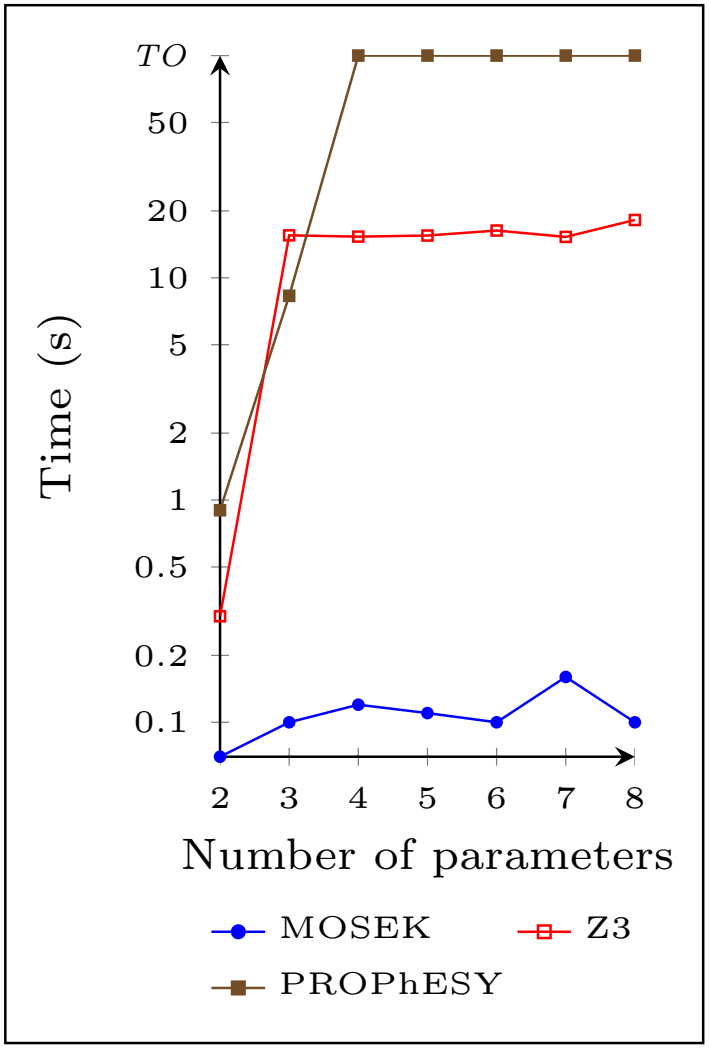
Correctness

The GP resulting from relaxation, lifting, and tightening computes a well-defined valuation of parameters and scheduler $\sigma \in Sched^{\mathcal{M}}$ such that

$$\mathcal{M}^{\sigma} \models \varphi \wedge \psi$$

Experimental Results

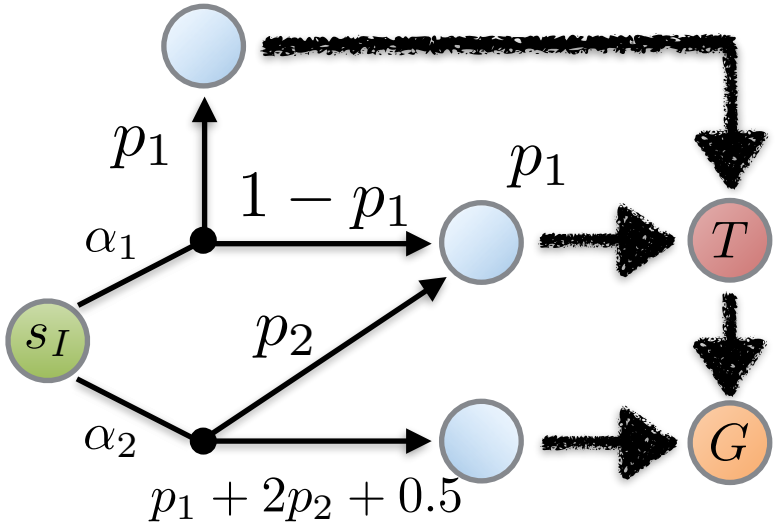




Outline

1. Parameter Synthesis for Markov Chains
2. Parameter Synthesis for MDPs
3. Sequential Convex Programming
4. The Convex-concave Procedure
5. Numerical Examples

Looking at the Functions Again



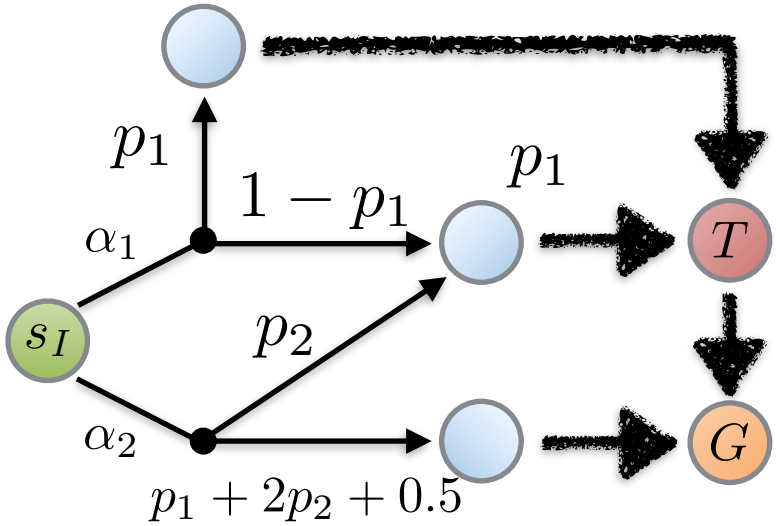
States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Looking at the Functions Again



States S

Actions $\alpha_1, \alpha_2 \in A$

Parameters $p_1, p_2, \ldots, p_n \in V$

Probability and cost functions over parameters

Types of functions

$$f = \sum_{k=1}^{n} c_k \cdot p_k + b \quad \text{with}$$

$c_k \in \mathbb{R}, b \in \mathbb{R}$: Affine Functions

Nonlinear Program

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act. && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act. && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act. && p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}
\end{aligned}
$$

Nonlinear Program

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act. && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act. && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act. && p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}
\end{aligned}
$$

graph-preserving parameter instantiations

probability computation

Nonlinear Program

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act. && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act. && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act. && p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}
\end{aligned}
$$

graph-preserving parameter instantiations

probability computation

upper bound on the probability

Nonlinear Program

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act. && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act. && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act. && p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}
\end{aligned}
$$

graph-preserving parameter instantiations

probability computation

upper bound on the probability

affine function

Nonlinear Program

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act. && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act. && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act. && p_s \geq \sum_{s' \in S} \mathcal{P}(s, \alpha, s') \cdot p_{s'}
\end{aligned}
$$

graph-preserving parameter instantiations

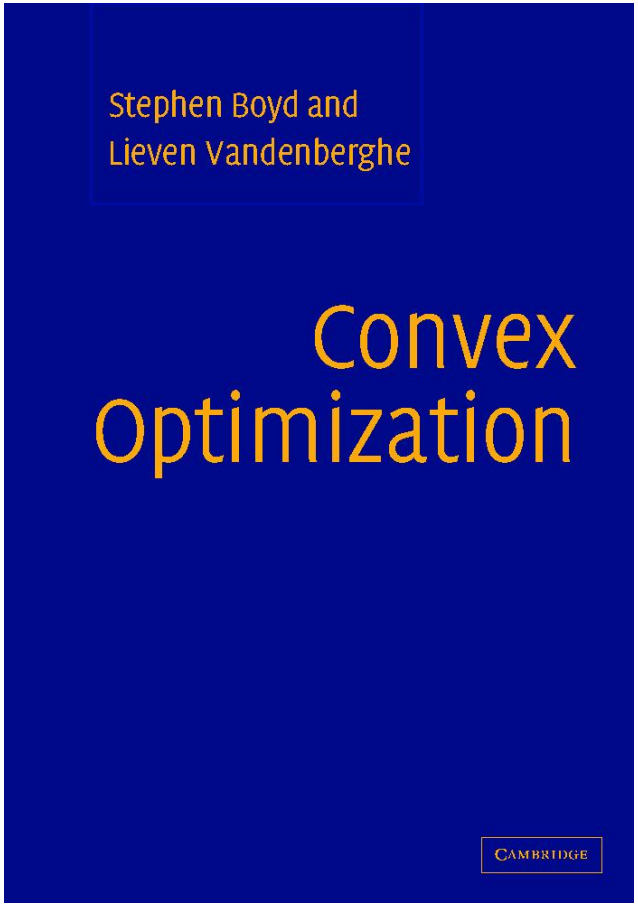
probability computation

upper bound on the probability

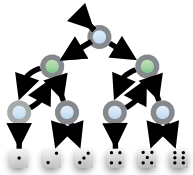
affine function

real-valued variable

Convexify it!



Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Split quadratic functions into convex and concave part

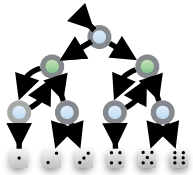
Quadratically Constrained Quadratic Program (QCQP)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Split quadratic functions into convex and concave part

Quadratically Constrained Quadratic Program (QCQP)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Quadratically Constrained Quadratic Programming (QCQP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{x}^T P_0 \mathbf{x} + q_0^T \mathbf{x} + r_0 \\ \text{subject to} \quad & \\ \forall i \in \{1, \ldots, m\} \quad & \mathbf{x}^T P_i \mathbf{x} + q_i^T \mathbf{x} + r_i \leq 0 \end{aligned}$$

$P_i = 0 \rightsquigarrow$ QCQP is a linear program

P_i positive semidefinite (PSD) $\rightsquigarrow$ QCQP convex

P_i not PSD $\rightsquigarrow$ QCQP nonconvex

Quadratically Constrained Quadratic Programming (QCQP)

$$\begin{aligned} \text{minimize} \quad & \mathbf{x}^T P_0 \mathbf{x} + q_0^T \mathbf{x} + r_0 \\ \text{subject to} \quad & \\ \forall i \in \{1, \ldots, m\} \quad & \mathbf{x}^T P_i \mathbf{x} + q_i^T \mathbf{x} + r_i \leq 0 \end{aligned}$$

real-valued coefficients

vector of variables

$P_i = 0 \rightsquigarrow$ QCQP is a linear program

P_i positive semidefinite (PSD) $\rightsquigarrow$ QCQP convex

P_i not PSD $\rightsquigarrow$ QCQP nonconvex

Construction of the QCQP

$$
\begin{aligned}
\text{minimize} \quad & p_{s_I} \\
\text{subject to} \quad & \\
\forall s \in T. \quad & p_s = 1 \\
\forall s, s' \in S. \forall \alpha \in Act(s). \quad & \mathcal{P}(s, \alpha, s') \geq \epsilon \\
\forall s \in S. \forall \alpha \in Act(s). \quad & \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& \lambda \geq p_{s_I} \\
\forall s \in S \setminus T. \forall \alpha \in Act(s). \quad & p_s \geq \mathbf{x}^T P_{s,\alpha} \mathbf{x} + q_{s,\alpha}^T \mathbf{x}
\end{aligned}
$$

Construction of the QCQP

$$
\begin{aligned}
&\text{minimize} && p_{s_I} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act(s). && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act(s). && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act(s). && p_s \geq \mathbf{x}^T P_{s,\alpha} \mathbf{x} + q_{s,\alpha}^T \mathbf{x}
\end{aligned}
$$

vector with parameter and probability variables

quadratic part $a \cdot v \cdot p_s$

affine part $b \cdot p_s$

$\mathcal{P}(s, \alpha, s') \cdot p_s = (a \cdot v + b) \cdot p_s$

Construction of the QCQP

$$
\begin{aligned}
\text{minimize} \quad & p_{s_I} \\
\text{subject to} \quad & \\
\forall s \in T. \quad & p_s = 1 \\
\forall s, s' \in S. \forall \alpha \in Act(s). \quad & \mathcal{P}(s, \alpha, s') \geq \epsilon \\
\forall s \in S. \forall \alpha \in Act(s). \quad & \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& \lambda \geq p_{s_I} \\
\forall s \in S \setminus T. \forall \alpha \in Act(s). \quad & p_s \geq \mathbf{x}^T P_{s,\alpha} \mathbf{x} + q_{s,\alpha}^T \mathbf{x}
\end{aligned}
$$

not PSD!

vector with parameter and probability variables

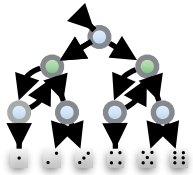
quadratic part $a \cdot v \cdot p_s$

affine part $b \cdot p_s$

$\mathcal{P}(s, \alpha, s') \cdot p_s = (a \cdot v + b) \cdot p_s$

The QCQP is nonconvex in general.

Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Quadratically Constrained Quadratic Program (QCQP)

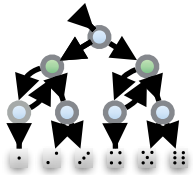
Split quadratic functions into convex and concave part (DC Problem)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Quadratically Constrained Quadratic Program (QCQP)

Split quadratic functions into convex and concave part (DC Problem)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Difference-of-Convex (DC) Problem

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) - g_0(\mathbf{x}) \\ &\text{subject to} && \\ &\forall i = 1, \ldots, m. && f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0 \end{aligned}$$

f_i, g_i are convex

$-g_i$ are concave

Difference-of-Convex (DC) Problem

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) - g_0(\mathbf{x}) \\ &\text{subject to} && \\ &\forall i = 1, \ldots, m. && f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0 \end{aligned}$$

f_i, g_i are convex

$-g_i$ are concave

large enough to render matrix PSD

$$P_{s,\alpha} = P^+_{s,\alpha} - P^-_{s,\alpha} \text{ with } P^+_{s,\alpha} = P_{s,\alpha} + tI \text{ and } P^-_{s,\alpha} = tI$$

Difference-of-Convex (DC) Problem

$$\begin{aligned} &\text{minimize} && f_0(\mathbf{x}) - g_0(\mathbf{x}) \\ &\text{subject to} && \\ &\forall i = 1, \ldots, m. && f_i(\mathbf{x}) - g_i(\mathbf{x}) \leq 0 \end{aligned}$$

f_i, g_i are convex

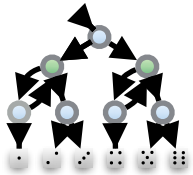
$-g_i$ are concave

large enough to render matrix PSD

$$P_{s,\alpha} = P^+_{s,\alpha} - P^-_{s,\alpha} \text{ with } P^+_{s,\alpha} = P_{s,\alpha} + tI \text{ and } P^-_{s,\alpha} = tI$$

$$\mathbf{x}^T P_{s,\alpha}\mathbf{x} + q^T_{s,\alpha}\mathbf{x} \rightsquigarrow \left(\mathbf{x}^T P^+_{s,\alpha}\mathbf{x} + q^T_{s,\alpha}\mathbf{x}\right) - \mathbf{x}^T P^-_{s,\alpha}\mathbf{x}$$

Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Quadratically Constrained Quadratic Program (QCQP)

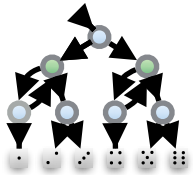
Split quadratic functions into convex and concave part (DC Problem)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Story - Convex-concave Procedure



PARAM

Specification

General parametric MDP

Parametric MDP restricted to affine functions

Nonlinear Program

Quadratically Constrained Quadratic Program (QCQP)

Split quadratic functions into convex and concave part (DC Problem)

Linearize concave part, introduce penalty for violation

Minimize violations

Feasible solution

Linearization

Linearization

$f_i(\mathbf{x}) - g_i(\mathbf{x})$ DC function

Linearization

$f_i(\mathbf{x}) - g_i(\mathbf{x})$ DC function

$\bar{g}_i(\mathbf{x}) = g_i(\hat{x}) + \nabla g_i(\hat{x})^T(\mathbf{x} - \hat{x})$ Linearization around initial point $\hat{x}$

Linearization

$f_i(\mathbf{x}) - g_i(\mathbf{x})$ DC function

$\bar{g}_i(\mathbf{x}) = g_i(\hat{x}) + \nabla g_i(\hat{x})^T(\mathbf{x} - \hat{x})$ Linearization around initial point $\hat{x}$

$f_i(\mathbf{x}) - \bar{g}_i(\mathbf{x})$ Convex over-approximation of the DC function

Convex-Concave Procedure (CCP)

$$\begin{aligned}
&\text{minimize} && p_{s_I} + \tau \sum_{\forall s \in S \setminus T} \sum_{\forall \alpha \in Act} k_{s,\alpha} \\
&\text{subject to} && \\
&\forall s \in T. && p_s = 1 \\
&\forall s, s' \in S. \forall \alpha \in Act(s). && \mathcal{P}(s, \alpha, s') \geq \epsilon \\
&\forall s \in S. \forall \alpha \in Act(s). && \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1 \\
& && \lambda \geq p_{s_I} \\
&\forall s \in S \setminus T. \forall \alpha \in Act(s) && k_{s,\alpha} + p_s \geq \mathbf{x}^T P^+_{s,\alpha} \mathbf{x} + q^T_{s,\alpha} \mathbf{x} - \hat{x}^T P^-_{s,\alpha} (2\mathbf{x} - \hat{x}) \\
&\forall s \in S \setminus T. \forall \alpha \in Act(s) && k_{s,\alpha} \geq 0,
\end{aligned}$$

Convex-Concave Procedure (CCP)

$$\text{minimize} \quad p_{s_I} + \tau \sum_{\forall s \in S \setminus T} \sum_{\forall \alpha \in Act} k_{s,\alpha}$$

minimize violation

$$\text{subject to}$$

$$\forall s \in T. \quad p_s = 1$$

$$\forall s, s' \in S. \forall \alpha \in Act(s). \quad \mathcal{P}(s, \alpha, s') \geq \epsilon$$

$$\forall s \in S. \forall \alpha \in Act(s). \quad \sum_{s' \in S} \mathcal{P}(s, \alpha, s') = 1$$

$$\lambda \geq p_{s_I}$$

penalty variable to account for over-approximation

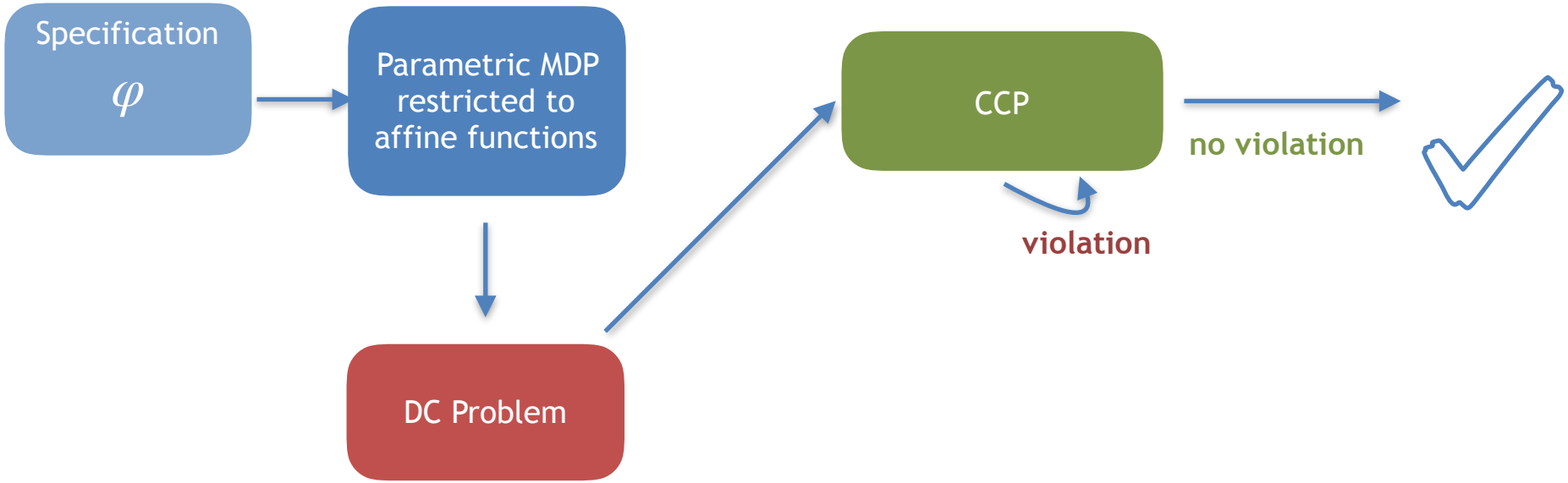
$$\forall s \in S \setminus T. \forall \alpha \in Act(s) \quad k_{s,\alpha} + p_s \geq \mathbf{x}^T P^+_{s,\alpha} \mathbf{x} + q^T_{s,\alpha} \mathbf{x} - \hat{x}^T P^-_{s,\alpha} (2\mathbf{x} - \hat{x})$$

$$\forall s \in S \setminus T. \forall \alpha \in Act(s) \quad k_{s,\alpha} \geq 0,$$

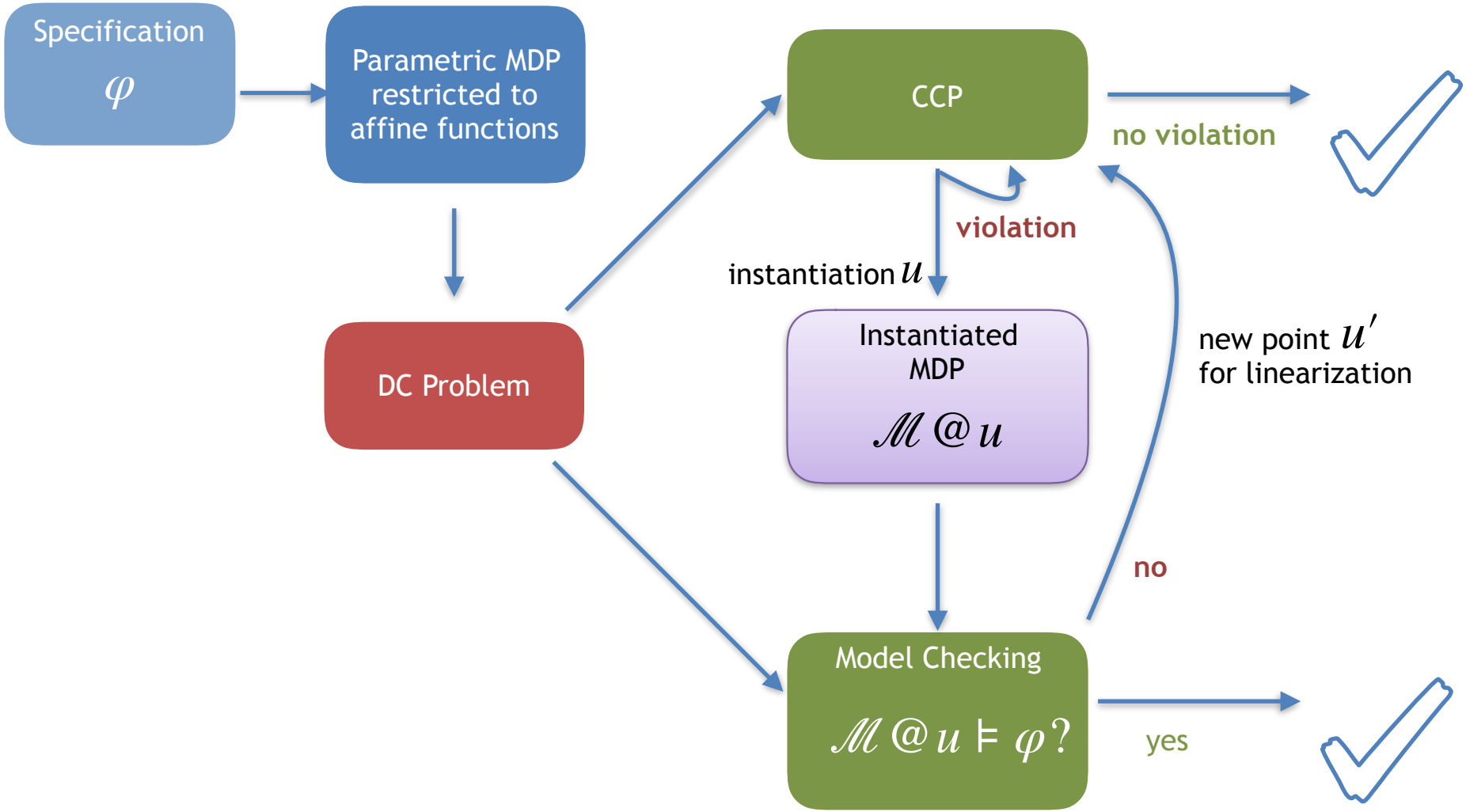
Solving the problem seeks to minimize the violation of the original DC constraints by minimizing the sum of the penalty variables.

If all penalty variables are assigned zero, the algorithm terminates as we have found a feasible solution.

CCP augmented with Model Checking



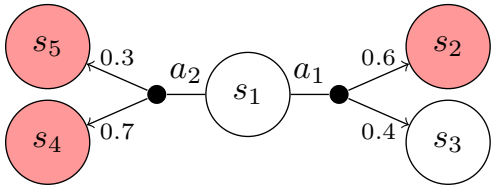
CCP augmented with Model Checking

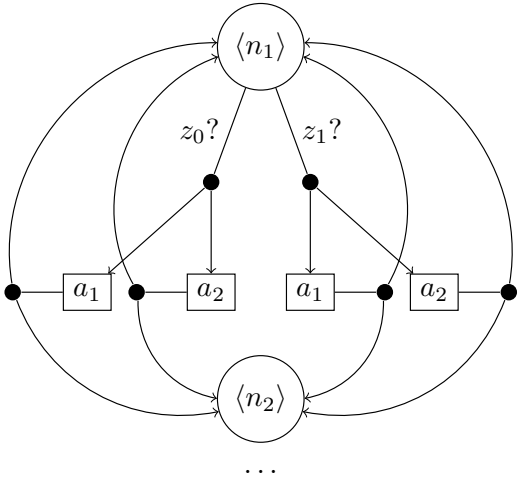


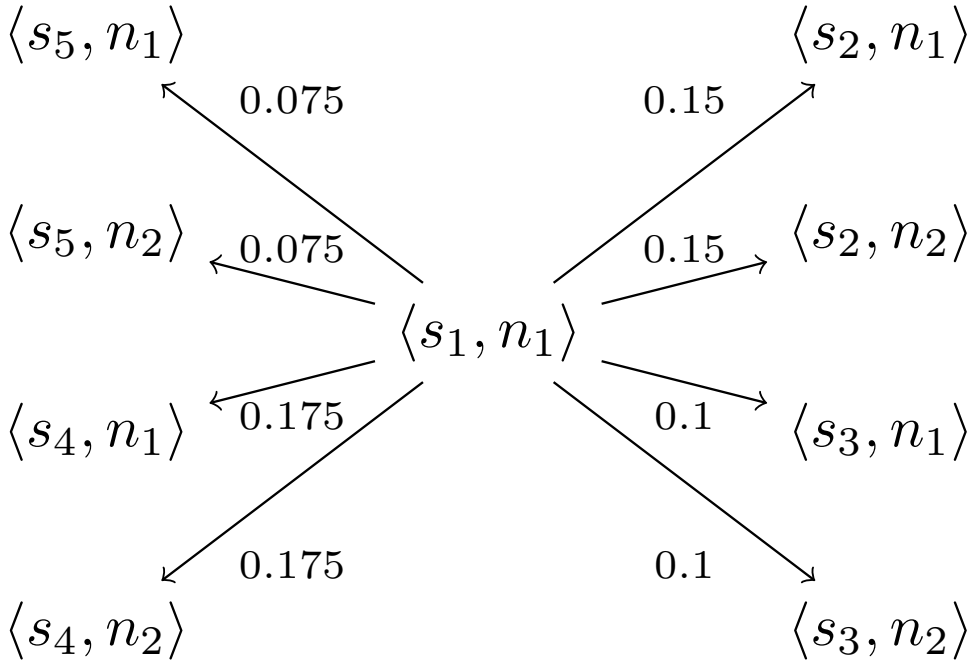
Outline

1. Parameter Synthesis for Markov Chains
2. Parameter Synthesis for MDPs
3. Sequential Convex Programming
4. The Convex-concave Procedure
5. Numerical Examples

Finite-memory Strategies for POMDPs

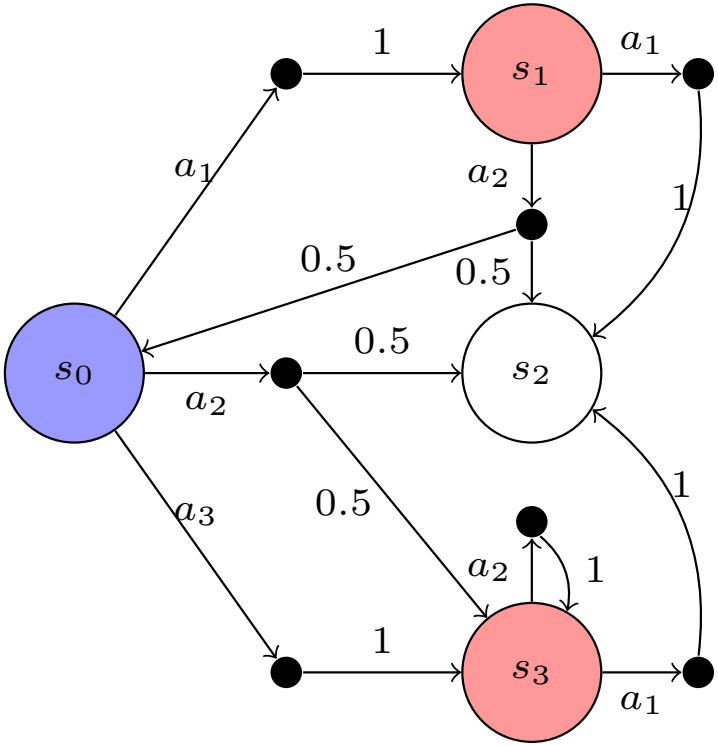


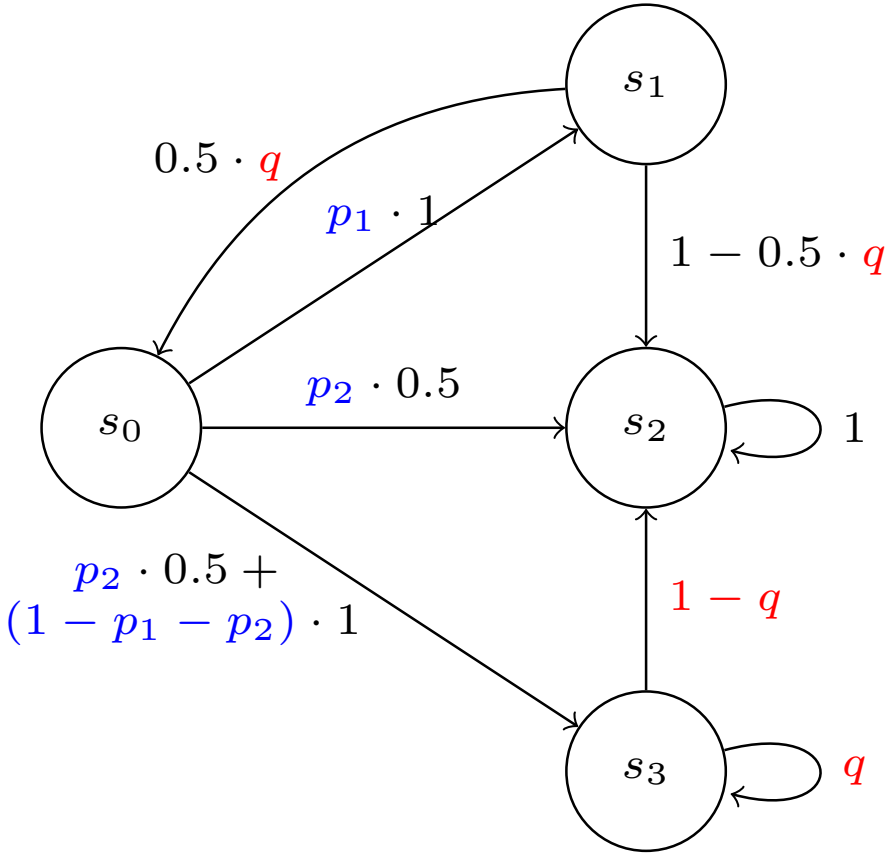




Solved by Parameter Synthesis

Induced Markov Chain with unknown probabilities





Empirical Results

Problem			Info			PSO			SMT	CCP		
Set	Inst	Spec	States	Trans.	Par.	tmin	tmax	**tavg**	**t**	**t**	solv	iter
Brp	16,2	$\mathbb{P}_{\leq 0.1}$	98	194	**2**	0	0	**0**	40	0	30%	3
Brp	512,5	$\mathbb{P}_{\leq 0.1}$	6146	12290	**2**	24	36	**28**	TO	33	24%	3
Crowds	10,5	$\mathbb{P}_{\leq 0.1}$	42	82	**2**	4	5	5	8	**4**	2%	4
Nand	5,10	$\mathbb{P}_{\leq 0.05}$	10492	20982	**2**	21	51	28	TO	**22**	21%	2
Zeroconf	10000	$\mathbb{E}_{\leq 10010}$	10003	20004	**2**	2	4	**3**	TO	57	81%	3
GridA	4	$\mathbb{P}_{\geq 0.84}$	1026	2098	**72**	11	11	**11**	TO	22	81%	11
GridB	8,5	$\mathbb{P}_{\geq 0.84}$	8653	17369	**700**	409	440	427	TO	**213**	84%	8
GridB	10,6	$\mathbb{P}_{\geq 0.84}$	16941	33958	**1290**	533	567	553	TO	**426**	84%	7
GridC	6	$\mathbb{E}_{\leq 4.8}$	1665	305	**168**	261	274	267	TO	**169**	90%	23
Maze	5	$\mathbb{E}_{\leq 14}$	1303	2658	**590**	213	230	219	TO	**67**	89%	8
Maze	5	$\mathbb{E}_{\leq 6}$	1303	2658	**590**	–	–	TO	TO	**422**	85%	97
Maze	7	$\mathbb{E}_{\leq 6}$	2580	5233	**1176**	–	–	TO	TO	**740**	90%	60
Netw	5,2	$\mathbb{E}_{\leq 11.5}$	21746	63158	**2420**	312	523	359	TO	**207**	39%	3
Netw	5,2	$\mathbb{E}_{\leq 10.5}$	21746	63158	**2420**	–	–	TO	TO	**210**	38%	4
Netw	4,3	$\mathbb{E}_{\leq 11.5}$	38055	97335	**4545**	–	–	TO	TO	MO	-	-
Repud	8,5	$\mathbb{P}_{\geq 0.1}$	1487	3002	**360**	16	22	18	TO	**4**	36%	2
Repud	8,5	$\mathbb{P}_{\leq 0.05}$	1487	3002	**360**	273	324	293	TO	**14**	72%	4
Repud	16,2	$\mathbb{P}_{\leq 0.01}$	790	1606	**96**	–	–	TO	TO	**15**	78%	9
Repud	16,2	$\mathbb{P}_{\geq 0.062}$	790	1606	**96**	–	–	TO	TO	TO	-	-

pMCs

POMDPs

Empirical Results

Problem			Info				PSO			SMT	CCP		
Set	Inst	Spec	States	Act	Trans.	Par.	tmin	tmax	**tavg**	**t**	**t**	solv	iter
BRP	4,128	$\mathbb{P}_{\leq 0.1}$	17131	17396	23094	**2**	45	47	46	TO	**39**	33%	4
Coin	32	$\mathbb{E}_{\leq 500}$	4112	6160	7692	**2**	117	119	**118**	TO	TO	-	-
CoinX	32	$\mathbb{E}_{\leq 210}$	16448	24640	30768	**2**	1196	1222	1208	TO	**32**	78%	3
Zeroconf	1	$\mathbb{P}_{\geq 0.99}$	31402	55678	70643	**3**	18	19	**19**	TO	79	82%	2
CSMA	2,4	$\mathbb{E}_{\leq 69.3}$	7958	7988	10594	**26**	n.s.	n.s.	n.s.	TO	**79**	86%	10
Virus	-	$\mathbb{E}_{\leq 10}$	809	3371	6741	**18**	113	113	113	TO	**13**	76%	4
Wlan	0	$\mathbb{E}_{\leq 580}$	2954	3972	5202	**15**	n.s.	n.s.	n.s.	TO	**7**	72%	2

pMDPs

Conclusion

- efficient method to solve realistic parameter synthesis problems
- new methods obtained by exploring the area of convex optimization

Future

- robust parameter instantiations
- more performance in proving absence of solutions
- support for multi-objective properties
- more applications

See also work by

- Benoit Delahaye, Laure Petrucci et al: Parametric Interval Markov Chains
- David Rosenblum et al: Applications
- Radu Calinescu et al: Applications
- Antonio Fillieri et al: Applications & Algorithms
- Milan Ceska: Algorithms for parametric Continuous time Markov chains