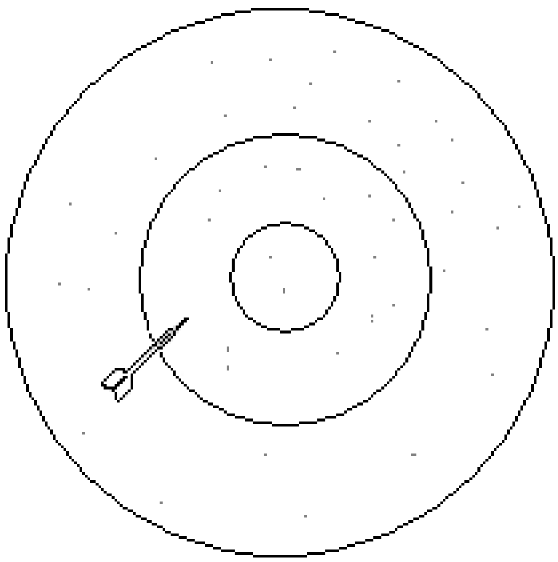
# Axioms of symmetry

*an intuitionistic investigation*



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# Contents

**Abstract**

In this thesis, we will discuss Freiling's axiom of symmetry from an intuitionistic point of view. Using Brouwer's Continuity Principle and fan theorem, we will find that the natural interpretation of this axiom holds in intuitionistic mathematics. The same is true for most of the extensions formulated by Freiling, in so far as they have a natural interpretation in intuitionism.

# Chapter 1

# The problem

## 1.1 The axiom of symmetry

In 1986, Freiling published an article entitled "*Axioms of Symmetry: throwing darts at the real number line*" [2]. This article, about the axiom of symmetry previously explored by Sierpinski aimed to give an intuitive argument to reject the continuum hypothesis.

What is the axiom of symmetry? It is a statement about functions that associate to each real number a countable set of reals. The statement claims that, however you choose such a function, there will always be a *symmetric pair*: a pair of points such that each is outside the set associated with the other.

Formally, this gives us the following formulation of the axiom:

$$\forall f : \mathbb{R} \to \mathbb{R}_{\aleph_0} \exists x_1, x_2 [x_2 \notin f(x_1) \wedge x_1 \notin f(x_2)]. \tag{1.1}$$

Here, $\mathbb{R}_{\aleph_0}$ denotes the class of subsets of $\mathbb{R}$ with cardinality $\aleph_0$ - so the class of countable subsets of $\mathbb{R}$.
In the future, I will refer to 1.1 as AS.

At first sight, most people would consider this statement to be quite plausible: choose any real number at random ("throw a dart at the real number line"). Pick any other point at random too. The chance that the latter lands in the set associated with the first is 0 (because there are uncountably many reals). But since we chose our points at random and not dependent on each other, the chance of the first hitting the set associated with the latter is 0 as well. It is certainly difficult to believe that it would somehow always happen that one is in the other's image!

The above is essentially what Freiling's argument comes down to. He brings it in a slightly different way: pick any number. Then choose another at random - the chance of this number being in the image of the other is so small we may assume that this does not happen. However, the situation is symmetric: "the real line does not know what dart was thrown first". So we may equally assume that the first number is not in the image of the second.
As such, practically all choices of two points will lead to a valid combination. Since we have no quantifiers for "nearly all", and it wouldn't give us too much extra anyway, we can suffice by saying that there must be at least one such combination.

Note that if there is always at least one such combination, there must always be uncountably many. To see this, suppose that, for some $f : \mathbb{R} \to \mathbb{R}_{\aleph_0}$ there are only countably many symmetric pairs $(x_i, y_i)$, that can be chosen in the quantifier of equation 1.1. Define the function $g : \mathbb{R} \to \mathbb{R}_{\aleph_0}$ such that, for $x \in \mathbb{R}$: $g(x) = f(x) \cup \{x_i | i \in \mathbb{N}\} \cup \{y_i | i \in \mathbb{N}\}$.
$g$ is also a function that maps $\mathbb{R}$ to $\mathbb{R}_{\aleph_0}$, so there will be a symmetric pair $(a, b)$. So $a \notin g(b) \wedge b \notin g(a)$, and therefore also $a \notin f(b) \wedge b \notin f(a)$. We can conclude that such a pair $(a, b)$ will coincide with some of the combinations $(x_i, y_i)$, and therefore $a \in g(b) \wedge b \in g(a)$. $\lightning$
So we see that, if there is always at least one combination that satisfies the axiom, then there always must be uncountably many. This makes the axiom at least closer to Freiling's actual statement, that "most" combinations will work (although an uncountable set may still have measure 0).

In the next few sections, I will assume a bit of knowledge about set theory, and about measurability. This part is useful to get an idea of the axiom of symmetry in its classical context, but is not required to understand the rest of this essay. I will not explain the concepts used here as they are not too important for this research, but they are relatively basic, and the interested reader should be able to find an explanation of them without too much searching.

## 1.2 The continuum hypothesis

However likely the statement looks, Sierpinski showed, in his *Hypothèse du continu* in 1934 [5], that under assumption of the axiom of choice, the axiom of symmetry is equivalent to the negation of the continuum hypothesis.

It is easy to see why AS implies the negation of CH (continuum hypothesis). Under assumption of the axiom of choice, the continuum hypothesis states that $\mathbb{R}$ is equipollent to $\aleph_1$. So assuming CH, we can make a welordering $<^*$ of the reals such that for each $x$: $\{y | y \in \mathbb{R} | y <^* x\}$ is countable. Taking $f(x) = \{y | y \in \mathbb{R} | y \leq^* x\}$ gives us a function that maps $\mathbb{R}$ to $\mathbb{R}_{\aleph_0}$ such that for all pairs $x, y$: $x \in f(y)$ or $y \in f(x)$ (since always $x \leq^* y$ or $y \leq^* x$).
So CH implies $\neg$AS, and thus AS implies $\neg$CH.

The converse ($\neg$CH implies AS) is also true (under assumption of the axiom of choice) - but since this is not too important for us, I will just refer the reader to the works of Freiling and Sierpinski for the proof of this.

## 1.3 Objections and counterarguments

The axiom of symmetry itself, and Freiling's reasoning to accept it, have been discussed in various (classical) literature. To indicate some of the background of this statement, I will sketch a few of the objections to AS and the ways they can be countered.

A first objection to Freiling's argument is brought up by Freiling himself: exactly the same reasoning can be applied to reject the axiom of choice. To see this, introduce a statement AS':

$$\forall f : \mathbb{R} \to \mathbb{R}_{<\mathbb{R}} \exists x_1, x_2 [x_2 \notin f(x_1) \wedge x_1 \notin f(x_2)].$$

where $\mathbb{R}_{<\mathbb{R}}$ denotes the collection of all sets of real numbers with cardinality smaller than $\mathbb{R}$; a subset $X$ of $\mathbb{R}$ is said to be of cardinality smaller than $\mathbb{R}$if $\mathbb{R}$ can not be embedded in it.

Now, we can apply the same intuitive argument that Freiling used to defend AS - but AS' contradicts the axiom of choice![1]

This reasoning isn't always used as an objection - Michiel van Lambalgen, for example, accepts this extended version of the axiom in his article "*Independence, Randomness and the Axiom of Choice*" [7] and uses it to reject the axiom of choice.

Freiling himself, however, chooses to defend the axiom of choice, and rejects AS', on the basis that it uses probability to reason about possibly non-measurable sets. For this reason, he believes the axiom of symmetry may be extended in a different way: $f$ can be any function that maps $\mathbb{R}$ to the class of subsets of $\mathbb{R}$ with measure 0.

A second objection to the argument Freiling uses to defend AS is that non-intuitive results often occur when using the axiom of choice: the most prominent example of this is the the Banach-Tarski paradox, where Banach and Tarski managed to cut up the 3-dimensional ball in a finite number of pieces, and, by only rotation and translation of these pieces, mapped it onto two 3-dimensional balls of the same radius.
Therefore, you cannot really assume the axiom of choice and use an argument that appeals to intuition. If our feeling wrongly suggests that the ball-doubling is wrong, then why should we trust it in the case of the axiom of symmetry? As before, the question arises whether the argument Freiling uses is not more an argument against the axiom of choice than the continuum hypothesis!

A third objection that is sometimes brought up is that reasoning with probability does not go too well with the axiom of choice - after all, invoking this theorem tends to bring up non-measurable sets.
However, at first sight it looks like this argument doesn't apply: all sets considered by Freiling's argument are measurable (being countable or co-countable).

Or are they? We are considering questions with two variables - "$x \in f(y)$", "$y \in f(x)$" and their conjunction. As such, we should not consider measurability of chosen sets in $\mathbb{R}$, but in $\mathbb{R}^2$. And there is no reason to assume that, for example, $\{(x,y)|(x,y) \in \mathbb{R}^2|x \in f(y)\}$ is measurable - when we are applying the axiom of choice, this may be rather bold!
If we do assume that both $\{(x,y) \in \mathbb{R}^2|x \in f(y)\}$ and $\{(x,y) \in \mathbb{R}^2|y \in f(x)\}$ are measurable and null, then indeed $\mathbb{P}(\{(x,y) \in \mathbb{R}^2|y \in f(x) \vee x \in f(y)\}) \leq \mathbb{P}(\{(x,y) \in \mathbb{R}^2|y \in f(x)\}) + \mathbb{P}(\{(x,y) \in \mathbb{R}^2|x \in f(y)\}) = 0$, so then AS holds.

However, this is not exactly what Freiling does, although the difference is philosophical: he does not choose two points at random at the same time (which would make it a two-variable-random choice, but he chooses the first number fixed, before even considering the second. Then $\mathbb{P}(y \in f(x))$ is welldefined and 0, by our choice of $f$. Then he applies the symmetry of the situation (we could just have chosen our second point first) to deduce that keeping $y$ fixed, the chance that $x \in f(y)$ is 0 too. This may, however, be abusing the methods of probability theory substantially.

[1] To see this, find a minimal wellordering of $\mathbb{R}$, and assign to each number $x$ the set $\{y|y \in \mathbb{R}|y \leq^* x\}$.

## 1.4 Freiling's extensions

In his article on the axiom of symmetry, Freiling also presented several extensions and variations of the axiom, and researched their plausibility. Some of the variations researched were:

- What happens if you let the function $f$ in 1.1 map to subsets of $\mathbb{R}$ of any cardinality smaller than $\mathbb{R}$ rather than just countable sets?
  This is the question we have seen explored in the arguments. If you accept the statement in this form, it leads to the rejection of the axiom of choice.

- Throwing more darts: given some cardinal $\kappa < \text{card}(\mathbb{R})$, and given a function that maps the real numbers onto countable subsets of $\mathbb{R}$ (as in standard AS), can we find not just $x, y$ that are not in each other's set, but $x_\alpha$ for each $\alpha \in \kappa$, with the property that if $\alpha \neq \beta$, then $x_\alpha \notin f(x_\beta)$?
  This statement ($\text{AS}_{\aleph_0}^{*\kappa}$), unfortunately, turns out to be equivalent to standard AS! So it doesn't give us anything new.

- A more interesting extension is to let our function $f$ take several arguments, say $n$. The axiom then becomes: for any function $f : \mathbb{R}_n \to \mathbb{R}_{\aleph_0}$ there are $x_0, \ldots, x_n$ such that $\forall k \leq n[x_k \notin f(\{x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\})]$.
  This variation, that we shall refer to as $\text{AS}_{\aleph_0}^n$, is more interesting. It is consistent with the usual axioms of set theory and the axiom of choice, but for different $n_1, n_2$, $\text{AS}_{\aleph_0}^{n_1}$ and $\text{AS}_{\aleph_0}^{n_2}$ are not equivalent.

- However, the stronger statement $\text{AS}_{\aleph_0}^{\aleph_0}$ ($\forall f : \mathbb{R}_{\aleph_0} \to \mathbb{R}_{\aleph_0} \exists X \subseteq \mathbb{R}[X \sim \mathbb{N} \wedge \forall x \in X[x \notin f(X \setminus \{x\})]]$), contradicts the axiom of choice.

- What happens if we assign each real to a null set rather than a countable set?
  We treated the need for this question in section 1.3. Freiling shows that this alternative axiom is reasonable as well - it follows from $\neg$CH combined with certain assumptions about null sets.

- What happens if we let our function $f$ map onto other negligible subsets of $\mathbb{R}$, for an entirely different notion of "negligible", like meagre?
  Meagre sets are countable unions of nowhere dense sets - a set is nowhere dense if every interval in $\mathbb{R}$ has a subinterval that is entirely outside the set. What makes this question particularly interesting, is that, although a notion of "small", meagre subsets of $[0, 1]$ might well have measure 1. [2]
  As such, Freiling does not have such a symmetry statement about meagre sets - although he leaves the question open under what assumptions it can hold, if any (and particularly whether it can hold together with AS).

[2] To see this, we can create, for any $k$, a nowhere dense set of measure at least $1 - 2^{-k}$. Let $q_n$ define an enumeration of the rationals. Let, for each $n$, $A_n = [q_n - 2^{-(n+k+2)}, q_n + 2^{-(n+k)}]$. The set $B_k = [0, 1] \setminus (\bigcup_{n \in \mathbb{N}} A_n)$ is nowhere dense, and has at least the promised measure. For a meagre set with measure 1, just take the union of $B_k$ for all $k \in \mathbb{N}$.

## 1.5 Research question

We have seen how Freiling defends the statement - but most mathematicians will not agree with a (dubious) probabilistic reasoning when we assume the axiom of choice.

The question, however, is also interesting when considered from other points of view - even more, perhaps, since we already know the conditions that allow the axiom of symmetry and most of its extensions to hold in the axiom system ZFC.

In this essay, I will investigate the axiom from an intuitionistic viewpoint. This does not just mean I will limit the arguments I use to constructive ones - indeed, this point of view would do us little good here, as any constructive proof holds classically, and thus the results would likely not be very exciting! More than this, I will use some of the special intuitionistic principles Brouwer introduced: the continuity principle and the fan theorem. Both of these will be treated in chapter 2

Even though I will use axioms that are false from a classical point of view, the results may have some classical relevance when considered in the right light. I will also try to limit the use of extra axioms where possible, to keep the results as general as they allow.

Apart from the axiom of symmetry, I will also examine some of the extensions Freiling presented, and add some extensions of my own.

## 1.6 Results

We will find that both the axiom of symmetry and many of its extensions can easily be interpreted from an intuitionistic point of view. More than that, most of these natural interpretations will turn out to be true!

Highlights will be the definition and proof of the standard axiom of symmetry (chapter 3), the extension where each point of the real number line is mapped to a set with small measure (6), and the proof that, despite Freiling's intuition, the variation with meagre sets (7) holds when interpreted in a natural, constructive way.
The other variations also give very nice results.

# Chapter 2

# Prerequisites and conventions

In the discussion of the original problem in the previous chapter, I have assumed some knowledge of basic set theory: the notions of ordinals, cardinals, the axiom of choice, and of course the continuum hypothesis. I have also used a bit of measure theory without a real explanation - because as I mentioned, the classical discussion of the problem is not too important for the rest of this essay. From an intuitionistic point of view, these notions are (still) meaningless.

In this chapter, I will state all the definitions, extra axioms and - without much proof - theorems I will use in the rest of the essay. Most of these are not mine, but originate from Brouwer or others who have studied his work.
I will also add some definitions of my own and note conventions in the use of variables.

## 2.1 Basics

### 2.1.1 Fundamental structures

We will assume the existence of two fundamental structures. The first, and most basic one, is the set $\mathbb{N}$, consisting of all the natural numbers. On this set, several operations and relations are defined: for any pair of elements of $\mathbb{N}$, we know what it means to multiply or add them, or to subtract one from the other. Apart from this, in this set it is always decidable whether $a = b$ or $a < b$.

The second fundamental structure, $\mathcal{N}$, is the set of countable sequences of natural numbers, or functions that map $\mathbb{N}$ to $\mathbb{N}$. We imagine that any sequence $\alpha$ in $\mathcal{N}$ is defined by some infinite procedure; that is, $\alpha$ can be given by supplying, one by one, $\alpha(0), \alpha(1), \alpha(2), \ldots$ This could be done by some fixed procedure, for example taking the decimals of $\pi$, but we could also just choose every place randomly.
In $\mathcal{N}$, equality is defined, but not decidable ($\alpha = \beta \Leftrightarrow \forall n \in \mathbb{N}[\alpha(n) = \beta(n)]$). There are no standard operations defined, although we will see some additional definitions later.
I will consider each of the members of $\mathbb{N}$ and $\mathcal{N}$ as a *basic object*.

### 2.1.2 Hereditarily countable sets

We can group these basic objects together in *sets*. I will be a bit careful here, to avoid controversy on the subject of what a set really is - I have no intention to build up a full axiom system for set theory. In practice, I will mainly use *hereditarily countable sets*, briefly *hsets*.
To create hsets, I want to allow the process of summing up an enumerable group of basic objects (this may be countable or finite, we don't have to decide beforehand) and taking them together in a set. To be exact, if we have a sequence of elements $A_n$, where each $A_n$ is either an already defined basic object, or $\langle$*undefined* $\rangle$, then we can form the set consisting of all the basic objects in this sequence (not including the $\langle$undefined $\rangle$ elements). Any set defined in this way is an hset.
Each hereditarily countable set is again a basic object. So we can iterate the procedure (as often as we want), and make sets of sets. I will only consider natural numbers, countable sequences of natural numbers and hsets basic objects.

Now, since each hset is given by a sequence, I will sometimes equate a hset with the sequence used to define it; so for every hset $A$, I have the sequence $(A_n)$ at my disposal. If I say that $a \in A$, then this means that we can find some $n$ such that $A_n = a$. As usual, two sets are considered equal to each other when they share all their elements. Note that for hsets $A$, $B$ indeed, if $A = B$, it doesn't have to be true that for all $n$ $A_n = B_n$ (we only know that for all $n$ there is some $m$ such that $A_n = B_m$, and the other way around). However, wherever I interpret a hereditarily countable set as a sequence, the choice of the representation should not make a difference.

On several places in this essay, I will use hsets as a replacement for the "countable sets" in classical mathematics. As such, I will simply speak of "an hset of reals" to denote an hset $A$ where all elements $A_n$ are real numbers.

### 2.1.3 Spreads

In order to define larger structuers than just the countable ones, let us consider *spreads*. Using a spread, we can form the (often uncountable) set of all sequences in $\mathcal{N}$ created place by place in a certain standard way.

To be exact, we first need to introduce finite sequences.
So let $p(n)$ be the $n^{\text{th}}$ prime for all $n \in \mathbb{N}$, and for any $n$ and $a_0, \ldots, a_{n-1} \in \mathbb{N}$, define $\langle a_0, \ldots, a_{n-1}\rangle$ as $p(0)^{a_0+1} \cdot p(1)^{a_1+1} \cdots p(n-1)^{a_{n-1}+1}$, also a natural number. We can enumerate all finite sequences, and therefore define the hset $\mathbb{N}^*$ that contains all of them. Then $\mathbb{N}^*$ is a decidable subset of $\mathbb{N}$, as for each number we can easily determine whether it encodes a finite sequence or not.
Due to the unicity of the prime decomposition of natural numbers, no number will ever encode more than one sequence. So we can define functions len and $(n)$: $\text{len}(\langle a_0, \ldots, a_{n-1}\rangle) = n$, and for $k < n$, $\langle a_0, \ldots, a_{n-1}\rangle(k) = a_k$.

For any sequence $\alpha \in \mathcal{N}$ and $n \in \mathbb{N}$, define $\overline{\alpha}n$ as the finite sequence $\langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$.
For sequences $a, b$, let $a * b = \langle a(0), a(1), \ldots a(\text{len}(a)-1), b(0), b(1), \ldots, b(\text{len}(b)-1)\rangle$.

Now we are able to define the notion of a spread.

**Definition 1.** spread
Let $A$ be a decidable set of finite sequences, such that:

- $\langle\rangle \in A$
- for each $a \in A$ there exists some $n \in \mathbb{N}$ such that $a * \langle n \rangle \in A$
- for each $a \in A$, the set of $n$ such that $a * \langle n \rangle \in A$ is decidable

Then we can form the set of infinite sequences $\alpha \in \mathcal{N}$ such that for all $n$, $\overline{\alpha}n \in A$. This set is called a *spread*.

As such, a spread is nothing more than a set of sequences formed in some constructive, inductively defined way; each step we have a simple procedure to define the next place, and never have to think about the future.

The structures we will introduce will almost always be either an hset or a spread, usually combined with one or more operations or relations.

### 2.1.4 Controversy?

Now, when I was introducing hereditarily countable sets, I suggested I was going to avoid controversy on the definition of a set. But it seems I have allowed myself to go into rather dangerous waters nevertheless - we could introduce quite impressive sets just using spreads. And several people would already disagree on calling $\mathcal{N}$ a set.
It wouldn't limit us at all if, instead of using the word "set", we called these structures *classes*. So $\mathbb{N}$ and $\mathcal{N}$ would be classes, as well as the groups of sequences defined by some spread. This isn't a problem, as we will never need to group these classes together in another class (or set). Hereditarily countable sets, however, would still be sets - they are basic objects that can be contained in other sets.

All in all, while I believe the definitions given in this section are grounded and unproblematic - we are staying within the bounds of normal analysis - the results in this essay should be acceptable even to those who disagree.

## 2.2 More complex structures

From these basic objects, we can form all the structures we will need:

We have already seen the definition of $\mathbb{N}^*$; similarly, we can define for any $n$ $\mathbb{N}^n$ as the set of finite sequences of length $n$. We use the term *ordered pairs* for the sequences of length 2.
Additionally, for a given set $A$, write $A^*$ for the set of finite sequences $a$ where all places $a(n)$ are in $A$. Write $A^n$ for those $a \in A^*$ with length $n$. I will mainly use this for the binary sequences $\{0,1\}^*$.
The main operations on finite sequences are $*$, len and $(n)$, as we have defined in section 2.1.3.

Let $\mathbb{Z} = \{\langle a, b\rangle | a, b \in \mathbb{N} | a = 0 \vee b = 0\}$. We will call the elements of this set the *integers*. $\mathbb{N}$ can be embedded into this set; we will identify the integer $\langle a, 0\rangle$ with the natural number $a$. Write $-a$ for the integer $\langle 0, a\rangle$.
On $\mathbb{Z}$ we can define the standard mathematical operations as well as the usual relations: multiplication, addition, substraction, ordering.

$\mathbb{Q}$ consists of the set of all ordered pairs of integers, and some relations defined on it: a decidable equivalence relation $\equiv$ giving equality of rationals ($\langle a, b\rangle \equiv \langle c, d\rangle$ if and only if $a \cdot d = b \cdot c$ in $\mathbb{Z}$) and a decidable inequality $<$ ($\langle a, b\rangle < \langle c, d\rangle$ if and only if $a \cdot d < b \cdot c$ in $\mathbb{Z}$). Addition, subtraction and multiplication are defined as usual.
There are only countably many rationals.

$S$ is the set of all rational intervals: a pair $\langle p, q\rangle$ is a rational interval if $p$ and $q$ are rationals, $p \leq q$. Standard operations on these intervals are "taking one of the elements": for some $\langle p, q\rangle = s \in S$, $s' = p,\ s'' = q$. We also have $|s| = s'' - s' \in \mathbb{Q}$.
Additionally, define, given two rational intervals $s$ and $t$: $s \cap t = \langle \max(s', \min(t', s'')), \min(s'', \max(s', t''))\rangle$, so the interval that covers the area covered by both $s$ and $t$. We say $s \subseteq t$ if $|s| = 0$ or $t' \leq s' \wedge s'' \leq t''$. It is clear that always $s \cap t \subseteq s$ and $s \cap t \subseteq t$.
There are only countably many rational intervals.

$\mathcal{C}$ is the subset of $\mathcal{N}$ whose elements only map to $\{0, 1\}$ (so for all $n$, $\alpha(n) \in \{0, 1\}$) - this is easily defined with a spread. Unfortunately, it is not decidable whether some sequence $\alpha$ in $\mathcal{N}$ is a member of $\mathcal{C}$.
By Cantor's diagonal argument [1] , we can easily see that $\mathcal{C}$ is not countable.

Now, to define the real numbers, we will use a relatively standard definition: a real number is a sequence of rational intervals that shrinks and shrivels.
That is, a real number is an element $x \in \mathcal{N}$ such that:

- for each $n \in \mathbb{N}$, $x(n) \in S$
- for each $n \in \mathbb{N}$, $x(n)' \leq x(n+1)'$ and $x(n+1)'' \leq x(n)''$ - the sequence shrinks
- for each $n \in \mathbb{N}$, we can find some $m \in \mathbb{N}$ such that $|x(m)| < 2^{-n}$ - the sequence shrivels

We write $x \equiv y$ for $\forall n \in \mathbb{N}[x(n)' \leq y(n)'' \wedge y(n)' \leq x(n)'']$. Write $x \# y$ for $\exists n \in \mathbb{N}[x(n)'' < y(n)' \vee y(n)'' < x(n)']$.

Unfortunately, this definition does not make a spread. We can introduce several variations that are spreads, but generally when I spreak of "the reals" I mean the class with those elements of $\mathcal{N}$ that can be shown to be real numbers given the above definition.

As a standard form of the set of real numbers, consider the set of $x \in \mathcal{N}$ such that:

- for each $n \in \mathbb{N}$, $x(n) \in S$
- for each $n \in \mathbb{N}$, $x(n)' \leq x(n+1)'$ and $x(n+1)'' \leq x(n)''$
- for each $n \in \mathbb{N}$, $|x(n+1)| \leq \frac{1}{2} \cdot |x(n)|$

We will use $\mathbb{R}$ to denote this set; it is not hard to see that the set is a spread. On $\mathbb{R}$, we will also use the relations $\equiv$ and $\#$. We could define the standard operations as well (addition, subtraction, multiplication etcetera), but we won't need these.

It is useful to note that any given real number $\alpha$ can be uniquely associated with some $\equiv$-equivalent element of $\mathbb{R}$: choose a real $x \in \mathbb{R}$ such that $x(0) = \alpha(0)$, and for each $n$, $x(n+1) = \alpha(m)$ where $m$ is the smallest number for which $|\alpha(m)| \leq \frac{1}{2} \cdot |x(n)|$. It is not too hard to see that we can always find such $m$. We don't need any kind of choice axiom for this association. As such, for most applications using the reals will be equivalent to using $\mathbb{R}$.

Further, for rationals $p < q$, we define $[p, q]$ as the subset of $\mathbb{R}$ that contains only those real numbers $x$ for which $x(0) = \langle p, q \rangle$.

We can imagine several alternative definitions that define the reals just as well: subsets of $\mathbb{R}$ with the nice property that each element is $\equiv$-equivalent to some element of $\mathbb{R}$. We could, for example, demand strict shrinking of the sequence $(x(n)' < x(n+1)' < x(n+1)'' < x(n)'')$, so $x$ will be an interior point of each of the intervals that defines it. Or we could allow only a limited number of choices for each $x(n)$: say $x(n)' \in \{\frac{a}{2^{n+1}} | a \in \mathbb{N}, |x(n)| = 2^{-n}\}$.
Depending on the application, some definitions may be easier to use than others; however, any class $\sigma$ we will consider as a form of the reals will have the property that $\sigma \subseteq \mathbb{R}$ and $\forall x \in \mathbb{R} \exists y \in \sigma [x \equiv y]$.

I also want introduce a somewhat more restricted version of the reals, as this will often make proofs easier.
To this end, let us define, for all $n$, a standardinterval of order $n$ to be a rational interval of the form $\langle \frac{a}{2^n}, \frac{a+1}{2^n} \rangle$ for some $a \in \mathbb{N}$. For each $n$ there are only countably many such intervals, so we can define the set $S_n$ that contains all of them.
Now define $\mathbb{R}^*$ to be the class of those $x \in \mathbb{R}$ such that for all $n$, $x(n) \in S_n$, and call its members "standard reals". For $a, b \in \mathbb{N}$, write $[a, b]^*$ for those standard real numbers $x$ with $x(0) = \langle a, b \rangle$.

It is no longer true that each $x \in \mathbb{R}$ is $\equiv$-equal to some member of $\mathbb{R}^*$. As an example, consider the floating number $\rho$:

$$\rho(n) = \begin{cases} \langle -2^{-n}, 2^{-n} \rangle & \text{if } k_{99} > n \\ \langle -2^{-k}, -2^{-k} \rangle & \text{if } k_{99} = k \wedge k < n \wedge k \text{ is odd} \\ \langle 2^{-k}, 2^{-k} \rangle & \text{if } k_{99} = k \wedge k < n \wedge k \text{ is even} \end{cases}$$

Here, $k_{99} > n$ stands for "no sequence of 99 nines occurs in the first $n + 98$ places of the decimal expansion of $\pi$". $k_{99} = k$ means "a sequence of 99 nines starts at the $k^{\text{th}}$ place of the decimal expansion of $\pi$".
$k_{99}$ is not actually a defined number - we use it only in the above forms, as an abbreviation.

Now, we can't tell whether $\rho \leq 0$, or $\rho \geq 0$ - it would be bold to claim either! But if we could define some $y \in \mathbb{R}^*$ such that $\rho \equiv y$, we would have to choose, at a certain point in time, $y(0)$. Then we could see that either $y(0) = (-1, 0)$ or $y(0) = (0, 1)$.
Therefore, while $\rho$ is a perfectly welldefined real number, it is bold to say there is some $y \in \mathbb{R}^*$ that is equivalent to it.
Despite this, we will often use $\mathbb{R}^*$ in existence proofs, as any standard real is also a real, and they are relatively easy to handle.

## 2.3 Definitions

To work with the structures we just defined, I will introduce some standard functions and relations that will help us write down constructions that we will need a few times.

To start, we define, for any $n \in \mathbb{N}$, $\underline{n}$ as the sequence with $n$ at every place ($\forall m \in \mathbb{N}[\underline{n}(m) = n]$). This allows us to create standard members of $\mathcal{N}$ (and $\mathcal{C}$) easily.

We have already defined $*$ to concatenate two finite sequences. Further, it will prove useful to define concatenation of finite and infinite sequences.
So let $a \in \mathbb{N}^*$ and $\beta \in \mathcal{N}$. Then $a * \beta$ is the sequence $\gamma$ with $\gamma(k) = a_k$ for $k < \text{len}(a)$, $\gamma(\text{len}(a) + n) = \beta(n)$ for any $n \in \mathbb{N}$.

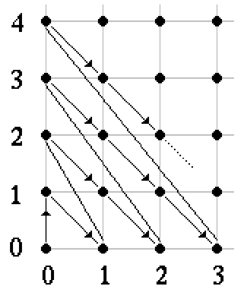
For a finite sequence $a$ and $n < \text{len}(a)$, let $\overline{a}n$ be the sequence consisting of the first $n$ elements of $a$ ($\langle a(0), \ldots, a(n-1) \rangle$). This is similar to the definition of $\overline{\alpha}n$ for infinite sequences $\alpha$.

If $a$ is some rational interval in $S$ and $x \in \mathbb{R}$, say that $x \in a$ if there is some $n$ such that $a' < x(n)' < x(n)'' < a''$ ($x$ is an interior point of $a$). We call $x$ an exterior point of $a$ if $x(n)'' < a' \vee a'' < x(n)'$. To indicate that $x$ is an exterior point of $a$, write $x \notin a$.
Note that not always $x \in a$ or $x \notin a$, even from a non-constructive point of view - reals that are $\equiv$-equal to $a'$ or $a''$ fall in neither category.

Given two basic objects $a, b$, write $(a, b)$ for the hset $\{a, \{a, b\}\}$ (this is just a simple method to pair any two objects). Define the (Cartesian) *product* of two sets $A$ and $B$ as the set $A \times B$ that contains pairs $(a, b)$ where $a \in A$ and $b \in B$. Note that if $A$ and $B$ are hsets, then so is $A \times B$.
With a little ambiguity to the ordered pairs of natural numbers defined in section 2.2, I will also refer to the elements of a product as *pairs*. It should be clear from context which kind of pair is intended.

Define a *function* that maps $A$ to $B$ to be a subset $F$ of $A \times B$, such that for each $a \in A$ we can find a unique $b \in B$ with $(a, b) \in F$. If $F$ maps $A$ to $B$, we write $F : A \to B$. For $a \in A$, write $F(a)$ for the unique $b \in B$ such that $(a, b) \in F$.
Note that I have also used the term function to define standard and less standard operations on some of the structures created in the previous section. However, I believe that this ambiguity again will not lead to problems, as both can be thought of as the same thing - each of the functions used earlier could also be a function of this kind!
Sometimes I will abuse the function notation to denote a mapping from one *class* of basic objects to another.

We have defined ordered pairs of natural numbers using finite sequences, but sometimes it can be useful to have a special kind of pairing. More specifically, we want to define a pairing function $\langle\!\langle \cdot, \cdot \rangle\!\rangle$: $N^2 \to \mathbb{N}$ which is bijective (in a strong way): for each $n \in \mathbb{N}$ we can find unique $a, b \in \mathbb{N}$ such that $n = \langle\!\langle a, b \rangle\!\rangle$.
As a standard example, we can take $\langle\!\langle a, b \rangle\!\rangle = \frac{1}{2}(a+b)(a+b+1)+a$. We can easily calculate this function, and it isn't too hard to find $a, b$ back from the function result - I will not formally prove this here, but I invite the reader to have a look at the picture below to get an indication of why this works.



We could use other functions for pairing, and will even do so later in this essay; any bijective map from $\mathbb{N}^2$ to $\mathbb{N}$ will suffice.
Having a pairing function, we define the functions $(\cdot)_1$ and $(\cdot)_2$ to give each part of a pair.

Using pairing functions, we can write down a countable sequence of sequences (members of $\mathcal{N}$) as a single sequence: define, for any $\alpha \in \mathcal{N}$, for any $n, m \in \mathbb{N}$: $\alpha^n(m) = \alpha(\langle\!\langle n, m \rangle\!\rangle)$. This easily shows that any countable list of members of $\mathcal{N}$ is itself a member of $\mathcal{N}$.

## 2.4 Conventions

For brevity, I will use certain conventions on the use of variables in quantifiers. Unless explicitly stated otherwise:

- $i, j, k, n, m$ will always be elements of $\mathbb{N}$.
- if not mentioned otherwise, $\alpha, \beta, \gamma$ will be members of $\mathcal{C}$. They will never be used for anything other than infinite sequences (members of $\mathcal{N}$).
- $x, y$ will always be elements of $[0, 1]$. Usually, they will also be taken in $[0, 1]^*$, but where this is important it will be explicitly mentioned.

## 2.5 Axioms of Countable Choice

While the classical axiom of choice is very disputable and, from a constructive point of view, meaningless, the axiom of countable choice - where we only have to make an arbitrary choice countably many times - seems fairly plausible.

As we defined $\mathcal{N}$, any enumerable sequence of natural numbers constructed over the course of time is a basic object. We do not have to specify beforehand what algorithm we're going to use to define the sequence - in fact, we do not need to have an algorithm at all!

Formalising our intuition, we get the following axiom:

**Axiom 1.** *First Axiom of Countable Choice*
*Suppose we have some relation $P \subseteq \mathbb{N} \times \mathbb{N}$ such that, for each natural number $n$, we can find at least one $m$ with $P(n, m)$.*
*Then there is some $\alpha \in \mathcal{N}$ such that $\forall n[P(n, \alpha(n))]$.*

Unfortunately, this will not always be enough. The first axiom of countable choice only allows us to make arbitrary choices of integers - and we will need a bit more than that occasionally. Mainly, the need will arise to map $\mathbb{N}$ to sequences of numbers:

**Axiom 2.** *Second Axiom of Countable Choice*
*Suppose we have some relation $Q \subseteq \mathbb{N} \times \mathcal{N}$ such that, for each natural number $n$, we can find at least one $\alpha \in \mathcal{N}$ with $P(n, m)$.*
*Then there is some $\beta \in \mathcal{N}$ such that $\forall n[P(n, \beta^n)]$.*

Unlike the first axiom of countable choice, this principle does not follow directly from the way we have set up our system; it is, however, a consequence of the way we think about things. For each $n$ we can, over the course of time, define some $\alpha_n$ with the property that $P(n, \alpha_n)$. So we can define a $\beta \in \mathcal{N}$ place by place as follows: at place $\langle\!\langle\, n, 0 \,\rangle\!\rangle$, start creating $\alpha_n$. Choose $\beta(\langle\!\langle\, n, 0 \,\rangle\!\rangle) = \alpha_n(0)$. Now postpone the rest of the creation process and go on with the next place of $\beta$. At place $\langle\!\langle\, n, m+1 \,\rangle\!\rangle$, continue the procedure to build $\alpha_n$, until we know the $m+1^{\text{th}}$ place, and then postpone the procedure again. In this way, we can build $\beta$ completely.

These axioms, as well as the principles treated in the next section, are presented and explained in slightly more depth in Wim Veldman's work [9].

## 2.6 The Continuity Principle

In this essay, I will repeatedly use Brouwer's Continuity Principle. This principle, while controversial, is a natural consequence of the way we think about things, and it will be interesting to see what results follow from it.
I accept that many intuitionists would prefer to avoid the special axioms Brouwer introduced, but in this case, there seems little need - the problem we intend to study, unless interpreted in a wildly different way than normal, has no solution without additional axioms. That is, if we restrict ourselves to the standard principles of set theory (in so far as these are constructively meaningful and valid), but with constructive reasoning, neither the axiom or its negation would hold!

The continuity principle states that, if we can assign some natural number to any sequence of numbers, we have to base this decision on only a finite part of the sequence. To be precise, for any class $P \subseteq \mathcal{N} \times \mathbb{N}$:

$$\text{if } \forall \alpha \in \mathcal{N} \exists n [P(\alpha, n)] \text{ then } \forall \alpha \in \mathcal{N} \exists m, n \forall \beta \in \mathcal{N} [\overline{\alpha} m = \overline{\beta} m \to P(\beta, n)]$$

The quantifiers in this formula should be read in a constructive way; so $\exists n[\ldots]$ means that we can *find* some natural for which the given property holds.

Why would we accept such a bold statement? Well, as we stated earlier in this chapter (section 2.1), any $\alpha \in \mathcal{N}$ can be defined as the result of an infinite procedure. The value at every place must be known at a certain point in time (as $\alpha$ is a function that maps to $\mathbb{N}$, and therefore must give a fixed result for any fixed input), but we may never have access to the full list.
Now, suppose someone has promised to assign some integer to any list that we give her. Being nasty, we take a second to decide on each place, so $\alpha(n)$ is only defined after $n$ seconds, and we never give any indication of our plans on how to continue - in fact, we may not have them! However, sticking to her promise, she will have to choose some integer $n$ such that $P(\alpha, n)$, at some point. At that point we will only have decided on the first part of the sequence, say the first $m$ places, and can continue it in whichever way we like! As such, any sequence that agrees with $\alpha$ on the first $m$ places can have the same $n$ assigned to it.

There is no reason why this principle should work only on $\alpha \in \mathcal{N}$: the same argument applies to any spread. So let us consider the following, general statement of the continuity principle:

**Axiom 3.** *Brouwer's Generalised Continuity Principle (CP)*
*Let $C$ be any spread, $P \subseteq C \times \mathbb{N}$ such that for all $\alpha \in C$ we have some $n \in \mathbb{N}$ with $(\alpha, n) \in P$.*

*Then we can find certain natural numbers $m$ and $n$ such that for all $\beta \in C$: $\overline{\alpha}m = \overline{\beta}m \rightarrow (\beta, n) \in P$.*

Many people, even some of those who are inclined to accept some additional axioms, will consider this a very strong statement. However, for most of the theorems in this essay, we can weaken it. To this end, we make the assumption to the axiom stronger, by adding a uniqueness condition: let $P$ be such that there is always exactly one $n$ that fits any given $\alpha$. So $P$ defines a *function* from $\mathcal{N}$, $\mathcal{C}$, $\mathbb{R}$ or any other spread to $\mathbb{N}$.
This will give us the following statement:

**Axiom 4.** *The Weak Continuity Principle (WCP)*
*Let $C$ be any spread, and $P$ a function that maps $C$ to $\mathbb{N}$.*
*Then, for all $\alpha \in C$, there exists some $m \in \mathbb{N}$ such that for all $\beta \in C$:*
$\overline{\alpha}m = \overline{\beta}m \rightarrow P(\alpha) = P(\beta)$.

There are various similar statements we could consider. However, the two treated here are the most important ones we will be using. The only other variation I will treat is the first axiom of continuous choice, which I will use in section 2.7 to derive a very common version of the fan theorem.
Wherever possible, I will stick to the weak version of the continuity principle; presumably some readers will feel more comfortable with it. The stronger principle shall sometimes be necessary, but only to prove that a statement does *not* hold. Since this version of the principle can be proved to be consistent with the other things we assume in intuitionistic mathematics, this still shows, even to those who will not accept the (strong) principle, that the contradicted theorem is unprovable.

Even in its weak form, the principle is still rather powerful: it allows us to show, for example, that all functions that map the reals to the reals are continuous - as long as they respect $\equiv$ (for the proof of this, see Brouwer's or Veldman's work [8]).

## 2.7 The Fan Theorem

However, just using the continuity principle will not always be enough. We will see some of the most exciting results when we study the variation of AS where each real number is mapped to an arbitrary nullset. These results are a consequence of the *fan theorem*. This beautiful theorem, while obvious in classical mathematics, is much contested. But before judging it, let us consider the theorem.

### 2.7.1 Fans

To discuss the Fan Theorem, we must define the notion of a *fan*. We have seen the definition of a spread: it is the class resulting from taking all sequences $\alpha \in \mathcal{N}$ for which each initial segment $\overline{\alpha}n$ is in some fixed set $A$ with properties:

- $\langle\rangle \in A$
- for each $a \in A$ there exists some $n \in \mathbb{N}$ such that $a * \langle n \rangle \in A$

- for each $a \in A$, the set of $n$ such that $a * \langle n \rangle \in A$ is decidable

A fan is a spread with the additional property that:

- for each $a \in A$ there are only finitely many $n \in \mathbb{N}$ such that $a * \langle n \rangle \in A$ - "finitely" in this is strong, meaning that we know the exact number of suitable $n$

So a fan is basically a spread with elements $\alpha$ such that there are only finitely many choices for each place $\alpha(n)$. It is easy to see that $\mathcal{C}$ is a fan, as is $[0, 1]^*$, but $\mathbb{R}$ and $\mathcal{N}$are not.

### 2.7.2 The Basic Fan Theorem

The fan theorem states that for each fan $F$, if we have some decidable set $B \subseteq \omega$ such that each $\alpha \in F$ has an initial segment $\overline{\alpha}n$ in $B$, then there is some integer $N$ such that $\forall \alpha \in F \exists n \leq N[\overline{\alpha}n \in B]$.

Why "theorem"? This result can not be found as the direct consequence of some set of axioms. But to Brouwer, that isn't necessary - axioms are a formalist's tools. He proves the fan theorem using a construction that appeals to intuition of what should be possible. I will not explain this proof here, as it is beyond the scope of this essay - for a full explanation I refer the reader either to Brouwer's work or Veldman's treatment of the theorem [8].

To get a better understanding of what the fan theorem really claims, note that, classically, it holds: the theorem is classically equivalent to the statement:
*Every infinite finitely-branching tree will have an infinite path.*

To see this, it helps to realise that every fan can be viewed as an infinite, finitely-branching tree: let the nodes be the elements of $A$ (where $A$ is the subset of $\mathbb{N}^*$ associated with the fan) and for $a, b \in A$, $a$ is a child of $b$ if and only if, for some $n$, $a = b * \langle n \rangle$. The fan consists of all infinite paths over the nodes, starting at $\langle\rangle$.
The fan theorem now states that if we cut each path at some point, we can find a certain depth ($N$) where all branches have been cut down. If, like classic mathematicians do, we take $p \Rightarrow q$ as equivalent to $\neg q \Rightarrow \neg p$, we come to the equivalent statement that if at any depth $N$ there are uncut branches ($\neg(\forall \alpha \in F \exists n \leq N[\overline{\alpha}n \in B])$), then not all paths can be finite ($\neg(\forall \alpha \in F \exists n[\overline{\alpha}n \in B])$).

The classical proof of this can be found in König's work [3]. The main idea of the proof is to create an $\alpha$, place by place, by always choosing $\alpha(n)$ such that the part of the tree that starts with $\overline{\alpha}(n+1)$ is still infinite. Since this involves taking a decision about infinitely many objects, infinitely often, and reasoning from contradiction as well, it should not come as a surprise that, intuitionistically, this result fails. The fan theorem gives us something similar - but has to be derived in a very different way!

However, Brouwer's formulation and proof of his fan theorem is not widely accepted either.

As such, I will not use the fan theorem in this essay except for chapter 6. In that chapter, we will use it to derive some of the most interesting results. In other places, the (weak) continuity principle will usually be enough.

### 2.7.3 the Extended Fan Theorem

We will generally use the fan theorem in combination with the following version of the continuity principle that looks a lot like an extension of the axiom of choice:

**Axiom 5.** *First Axiom of Continuous Choice*
*Let $C$ be a spread and $A \subseteq C \times \mathbb{N}$ a property such that $\forall \alpha \in C \exists n[A(\alpha, n)]$.*
*Then there exists some $\gamma \in \mathcal{N}$ such that for each $\alpha \in C$: there is a unique $n$ with $\gamma(\overline{\alpha}n) \neq 0$, and $A(\alpha, \gamma(\overline{\alpha}n) - 1)$.*

This axiom can be defended in a way very similar to the way we defended the continuity principle: we just build $\gamma$, place by place, and at each place that represents a finite sequence $a$ we decide whether we now know enough of the sequence $\alpha$ that starts with $a$ to find some $n$ with $A(\alpha, n)$; if so, and we haven't found any such $n$ for shorter initial segments of $\alpha$ (and $a$), we choose $\gamma(a) = n + 1$.
This was just a sketch of the reasoning though, with hopefully enough argumentation that the reader himself may conclude that the axiom is not unreasonable. For a more complete explanation, I refer once more to [9].
Note that the generalised axiom of choice really gives us a choice axiom for (uncountable) spreads: to define a choice function $F$ such that always $A(\alpha, F(\alpha))$, simply write $F(\alpha) = \gamma(\overline{\alpha}n)$ for the unique $n$ with $\gamma(\overline{\alpha}n) \neq 0$.

Combining the fan theorem with the first axiom of continuous choice, we arrive at a statement that claims that if, for any element of a fan, we can find a natural number with a given property, then we can choose all these numbers below some upper bound.

**Theorem 1.** *Extended Fan Theorem*
*Suppose, for a given fan $F$, we have some class $A \subseteq F \times \mathbb{N}$ with the property that for each $\alpha \in F$ there exists some $n$ such that $(\alpha, n) \in A$.*
*Then we can find some $N \in \mathbb{N}$ such that $\forall \alpha \in F \exists n[n \leq N \wedge (\alpha, n) \in A]$.*

*Proof.* Let $F$ be a fan, $A$ as in the theorem.
Using the generalised axiom of choice, determine $\gamma \in \mathcal{N}$ with the property that for each $\alpha \in F$ exists a unique $n$ such that $\gamma(\overline{\alpha}n) \neq 0$, and $A(\alpha, \gamma(\overline{\alpha}n) - 1)$.
Define $B$ as the set of finite sequences $a$ with $\gamma(a) \neq 0$; $B$ can be enumerated (and therefore is a set) and is decidable. It is also given that $\forall \alpha \in F \exists n[\overline{\alpha}n \in B]$.
Using the normal fan theorem, we can now find some $M$ such that $\forall \alpha \in F \exists n \leq M[\overline{\alpha}n \in B]$.
And because $F$ is a fan, there are only finitely many initial segments of length at most $N$.
Enumerate all these finite sequences $a$, and take $N$ to be the maximum of all $\gamma(a)$. It is easy to see that $N$ fits the condition in the theorem. □

The fan theorem is a powerful addition to constructive mathematics. It allows us to prove, for example, that every function defined on a closed interval is not only continuous (as follows from the continuity principle), but even uniformly continuous [8]. In measure theory, it's a fairly standard assumption, because it is needed to derive that every covering of the unit interval has the same measure.

# Chapter 3

# Solution to the basic problem

Now, we have all the tools available to research the validity of an intuitionistic interpretation of Freiling's axiom. In this chapter, I will discuss the solution to the basic problem. Later chapters will treat the natural extensions.

## 3.1 Original formulation and reformulation

The axiom of symmetry, as stated by Freiling, claims the following:

$$\forall f : \mathbb{R} \to \mathbb{R}_{\aleph_0} \exists x, y[x \notin f(y) \wedge y \notin f(x)]. \tag{3.1}$$

However, in our setup, it is not really clear what this statement means. For what, for example, do we mean by a function that maps $\mathbb{R}$ to $\mathbb{R}_{\aleph_0}$? We know what a function is, and can define $\mathbb{R}_{\aleph_0}$ as the class of all countable sets (hsets) of reals. But should such a function respect the $\equiv$-relation defined on the real numbers?
Another question is how we should interpret the $\notin$ constructively. To show that $y \notin f(x)$, do we only need to prove that it can't hold that $y$ equals some $f(x)_n$? Or do we have to see that, for each $n$, there is some place where $y$ differs from $f(x)_n$?

So let us reformulate the axiom. To any of the questions above, I will choose the constructively strongest answer. That is, the function $f$ does not have to respect $\equiv$. It must only be a statement that associates, to each $x \in \mathbb{R}$, some unique countable sequence of other real numbers (where each element in this sequence may be $\langle$ undefined $\rangle$) - as any hset of real numbers is defined by such a sequence. Having such $f$, we must then define $x, y$ such that, for all $n$, either $f(x)_n$ is undefined, or $y(m)$ is strictly separated from $f(x)_n(m)$ ($y(m)'' < f(x)_n(m)' \vee f(x)_n(m)'' < y(m)'$) for some $m$. Note that this implies that always $y \# f(x)_n$. We take the same strong interpretation for $x \notin f(y)$.

Even in this highly constructive form, the axiom turns out to be true! We will only have to accept the continuity principle to come to this conclusion.

A last question could be: what definition of the real numbers do we want to consider? We could use the general definition where a real is any sequence of rational intervals that shrinks and shrivels (or the variation that is a spread, $\mathbb{R}$). But we can avoid problems by choosing a more restricted version, without really limiting ourselves.

To start, let us consider a slightly different statement from 3.1: the statement obtained by replacing the set $\mathbb{R}$ of the real numbers by Cantor space $\mathcal{C}$. Cantor space ($\mathcal{C}$) is easier to reason with and to show the basic idea of the proof, as it avoids all considerations of $\equiv$. It has the additional nice property that any hset $A$ of elements of $\mathcal{C}$, where at least one element $\alpha$ of $A$ is known, can be written as another member $\beta$ of Cantor space (for any $n$, if $A_n = \langle\text{undefined}\rangle$, make $\beta$ such that $\beta^n = \alpha$, otherwise take $\beta^n = A_n$).

This leads us to the following statement of the "axiom":

**Theorem 2.** $A_\omega$
*Let $F$ be a function that maps $\mathcal{C}$ to $\mathcal{C}$.*
*Then there exist $\alpha$, $\beta$ such that $\forall n \exists m[\alpha(m) \neq F(\beta)^n(m)] \wedge \forall n \exists m[\beta(m) \neq F(\alpha)^n(m)]$.*

## 3.2 Idea of the proof

Having the continuity principle, we can imagine a proof along the following lines:

Suppose we have such a function $F$. We construct $\alpha$ step by step, choosing the next place 0 every second. After some time, we should know enough of $\alpha$ to determine $F(\alpha)^0(0)$. $\alpha$ has then only been defined on a finite number of places, say $N_0$.
Now take a break from $\alpha$ and continue working with $\beta$. First of all, take $\beta(0)$ different from $F(\alpha)^0(0)$, so at least we can be sure: $\beta \neq F(\alpha)^0$ (in a strong way). For the rest, keep choosing each place of $\beta$ equal to 0, until we know enough to decide on $F(\beta)^0(N_0)$. $\beta$ has then been defined on, say, $M_0$ places.
We stop working on $\beta$ for a bit, and continue with $\alpha$. Of course we take $\alpha(N_0)$ unequal to $F(\beta)^0(N_0)$, so we already have the safe knowledge that $\alpha \neq F(\beta)^0$. Then we just continue to add zeroes to $\alpha$, until we know $F(\alpha)^1(M_0)$. $\alpha$ has then only been defined on, say $N_1$ places.

The reader will easily see how this procedure is supposed to continue. We just keep interleaving working on $\alpha$ and $\beta$, each step disabling either $\alpha = F(\beta)^n$ or $\beta = F(\alpha)^n$.

However, this was merely a sketch to give the idea of what we're going to do. I did not mention the continuity principle, even though I used it - or at least, I used the idea with which I also defended the continuity principle. To make it clear what exactly happened, I will give a formal proof of the theorem of symmetry in the next section.

## 3.3 Formal proof

### 3.3.1 Observations

$F$ is a function, so we know that $\forall n \forall \alpha \exists! k[F(\alpha)(n) = k]$.
With the (Weak) Continuity Principle we can conclude: $\forall n \forall \alpha \exists m, k \forall \beta[\overline{\alpha}m = \overline{\beta}m \rightarrow F(\beta)(n) = k]$, or: $\forall n \forall \alpha \exists m \forall \beta[\overline{\alpha}m = \overline{\beta}m \rightarrow F(\alpha)(n) = F(\beta)(n)]$.

Now it would be convenient for the proof if we had $\forall n \exists m \forall \alpha, \beta[\overline{\alpha}m = \overline{\beta}m \rightarrow F(\alpha)(n) = F(\beta)(n)]$ instead, so we could always find a place where we "know enough" of $\alpha$ to continue $\beta$ (or the other way around). However, this would be a rather big step (it is possible to take this step with the fan theorem, but let us not assume such bold statements as long as we can avoid them). Fortunately, it will prove to be enough that something similar holds for all $\alpha$ of a certain form - as we saw in the idea of the proof, we only have to consider $\alpha$ that have some fixed start, but then just ends in an infinite sequence of zeroes.

So let $n \in \mathbb{N}$.
Now for $a \in \{0,1\}^*$ we still know: $\exists m \forall \beta [\overline{a * \underline{0}} m = \overline{\beta} m \to F(a * \underline{0})(n) = F(\beta)(n)]$. Using the Axiom of Countable Choice we can now define a function $\mathcal{A} : \mathbb{N} \times \{0,1\}^* \to \mathbb{N}$ such that

$$\forall n \forall a \in \{0,1\}^* \forall \beta [\overline{a * \underline{0}} \mathcal{A}(n,a) = \overline{\beta} \mathcal{A}(n,a) \to F(a * \underline{0})(n) = F(\beta)(n)] \tag{3.2}$$

We may choose $\mathcal{A}$ in such a way that it is increasing in both parameters; that is, $\forall n, m \forall a, b \in \mathbb{N}^* [n < m \wedge \mathrm{len}(a) < \mathrm{len}(b) \wedge \exists c \in \mathbb{N}^* [b = a * c] \to \mathcal{A}(n,a) < \mathcal{A}(m,a) \wedge \mathcal{A}(n,a) < \mathcal{A}(n,b)]$.

### 3.3.2 Construction

We will define recursively, in each step $n$, natural numbers $X_n$, $Y_{n+1}$ and $\alpha(X_{n-1} + 1), \ldots, \alpha(X_n)$, $\beta(Y_n), \ldots, \beta(Y_{n+1} - 1)$.
To make this work for all $n$, first choose $X_{-1} = -1$ and $Y_0 = 0$.

Now in each step $n$, define the values we need to choose as follows:

$$X_n = \mathcal{A}(\langle\!\langle n, Y_n \rangle\!\rangle, \langle \alpha(0), \ldots \alpha(X_{n-1}) \rangle)$$

$$\alpha(X_{n-1} + 1) = \ldots = \alpha(X_n - 1) = 0$$

$$\beta(Y_n) = 1 - F(\overline{\alpha} X_n * \underline{0})(\langle\!\langle n, Y_n \rangle\!\rangle)$$

$$Y_{n+1} = \mathcal{A}(\langle\!\langle n, X_n \rangle\!\rangle, \langle \beta(0), \ldots, \beta(Y_n) \rangle)$$

$$\beta(Y_n + 1) = \ldots = \beta(Y_{n+1} - 1) = 0$$

$$\alpha(X_n) = 1 - F(\overline{\beta} Y_{n+1} * \underline{0})(\langle\!\langle n, X_n \rangle\!\rangle)$$

It is easy to see that this defines $\alpha$ and $\beta$ completely and unambiguously.

### 3.3.3 Correctness proof

To prove the theorem, we must see:

$$\forall n \exists m [\alpha(m) \neq F(\beta)^n(m)] \text{ and } \forall n \exists m [\beta(m) \neq F(\alpha)^n(m)]$$

So let $n \in \mathbb{N}$.
Then, due to our construction, $\alpha(X_n) \neq F(\overline{\beta} Y_{n+1} * \underline{0})^n (X_n)$.
But let $b = \langle \beta(0), \ldots, \beta(Y_n) \rangle$. Checking our definitions, we trivially see that $\overline{\beta} Y_{n+1} * \underline{0} = b * \underline{0}$ and that $\overline{b * \underline{0}} \mathcal{A}(\langle\!\langle n, X_n \rangle\!\rangle, b) = \overline{\beta} \mathcal{A}(\langle\!\langle n, X_n \rangle\!\rangle, b)$. Thus, using the definition of $\mathcal{A}$ (3.2), we can conclude that $F(\beta)(\langle\!\langle n, X_n \rangle\!\rangle) = F(\overline{\beta} Y_{n+1} * \underline{0})(\langle\!\langle n, X_n \rangle\!\rangle)$ and therefore $\alpha(X_n) \neq F(\beta)^n(X_n)$, which gives the first of the two things we had to prove.

Similarly, for $n \in \mathbb{N}$, $\beta(Y_n) \neq F(\overline{\alpha} X_n * \underline{0})(\langle\!\langle n, Y_n \rangle\!\rangle)$. We can apply 3.2 again with $a = \langle \alpha(0), \ldots \alpha(X_{n-1}) \rangle$ and $\alpha$, to conclude that $F(\overline{\alpha} X_n * \underline{0})(\langle\!\langle n, Y_n \rangle\!\rangle) = F(\alpha)(\langle\!\langle n, Y_n \rangle\!\rangle)$ and thus $\beta(Y_n) \neq F(\alpha)^n(Y_n)$, which completes our proof. $\square$

## 3.4 Adaptation for the real numbers

In a very similar way, we can prove:

**Theorem 3.** *$A_\omega$ for real numbers*
*Let $F$ be a function that maps $\mathbb{R}$ to $\mathcal{N}$, such that for each $x, n$: $F(x)^n \in \mathbb{R}$. $F$ does not have to respect $\equiv$.*
*Then there exist $x, y \in \mathbb{R}$ such that, for each $n$, $x \# F(y)^n$ and $y \# F(x)^n$.*

The construction is slightly trickier than in the previous theorem, but in a very similar way as before we can create a symmetric pair $x, y \in \mathbb{R}^*$ (for all $n$, $x \# F(y)^n$, $y \# F(x)^n$). So first pick $x(0) = \langle 0, 1 \rangle = y(0)$. Now start making $x$ by choosing $x(n+1)$ as the left half interval of $x(n)$ in each step $n$.
Since each $F(x)^n$ is in $\mathbb{R}$, we can find some $m$ with $|F(x)^0(m)| \leq 4^{-1}$ - using the continuity principle, we can find this $m$ at some point in time, even though we will not completely have defined $x$. And then, some time later, or perhaps immediately, we will know enough of $x$ to determine $F(x)^0(m)$. Let's say we have defined $x$ up to (and including) place $N_0$ by the time this happens.

Now stop working on $x$ for a bit. Choose a standardinterval $s$ of order 2 such that $s'' < F(x)^0(m)'$ or $F(x)^0(m)'' < s'$. Such $s$ always exists, because $|F(x)^0(m)| \leq 4^{-1}$, so it excludes at most 3 standardintervals with the same length (out of the 4 elements of $S_2$). This choice secures that $y \# F(x)^0$. Let $y(1) \in S_1$ be the standardinterval of order 1 that contains $s$ and take $y(2) = s$.
Now continue working on $y$, by choosing the left half interval each step. We let $m = N_0 + 2$ (so $|F(y)^0(m)| \leq 4^{-1} \cdot 2^{-N_0}$), and at some point in time, when we have chosen, say, $M_0 + 1$ places of $y$ (so we just know $y(M_0)$), we can determine $F(y)^0(m)$.

We stop working on $y$ and return our attention to $x$. To start, let $s$ be the first standardinterval of order $m$ that is inside $x(N_0)$ but strictly separated from $F(y)^0(m)$. Take $x(N_0+1) \in S_{N_0+1}$ such that it contains $s$, $x(N_0 + 2) = s$. Then we can be sure that $x \# F(y)^0$.
Now let $m = M_0 + 2$, and continue the process of creating $x$ by choosing the left half interval each step, until we know enough to find $F(x)^1(m)$. This happens after we have defined, say, $N_1 + 1$ places of $x$.

And thus we can go on. Of course, this was merely the idea of how the proof should go (as done in section 3.2). The formal proof would take a bit more effort. However, I will not treat that here - I believe the idea, and the previous formal construction give enough information to make the proof straightforward, and in section 3.5 we will treat a stronger version of this theorem formally.

## 3.5 Classical relevance

Classical mathematicians would, of course, reject the proofs presented here: we are using the continuity principle which does not hold in classic mathematics!

However, this does not make the proof (classically) meaningless - it merely puts a restriction on the functions we consider. If we choose the right interpretation of *continuous*, the above can be read as a constructive proof that any continuous function will have a symmetric pair.

So how would we define continuity of the functions involved, when we are discussing functions that map the real number line to countable sets of real numbers? The straightforward way is to introduce a topology on $\mathbb{R}_{\aleph_0}$ and derive continuity from this. However, since searching for the right topology in this situation is beyond the scope of this essay, I will not do that here; instead, I simply view a function to a set as a set of functions. That is, I define a continuous function to $\mathbb{R}_{\aleph_0}$ to be a countable sequence $F_0, F_1, F_2, \ldots$ of continuous functions that map $\mathbb{R}$ to itself.

Now, with this definition we can use a proof very similar to what we did in theorem 2 and 3 to come to the conclusion:
For any continuous $F : \mathbb{R} \to \mathbb{R}_{\aleph_0}$, there are some $x, y \in \mathbb{R}$ such that $x \notin F(y)$ and $y \notin F(x)$.

To get this conclusion, it would be nice to just use theorem 3. So define, given $F : \mathbb{R} \to \mathbb{R}_{\aleph_0}$, a function $G : \mathbb{R} \to \mathcal{N}$ such that $\forall n, m$: $G(\langle\!\langle n, m \rangle\!\rangle) = F_n(x)(m)$. We have a slightly different "continuity principle": instead of

$$\forall n \forall m \forall x \exists k \forall y [\overline{x}k = \overline{y}k \to G(x)^n(m) = G(y)^n(m)]$$

we have

$$\forall n \forall m \forall x \exists k \forall y [\overline{x}k = \overline{y}k \to |G(x)^n - G(y)^n| < 2^{-m}]$$

What we would like to do now is apply theorem 3, replacing the assumption of the continuity principle with this alternative. Of course, this alternative is slightly different from the real continuity principle, and I have skipped the full details of the proof of theorem 3 in the first place. As such, I can imagine the cautious reader getting upset with me if I were to skip this proof as well.
Therefore I will now supply the classical theorem and its formal proof separately. The idea of the proof will be the same as in section 3.4. As a nice bonus, theorem 3 will follow as a result from this theorem (we only reason constructively).

**Theorem 4.** *Classical theorem of symmetry*
*Let $F_0, F_1, F_2, \ldots$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, that is, $\forall n \forall m \forall x \exists k \forall y [x(k) = y(k) \to |F_n(x) - F_n(y)| < 2^{-m}]$.*
*Then there exist $x, y \in \mathbb{R}$ such that, for each $n$, $x \# F_n(y)$ and $y \# F_n(x)$.*

For easier reasoning, I will only consider $x, y$ in $\mathbb{R}^*$. However, to keep this proof somewhat more general, I will not assume the $F_n$ map to $\mathbb{R}^*$. Even though classically every real has a standard form, not using this will allow us to keep the proof correct from an intuitionistic point of view as well!

To start, define, for any standard interval $a$, $\widetilde{a}$ as the standardreal that gives the left bound of this interval. That is, if $a$ is a standardinterval of order $n$ and $k \geq n$, then $\widetilde{a}(n) = a$, $\widetilde{a}(k+1)$ is the left half interval of $\widetilde{a}(k)$.

Using the requirement on the $F_n$, the axiom of countable choice and the way we build the standard intervals, we can define a function $\mathcal{A} : \mathbb{N}^2 \times S \to \mathbb{N}$ such that $\forall n, m, k \forall a \in S_k \forall y \in \mathbb{R}^* [\widetilde{a}(\mathcal{A}(n, m, a)) = y(\mathcal{A}(n, m, a)) \to |F_n(x) - F_n(y)| < 2^{-m}]$. We may assume that $\mathcal{A}$ is strictly increasing in the first two arguments and that whenever $a$ is a subinterval of $b$, $\mathcal{A}(n, m, a) < \mathcal{A}(n, m, b)$.

We will define recursively, in each step $n$, natural numbers $X_n$, $Y_{n+1}$, and standardintervals $x(X_{n-1} + 1), \ldots, x(X_n)$ and $y(Y_n - 1), \ldots, y(Y_{n+1} - 2)$.
To make this work, first choose $X_{-1} = 0$, $Y_0 = 2$.

In each step $n$, we can assume that we already know $x(X_{n-1})$ and $y(Y_n-2)$ (and trivially earlier places of $x$ and $y$). Now define $X_n,\ Y_{n+1},\ x(X_{n-1}+1),\ldots,x(X_n)$ and $y(Y_n-1),\ldots,y(Y_{n+1}-2)$ as follows:

Choose $X_n = \mathcal{A}(n, Y_n+1, x(X_{n-1}))$.

Let $a = x(X_{n-1})$, and choose $x(X_{n-1}+1) = \widetilde{a}(X_{n-1}+1),\ \ldots,\ x(X_n-2) = \widetilde{a}(X_n-2)$ - so for $x(X_{n-1}+1),\ldots,x(X_n-2)$ we just choose the left interval every time.

Now consider $F_n(\widetilde{a})$. Since this is a real number, there is some $v \in \mathbb{N}$ such that $|F_n(\widetilde{a})(v)| < 2^{-(Y_n+1)}$. Find some standard interval $b$ of order $Y_n$ which is contained in $y(Y_n-2)$ and has at least $2^{-(Y_n+1)}$ distance from $F_n(\widetilde{a})(v)$. This is always possible: at most $F_n(\widetilde{a})(v)$ forces us to exclude an area of size $3\cdot 2^{-(Y_n+1)} < 2\cdot 2^{-Y_n}$. So of the 4 standardintervals of order $Y_n$ that are inside $y(Y_n-2)$, at least 1 of them must be strictly separated from $F_n(\widetilde{a})(v)$.
Choose $y(Y_n-1)$ such that it contains $b$, $y(Y_n)=b$.
Given the definition of $X_n$ and $\mathcal{A}$, we can be sure that, however we continue $x$, $|F_n(x) - F_n(\widetilde{a})| < 2^{-(Y_n+1)}$. This will grant us that, whatever $y$ will finally turn out to be, certainly $y \# F_n(x)$.

Define $Y_{n+1} = \mathcal{A}(n, X_n+1, y(Y_n))$.

Recall that $b = y(Y_n)$ and choose $y(Y_n+1) = \widetilde{b}(Y_n+1),\ \ldots,\ y(Y_{n+1}-2) = \widetilde{b}(Y_{n+1}-2)$ - so for $y(Y_n+1),\ldots,y(Y_{n+1}-2)$ we again just choose the left interval every time.

Then consider $F_n(\widetilde{b})$. This is a real number, so we can find a number $w \in \mathbb{N}$ with $|F_n(\widetilde{b})(w)| < 2^{-(X_n+1)}$. Like before, we can easily find some standard interval $c$ of order $X_n$ which is covered by $x(X_n-2)$ and has a distance of at least $2^{-(X_n+1)}$ from $F_n(\widetilde{b})(w)$. Let $x(X_n)=c$ and $x(X_n-1)$ the standard interval of order $X_n-1$ that covers it. This choice will guarantee us that, however we go on, $x \# F_n(y)$.

To see that really $y \# F_n(x)$ for every $n$, we must be able to find some $m$ where $y(m)$ is separated from $F_n(x)(m)$. We have seen, in step $n$, that $y(Y_n)$ has distance greater than $2^{-(X_n+1)}$ from $F_n(\widetilde{a})(v)$ (for the $a$ and $v$ used in that step). Let $\epsilon$ be this distance minus $2^{-(X_n+1)}$; then $\epsilon > 0$. Therefore, we can find some $m' \in \mathbb{N}$ with $|F_n(x)(m')| < \epsilon$ (as $F_n(x)$ is a real).
We have also seen that $|F_n(x) - F_n(\widetilde{a})| < 2^{-(Y_n+1)}$. So, using the triangle inequality (for $p,q,r$ the most relevant borders of $F_n(\widetilde{a})(v)$, $y(Y_n)$ and $F_n(x)(m')$ respectively: $|q-r| \geq |p-q| - |p-r| > (2^{-(Y_n+1)}+\epsilon) - (2^{-(Y_n+1)}+\epsilon)$), we find that $F_n(x)(m')$ has positive distance from $y(Y_n)$.
If we then take for $m$ the maximum of $m'$ and $Y_n$, we are done.

We can apply the same reasoning to see that $x \# F_n(y)$. □

Restricting the functions under consideration is not unique: the same is done, for example, by Galen Weitkamp in [10] , where he discusses what kind of functions will always have a symmetric pair.
In fact, Weitkamp proves that such a pair exists for every Borel function (and therefore every continuous function). However, I believe the constructive proof here is easier to understand. Additionally, it is meaningful from an intuitionistic point of view as well.

## 3.6 Discussion

Okay. That was easy! In this section, we have proved that the axiom of symmetry, which we set out to research, is true. What more could we possibly want?

The answer is obvious - let us consider some of the extensions Freiling introduced. In its basic form, the axiom of symmetry holds - but this is also one of the weakest versions, as most related questions imply this standard form.
We have yet to find out whether the theorem still holds if we throw multiple darts, associate bigger sets to each point (for example arbitrary nullsets, or meagre ones), or consider orderable sets rather than the real numbers.

In the following chapters, I will sometimes use $\mathcal{C}$ rather than $\mathbb{R}$. As in this chapter, the proofs can be adapted for $\mathbb{R}$. The reason to use Cantor space rather than the real numbers is just that it allows us to worry slightly less about the details, and thus gives more understandable proofs.

Some of this will have classically relevant interpretations, other things not very much so. I will not go into this often - this is a separate subject for research.

# Chapter 4

# Countable Ordinals

In the classical proof that the axiom of symmetry is equivalent to the negation of the continuum hypothesis, one has to use the fact that the real numbers are wellorderable, a consequence of the axiom of choice. In a constructive environment, the reals can certainly not be assumed to be wellorderable. We can, however, try to construct a similar result if we *start out* with some wellorderable set.
Classically, even without the axiom of choice, we have the theorem:

**Theorem 5.** *Theorem of Non-Symmetry for countable ordinals in classical mathematics*
*There exists a function $F$ that assigns to each element of $\aleph_1$ a countable subset of $\aleph_1$, such that for every $\alpha, \beta \in \aleph_1$, either $\alpha \in F(\beta)$ or $\beta \in F(\alpha)$.*

In this, $\aleph_1$ is the set of countable ordinals.

Under assumption of the axiom of choice, the continuum hypothesis implies that $\mathbb{R}$ is equipollent to $\aleph_1$, and thus that the axiom of symmetry is in contradiction with theorem 5. Constructively, we can't take this step - but it may still be interesting to research whether some form of theorem 5 holds.

However, to reason about this, we must first be able to define the concept of an ordinal. This proves trickier than one might immediately expect. In her master's thesis, Anne-Marie van Berkel has researched various ways of defining the countable ordinals, but the resulting orderings generally failed to be total [6].

To demonstrate the problems and find whatever results we can, I will give a relatively natural definition of the countable ordinals.

## 4.1 Definition

We define the countable ordinals inductively.

**Definition 2.** Countable Ordinals
Suppose that we have a sequence $\tau_0, \tau_1, \tau_2, \ldots$ of hsets and the $\langle$undefined$\rangle$ element, such that each $\tau_i$ that is not $\langle$undefined$\rangle$ is an already defined countable ordinal. Additionally, suppose that for each $\tau_i$ in the list that is not $\langle$undefined$\rangle$, all its elements are also in the list.

Then the hset $\tau$ that is made up of all the sets in this list is also a countable ordinal.
All countable ordinals can be obtained through this construction.

Since we are only considering these countable ordinals, I will just write *ordinal* for such a set.

We write $O$ for the class of all such ordinals, and will write $\alpha \in O$ to denote that some hset is an ordinal. It would be quite bold to speak of $O$ as a set; but it doesn't have to be one for our purposes.
We will, however, need to speak of some form of functions that work on $O$. So think of a *function* that maps $O$ to some class $B$ as a "process" that assigns to each ordinal a unique object in $B$. We can find, given any ordinal $\alpha$, its value $F(\alpha) \in B$.

Because of the inductive definition, we can define a principle of induction on this class:
Let $P$ be some property of ordinals. Suppose that for all ordinals $\alpha$ we know: $\forall \beta \in \alpha[P(\beta)] \rightarrow P(\alpha)$. Then for all ordinals $\alpha$: $P(\alpha)$.

Using this principle of induction, we trivially find that the ordinals are *wellfounded*: each descending chain of ordinals (a sequence $\sigma_i$ such that always $\sigma_{i+1} \in \sigma_i$) is finite. To get this result, consider a fixed ordinal $\alpha$, and assume that for each $\beta \in \alpha$ every descending chain starting with $\beta$ is finite. Let $(\alpha, \sigma_1, \sigma_2, \ldots)$ be a descending chain starting with $\alpha$. Then $\sigma_1 \in \alpha$ and $(\sigma_1, \sigma_2, \ldots)$ is a decending chain, and therefore finite. Thus the original chain must have been finite as well!
From this we can trivially conclude that no ordinal is an element of itself. We will occasionally need this result.

## 4.2 The Classical Result

We will now see that the classical result - the axiom of symmetry does not hold for functions that map the ordinals to countable sets of ordinals - is false when interpreted constructively, in a direct way.

**Theorem 6.** *The statement "there is some $F$ that assigns each ordinal to a countable set of ordinals such that for all $\sigma, \tau \in O$, either $\sigma \in F(\tau)$ or $\tau \in F(\sigma)$" does not hold constructively.*

*Proof.* Define inductively, for $n \in \mathbb{N}$, $\widetilde{n}$ as the hset that contains all $\widetilde{k}$ for $k < n$. It is easy to see that $\widetilde{n}$ is an ordinal.
With this, define, for $\alpha \in \mathcal{C}$, the hset $\sigma_\alpha$ such that for all $n$: $\sigma_{\alpha,n} = \begin{cases} \widetilde{n} & \text{if } \overline{\alpha}n = \underline{0}n \\ \langle\text{undefined}\rangle & \text{otherwise} \end{cases}$

Now suppose $F$ is a function that assigns to each ordinal an hset of ordinals and suppose the statement holds for this $F$. For brevity, write $\phi(\alpha, \beta, n)$ for the statement $\sigma_\alpha = (F(\sigma_\beta))_n$. Then the statement would imply that for any $\alpha, \beta \in \mathcal{C}$ there is some $n \in \mathbb{N}$ such that $\phi(\alpha, \beta, n) \vee \phi(\beta, \alpha, n)$ - this is the way we defined $\in$ for hsets as discussed in chapter 2.

I can take the $\vee$ in there as a constructive disjunction too, and apply the continuity principle not only to the choice of $n$, but also to the side of the disjunction that will be chosen. However, even if we do not interpret the disjunction constructively, the statement will fail to hold (although, admittedly, we will have to argue in a slightly more subtle way).

Suppose that $\forall\alpha\exists n[\phi(\alpha^0,\alpha^1,n)\vee\phi(\alpha^1,\alpha^0,n)]$.
The (strong) continuity principle then gives us: $\forall\alpha\exists n,m\forall\beta[\overline{\alpha}m=\overline{\beta}m\rightarrow(\phi(\beta^0,\beta^1,n)\vee\phi(\beta^1,\beta^0,n))]$.
So choose $\alpha=\underline{0}$ and determine $n,m$.
Then for any $\gamma,\delta\in\mathcal{C}$ (choosing these as replacements for $\beta^0$ and $\beta^1$): if $\overline{\gamma}m=\overline{\delta}m=\overline{\underline{0}}m$, then certainly $\phi(\gamma,\delta,n)\vee\phi(\delta,\gamma,n)$.
Choosing $\gamma=\delta=\underline{0}$, and writing out the definition of $\phi$, we find that $(F(\sigma_{\underline{0}}))_n=\sigma_{\underline{0}}$.
Choosing $\gamma=\overline{\underline{0}}m*\underline{1}$ and $\delta=\underline{0}$, it would give a contradiction with the previous if $\phi(\gamma,\delta,n)$ holds (as $\sigma_{\overline{\underline{0}}m*\underline{1}}=\overline{m}\in\sigma_{\underline{0}}$). So we may conclude: $(F(\sigma_{\underline{0}m*\underline{1}}))_n=\sigma_{\underline{0}}$.
To conclude, choose $\gamma=\delta=\overline{\underline{0}}m*\underline{1}$. This gives us that $(F(\sigma_{\overline{\underline{0}}m*\underline{1}}))_n=\sigma_{\overline{\underline{0}}m*\underline{1}}$, which is in direct contradiction with the previous. $\lightning$

Thus, we must conclude that the statement does not hold. $\square$

## 4.3 The Less Constructive Classical Result

So we have seen that the straightforward constructive interpretation of the classical refutation of the symmetry axiom is untrue. However, this does not have to be a complete killer to all our ambitions to find a parallel to the classical theorem that contradicts the "axiom".
In the theorem above, I took a very constructive interpretation of the statement. I did not use the strong interpretation of the disjunction in that, but I did interpret "is an element of" ($\in$) very strongly. What would happen, we could ask ourselves, if we merely required that there exists no symmetric pair? Then we would not have to explicitly indicate a point in the definition of $F(\tau)$ where $\sigma$ is included (or the other way around), but merely state that the assumption that there is no such point leads to a contradiction.

Well, it turns out that the answer to the resulting question is not straightforward. In fact, if we follow the idea of the classic proof, we arrive at a question equivalent to a famous non-result: Kuroda's conjecture of double negation shift. That is the following axiom sheme:

$$\forall n\in\mathbb{N}[\neg\neg\phi(n)]\rightarrow\neg\neg\forall n\in\mathbb{N}[\phi(n)]$$

This principle, in its general form ($\forall x[\neg\neg A(x)]\rightarrow\neg\neg\forall x[A(x)]$) is independant from the standard axioms of logic [4]. When quantifying over the reals other than the natural numbers, the principle contradicts the Brouwer-Kripke axiom. The countable version is often accepted.

So let us consider this bold conjecture:

**Conjecture 1.** *There exists a function $F$ that maps the ordinals to the class of countable sets of ordinals such that for any $\sigma,\tau\in O$ it can not hold that neither is an element of the other's $F$-image.*
*In formula:* $\exists F:O\rightarrow O_\omega\forall\sigma,\tau\in O[\neg\neg\exists n[\alpha=(F(\beta))_n\vee\beta=(F(\alpha))_n]]$

Suppose we follow the classical reasoning, and define, for any ordinal $\sigma$, $F(\sigma)=\sigma\cup\{\sigma\}$.
Then the conjecture comes down to: $\forall\sigma,\tau\in O[\neg\neg(\sigma\in\tau\vee\sigma=\tau\vee\tau\in\sigma)]$.
How would we prove such a statement? The answer seems obvious: by induction.

So take some ordinal $\sigma$. By induction hypothesis, assume:

$$\forall\tau\in O\forall\alpha\in\sigma[\neg\neg(\alpha\in\tau\vee\alpha=\tau\vee\tau\in\alpha)]\tag{4.1}$$

We must see that this implies that $\forall \tau \in O[\neg\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma)]$.

So fix any $\tau \in O$. We need to show that $\neg\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma)$, and will do this by a further induction, over $\tau$. By induction hypothesis, assume:

$$\forall \beta \in \tau[\neg\neg(\sigma \in \beta \vee \sigma = \beta \vee \beta \in \sigma)] \tag{4.2}$$

Towards a contradiction, further assume that

$$\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma) \tag{4.3}$$

Suppose we can find some $\alpha \in \sigma$ such that $\tau \in \alpha$. Then, since ordinals are transitive by definition, $\tau \in \sigma$, which gives a contradiction with assumption 4.3.
Now suppose there is some $\alpha \in \sigma$ with $\tau = \alpha$. We directly have that $\tau \in \sigma$.
From these two remarks and induction hypothesis 4.1, we find that for any $\alpha \in \sigma$: $\neg\neg(\alpha \in \tau)$.

Similarly, if we can find some $\beta \in \tau$ with the property that either $\sigma \in \beta$ or $\beta = \sigma$, we have another contradiction. Thus we can conclude from 4.2 that for all $\beta \in \tau$: $\neg\neg(\beta \in \sigma)$.

We have now seen that both $\forall \alpha \in \sigma[\neg\neg(\alpha \in \tau)]$ and $\forall \beta \in \tau[\neg\neg(\beta \in \sigma)]$. Since $\sigma$ and $\tau$ are both countable, the DNS-principle would give us that $\neg\neg\forall \alpha \in \sigma[\alpha \in \tau] \wedge \neg\neg\forall \beta \in \tau[\beta \in \sigma]$, which implies that $\neg\neg(\sigma = \tau)$ and thus gives another contradiction. □

What exactly is the extra principle we need? Surely, we wouldn't really have to assume the principle $(\forall n[\neg\neg Q(n)] \to \neg\neg\forall n[Q(n)])$ for *every* property $Q$ just to make this proof work? And indeed, if we investigate the proof a little, we see that we only have to assume the statement for $Q(n)$ of the form $\sigma_n = \langle \text{undefined} \rangle \vee \sigma_n \in \tau$, for arbitrary ordinals $\sigma$ and $\tau$.

If we want to see that always $\neg\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma)$, it is impossible to avoid the assumption in this form. We can see this because it follows from this statement!

**Theorem 7.** *Suppose: for all ordinals $\sigma$ and $\tau$: $\neg\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma)$.*
*Then: for all ordinals $\sigma, \tau$: if $\forall \alpha \in \sigma[\neg\neg(\alpha \in \tau)]$, then $\neg\neg\forall \alpha \in \sigma[\alpha \in \tau]$.*

*Proof.* To prove this, suppose we are given some ordinals $\sigma$ and $\tau$, satisfying the following conditions:

1. $\neg\neg(\sigma \in \tau \vee \sigma = \tau \vee \tau \in \sigma)$
2. $\forall \alpha \in \sigma[\neg\neg(\alpha \in \tau)]$
3. $\neg\forall \alpha \in \sigma[\alpha \in \tau]$

If we can obtain a contradiction from these assumptions, it is clear that the theorem holds.

So suppose that $\sigma \in \tau$. Since $\tau$ is transitive, we see that all elements of $\sigma$ are also elements of $\tau$, which contradicts assumption 3. The same problem occurs if $\sigma = \tau$.
Using assumption 1, we can thus conclude that $\neg\neg(\tau \in \sigma)$. But from section 4.1 we know that $\neg(\tau \in \tau)$, which, combined with assumption 2, shows that $\neg(\tau \in \sigma)$. This gives us the required contradiction. $\lightning$ □

Thus, it is clear that we *have* to make a rather strong assumption to get relative comparability of ordinals. Note that, even though we restricted the DNS-principle to some very specific statements, there are many statements that can be converted to this form; that is, for statements $Q(n)$ of a given form, we can find ordinals $\sigma, \tau$ such that always $Q(n)$ is equivalent to $\sigma_n \in \tau$. Then "$\forall n[\neg\neg Q(n)] \to \neg\neg\forall n[Q(n)]$" becomes equivalent to "$\forall \alpha \in \sigma[\neg\neg(\alpha \in \tau)] \to \neg\neg\forall \alpha \in \sigma[\alpha \in \tau]$". Examples of such statements $Q$ are all $\Pi^1_0$ and $\Sigma^1_0$ formulas ($Q(n) = \forall m[P(n,m)]$ or $Q(n) = \exists m[P(n,m)]$ with $P(n,m)$ a decidable property).

Of course, it might be that we don't actually need comparability of the ordinals to prove the conjecture - we only need it to apply the proof that works in classical mathematics. Choosing a different function $F$, there may well be another possibility to confirm conjecture 1.
However, I haven't yet been able to find such a proof, or a way to see that it can not exist. So I will leave this question open for further research!

## 4.4 An Axiom of Symmetry?

So far we have asked ourselves what results we could obtain when we try to imitate, in some sense, the classical proof. Alternatively, one may wonder if the axiom of symmetry, interpreted over ordinals rather than reals, might become true in our constructive setup.

This question is very related to what we have done so far. The axiom of symmetry on the ordinals,

$$\forall F \exists \sigma, \tau \in O[\sigma \notin F(\tau) \wedge \tau \notin F(\sigma)]$$

is in direct contradiction with conjecture 1. Of course, that doesn't mean it can not be true, but it becomes less plausible.
Unfortunately, I have not (yet) found another way to refute this version of the axiom, so this will also stay an open question for the moment.

## 4.5 Discussion

In this chapter we have considered a constructive form of ordinals, and the axiom of symmetry in this setting. We have seen that, to all likeliness, the axiom itself does not hold in the intuitionistic ordinal setting - although the proof I presented for this needs the additional assumption of double negation shift. However, the constructive form of the refutation of the axiom doesn't work either. In fact, even the not-really-constructive version of the converse of the axiom can't easily be proven!

The questions in this chapter are not all completely answered. Further research could reveal whether it is really possible to refute the axiom of symmetry in an ordinal setting, or, even stronger, whether we can find any $F$ that does not have a symmetric pair.
Alternatively, more research could be spent on defining the ordinals in a constructive way, which is mostly skipped over here. Such research has been done before, but might still lead to more results.

# Chapter 5

# Throwing more darts

In chapter 3 we have seen how to prove the basic version of the theorem of symmetry. We can extend this method to show that the more complicated looking extensions - throwing multiple darts, or basing the functions on multiple points - hold as well!

## 5.1 Throwing many darts

To start, let us consider what happens if, in the setup of the standard axiom of symmetry (every point is mapped to a countable set), we want to choose not just two, but any finite number of points. Can we choose them in such a way that none of them is in the set associated with any of the others?

In the thought experiments Freiling conducted, we would consider throwing a dart at the real number line (where each number is mapped to a countable set of reals). First throw a dart, to fix some number. Throw the next one - the chance it will land in the set associated with the first one is 0, and, by the symmetry of the situation, the first one is not likely to be in the set from the second one either.
Now throw a third dart. The chance that it lands in the sets that belong to either the first or the second is, again, 0. And, by symmetry, so is the chance that any of the others is in the set associated with the third.
And thus we can go on. Throwing any finite number of darts, we see that we could expect that none of the darts will belong to one of the sets associated with the others. Even if we throw countably many darts, the probability that one of them belongs to a set associated with one of the others is still 0 (by the usual symmetric argument!). So, if we allow ourselves to be led by this intuition, we would expect that we can even find countably many points such that none of them belongs to the set associated with any of the others.

This reasoning is essentially equal to the way the standard axiom of symmetry was defended - we just continue on the same principle. For those who (classically) want to accept the statement it leads to ("we can find a countable set $X$ of points, so that for each pair $(x, y)$ of points in $X$, $x$ is not in the set associated to $y$ and $y$ not in the set associated to $x$"), it is therefore quite nice that this seemingly stronger statement turns out to be equivalent to the symmetry axiom itself. Freiling even shows that, if one accepts the negation of the continuum hypothesis, they may extend the symmetry axiom even further: for any cardinal number $\kappa$

less than the cardinality of $\mathbb{R}$, there is a set $X \subseteq \mathbb{R}$ of cardinality $\kappa$ such that for each pair of elements $x, y \in X$, neither is in the set associated with the other.

Of course, in an intuitionistic setup, we can not copy his proof (which assumes wellorderability of the reals and other such dangerous clauses). But if we assume the weak continuity principle again, both the finite and countable versions of the statement hold - we can prove this in what is basically exactly the same way as theorem 2 - we just have to work on finitely or countably many points at the same time.

As before replacing the set $\mathbb{R}$ of the real numbers by Cantor space $\mathcal{C}$, we obtain the following two extensions of the standard axiom of symmetry:

**Theorem 8.** *($A^{*k}$)*
*Let $F$ be a function that maps $\mathcal{C}$ to $\mathcal{C}$.*
*Then there exist $\alpha_1, \ldots, \alpha_k$ such that $\forall n, m \in \{1, \ldots, k\}[n \neq m \rightarrow \forall i \exists j[\alpha_n(j) \neq F(\alpha_m)^i(j)]]$.*

**Theorem 9.** *($A^{*\omega}$)*
*Let $F$ be a function that maps $\mathcal{C}$ to $\mathcal{C}$.*
*Then there exists $\alpha$ such that $\forall n, m \in \mathbb{N}[n \neq m \rightarrow \forall i \exists j[\alpha^n(j) \neq F(\alpha^m)^i(j)]]$.*

I will only prove the second one of these two theorems - theorem 8 follows trivially from 9 and isn't that much easier, so a separate proof would not add anything.

*Construction:*

Since $F$ is still a function that maps $\mathcal{C}$ to $\mathcal{C}$, we can use the same observations as in section 3.3.1. So we may assume that we again have a function $\mathcal{A} : \mathbb{N} \times \{0,1\}^* \rightarrow \mathbb{N}$ such that it is strictly increasing in its parameters, and the properties named in equation 3.2 hold:

$$\forall n \forall a \in \{0,1\}^* \forall \beta[\overline{a * \underline{0}}\mathcal{A}(n,a) = \overline{\beta}\mathcal{A}(n,a) \rightarrow F(a * \underline{0})(n) = F(\beta)(n)]$$

We will define recursively, in each step $q = \langle\!\langle m, k \rangle\!\rangle$, a natural number $X_{q+1}$ and $\alpha(\langle\!\langle n, X_q \rangle\!\rangle), \ldots, \alpha(\langle\!\langle n, X_{q+1} - 1)$ for every $n \in \mathbb{N}$.
To make this work for all $q$, first choose $X_0 = 0$.

Now, in step $q = \langle\!\langle m, k \rangle\!\rangle$, we may assume that each $\alpha^n$ has been defined in the first $X_q$ places. Now define:

$$X_{q+1} = \mathcal{A}(\langle\!\langle k, X_q \rangle\!\rangle, \langle \alpha^m(0), \ldots, \alpha^m(X_q - 1) \rangle)$$

$$\alpha(\langle\!\langle m, X_q \rangle\!\rangle) = \ldots = \alpha(\langle\!\langle m, X_{q+1} - 1 \rangle\!\rangle) = 0$$

$$\text{for each } n \in \mathbb{N},\ n \neq m\text{: } \alpha(\langle\!\langle n, X_q \rangle\!\rangle) = 1 - F(\overline{\alpha^m}X_q * \underline{0})(\langle\!\langle k, X_q \rangle\!\rangle)$$

$$\text{for each } n \in \mathbb{N},\ n \neq m\text{: } \alpha(\langle\!\langle n, X_q + 1 \rangle\!\rangle) = \ldots = \alpha(\langle\!\langle n, X_{q+1} - 1 \rangle\!\rangle) = 0$$

It is easy to see that this process defines $\alpha$ completely and unambiguously.

*Correctness proof:*

To prove the theorem, we must see: $\forall n, m[n \neq m \rightarrow \forall i \exists j[\alpha^n(j) \neq F(\alpha^m)^i(j)]]$.
So let $n \neq m,\ i \in \mathbb{N}$. Take $j = X_{\langle\!\langle m,i \rangle\!\rangle}$ and define $a = \overline{\alpha^m}j$.
Then $X_{\langle\!\langle m,i \rangle\!\rangle+1} = \mathcal{A}(\langle\!\langle i, j \rangle\!\rangle, a)$ and $\overline{a * \underline{0}}X_{\langle\!\langle m,i \rangle\!\rangle+1} = \overline{\alpha^m}X_{\langle\!\langle m,i \rangle\!\rangle+1}$. So we can apply 3.2 to obtain: $F(a * \underline{0})(\langle\!\langle i, j \rangle\!\rangle) = F(\alpha^m)(\langle\!\langle i, j \rangle\!\rangle)$.
Thus, $\alpha(\langle\!\langle n, j \rangle\!\rangle) = 1 - F(\alpha^m)(\langle\!\langle i, j \rangle\!\rangle)$, which gives us what we had to prove. □

## 5.2 Multi-dart based functions

So far, we have considered functions that associate to each point in $\mathbb{R}$or $\mathcal{C}$ a countable subset of $\mathbb{R}$ or $\mathcal{C}$ respectively. But there is little reason why we should only limit ourselves to functions in one variable - with similar probabilistic reasonings as before, we can defend extensions of AS with multi-variable functions.

Suppose, for example, that we have a function $F$ that maps (unordered) pairs of reals to countable sets of reals. Pick any two points on the real line. Pick a third. Using the same probabilistic reasoning as done in chapter 1, note that the chance that the third point falls into the image of the first two points is 0. By symmetry, it is equally implausible that the first falls into the image of the latter two, or that the second falls into the set that comes with the others.
So, surely, there will exist 3 points such that each of them falls outside the set associated to the pair consisting of the other two?

The same argument applies for functions in $n$ variables, for any natural number $n$. If we want the infinite extension, we will have to be a bit more careful.
So suppose we have a function that maps countable sets of reals to countable sets of reals. Choose, at random, a countable set $X \subseteq \mathbb{R}$ (we can do countably many random throws, as we have the (countable) axiom of choice!). Since we have chosen at random, the chance that an element of $X$ belongs to the set $F$ maps the rest to is still 0.
But this argument is different from the previous ones. The symmetry reasoning is dropped, and replaced by simply fixing the other points in the set - countably many times. Could it be that we are crossing a bound?

And indeed, classically this last, infinitary statement fails. That is, by the axiom of choice, we have wellorderability of the class of countable subsets of $\mathbb{R}$. Using this, Freiling shows that there exists a function $F : \mathbb{R} \to \mathbb{R}_\omega$ such that there is no infinite set $X$ with the property: $\forall x \in X[x \notin F(X \setminus \{x\})]$ [2].
However, this only applies to the infinite case. With some extra assumptions about the cardinality of the reals, the classical mathematician can show that for any function $F$ there are such sets $X$ of arbitrary, but finite size. However, unlike theorem 8, the resulting statements are not equivalent to the standard axiom of symmetry: extra assumptions are really necessary for their proof.
I will not go into the classical proof here; the interested reader may look it up in Freiling's article.

To study these extensions from an intuitionistic viewpoint, we must be careful. For what, exactly, is a function on countable sets of reals? Can we assume that the function "knows" the enumeration of such a set?
Sticking to the method of associating an hset directly with its enumeration, I will allow functions that depend on the enumeration of a given countable set. Doing this, both the finite and infinite extension will hold. Again, we only need the weak continuity principle to see this. Formally, the two extensions become:

**Theorem 10.** $A^n$
*Let $F$ be a function that assigns to each subset of $\mathcal{C}$ with $n$ elements an hset of sequences in $\mathcal{C}$. Then there are $\alpha_0, \ldots, \alpha_n \in \mathcal{C}$ such that for each $k \leq n$: $\alpha_k \notin F(\{\alpha_i | i \in \{0, \ldots, n\} | i \neq k\})$.*

**Theorem 11.** $A^\omega$
*Let $F$ be a function that maps the class of hsets of sequences in $\mathcal{C}$ to itself. Then there is an hset $X \subseteq \mathcal{C}$ such that for each $\alpha \in X$: $\alpha \notin F(X \setminus \{\alpha\})$.*

To make the proof both easier and stronger, let us think of hsets purely as infinite sequences. So we think of $F$ as a function that maps sequences of $\alpha \in \mathcal{C} \cup \{\langle\text{undefined}\rangle\}$ to other sequences, with some restrictions that guarantee equivalent results on equivalent sets.
Since the statement only becomes stronger if we make the sets mapped to by $F$ bigger, we may replace every occurence of $\langle$undefined$\rangle$ by $\underline{0}$. We can also choose not to enforce the restrictions on $F$; this, too, only makes the statement stronger.[1]

To formalise the theorems in this way, define for any sequence $\alpha$, any $n$: $\alpha \setminus \alpha^n$ is the sequence such that for all $m, k$: if $m < n$, $\alpha \setminus \alpha^n(\langle\!\langle m, k \rangle\!\rangle) = \alpha(\langle\!\langle m, k \rangle\!\rangle)$; if $m \geq n$, $\alpha \setminus \alpha^n(\langle\!\langle m, k \rangle\!\rangle) = \alpha(\langle\!\langle m+1, k \rangle\!\rangle)$.
If we also define $\mathcal{C}_n$ as the class of those elements $\alpha$ of $\mathcal{C}$ for which each $\alpha^k = \underline{0}$ if $k \geq n$, we can summarise the two theorems above as follows:

$$\mathrm{A}^i: \quad \forall F : \mathcal{C}_i \to \mathcal{C} \exists \alpha \in \mathcal{C}_i \forall n \leq i \forall m \exists k [\alpha^n(k) \neq F(\alpha \setminus \alpha^n)^m(k)] \tag{5.1}$$

$$\mathrm{A}^\omega: \quad \forall F : \mathcal{C} \to \mathcal{C} \exists \alpha \forall n \forall m \exists k [\alpha^n(k) \neq F(\alpha \setminus \alpha^n)^m(k)] \tag{5.2}$$

It is easy to see that 5.2 implies 5.1. As such, I will only give the proof for the latter of the two here; this will give us both theorem 10 and theorem 11.

*Construction:*

So let $F$ be a function that maps $\mathcal{C}$ to itself. The goal is to construct some $\alpha$ such that $\forall k \forall i \exists m [\alpha^k(m) \neq F(\alpha \setminus \alpha^k)^i(m)]$.

As before, we may assume that we have $\mathcal{A} : \mathbb{N} \times \{0,1\}^* \to \mathbb{N}$ such that it is strictly increasing in both parameters, and 3.2 holds:

$$\forall n \forall a \in \{0,1\}^* \forall \beta [\overline{a * \underline{0}}\mathcal{A}(n,a) = \overline{\beta}\mathcal{A}(n,a) \to F(a * \underline{0})(n) = F(\beta)(n)]$$

To use a construction similar to the one before, we will choose, in each step $q = \langle\!\langle n, m \rangle\!\rangle$ a natural number $X_{q+1}$ and for each $n$ binary values $\alpha(\langle\!\langle n, X_q \rangle\!\rangle), \ldots, \alpha(\langle\!\langle n, X_{q+1} - 1 \rangle\!\rangle)$. The choices in step $q = \langle\!\langle n, m \rangle\!\rangle$ will be made in such a way as to secure that $\alpha^n(X_q) \neq F(\alpha \setminus \alpha^n)^m(X_q)$.
First let $X_0 = 0$.

Now, in each step $q = \langle\!\langle n, m \rangle\!\rangle$ we can inductively assume that we have $X_q$ and for each $k$ $\overline{\alpha^k}X_q$. Define $X_{q+1}$ and the next parts of $\alpha$ as follows:

$X_{q+1} = \mathcal{A}(\langle\!\langle m, X_q \rangle\!\rangle, \langle a_0, \ldots, a_{\langle\!\langle q, X_q \rangle\!\rangle - 1} \rangle)$, where $a_i = (\alpha \setminus \alpha^n)(i)$ if this number is already defined, 0 otherwise.

For each $k \neq m$, let $\alpha(\langle\!\langle k, X_q \rangle\!\rangle) = \ldots = \alpha(\langle\!\langle k, X_{q+1} - 1 \rangle\!\rangle) = 0$.

To conclude, choose $\alpha(\langle\!\langle n, X_q \rangle\!\rangle) = 1 - F(\langle a_0, \ldots, a_{\langle\!\langle q, X_q \rangle\!\rangle - 1} \rangle * \underline{0})(\langle\!\langle m, X_q \rangle\!\rangle)$ and $\alpha^n(X_q + 1), \ldots, \alpha^n(X_{q+1} - 1) = 0$.

[1] The statement becomes *significantly* stronger if we drop this requirement - with an eye on the continuity principle, one might imagine this requirement would be very limiting when trying to construct interesting functions.

*Correctness proof:*

To see that $\alpha$ suffices, we must show that $\forall n \forall m \exists k[\alpha^n(k) \neq F(\alpha \setminus \alpha^n)^m(k)]$.
So let $n, m \in \mathbb{N}$, define $q = \langle\!\langle n, m \rangle\!\rangle$ and choose $k = X_q$. Also say $a = \langle a_0, \ldots, a_{\langle\!\langle q, X_q \rangle\!\rangle - 1} \rangle$, where the $a_i$ are the ones defined in step $q$.
From the construction it trivially follows that $\alpha(\langle\!\langle n, k \rangle\!\rangle) \neq F(a * \underline{0})(\langle\!\langle m, k \rangle\!\rangle)$. We also have that $X_{q+1} = \mathcal{A}(\langle\!\langle m, k \rangle\!\rangle, a)$.

So if only we had that $\overline{a * \underline{0}} X_{q+1} = \overline{\alpha \setminus \alpha^n} X_{q+1}$, we could apply 3.2 to see that $F(a * \underline{0})(\langle\!\langle m, k \rangle\!\rangle) = F(\alpha \setminus \alpha^n)^m(k)$, which gives us the needed result.

To prove that indeed $\overline{a * \underline{0}} X_{q+1} = \overline{\alpha \setminus \alpha^n} X_{q+1}$, let $l := \langle\!\langle i, j \rangle\!\rangle < X_{q+1}$. We must see that $a * \underline{0}(l) = (\alpha \setminus \alpha^n)(l)$.
Consider the possibilities: either $l < \langle\!\langle q, X_q \rangle\!\rangle$ or $\langle\!\langle q, X_q \rangle\!\rangle \leq l$.
In the first case, if $j < X_q$ then $a_l = (\alpha \setminus \alpha^n)(l)$. Otherwise $X_q \leq j < X_{q+1}$ and $(\alpha \setminus \alpha^n)(l) = 0$, as is $a_l$.
In the second case, we only have to prove that $(\alpha \setminus \alpha^n)(l) = 0$.

So suppose that $\langle\!\langle q, X_q \rangle\!\rangle \leq \langle\!\langle i, j \rangle\!\rangle$. Then either $i \geq q$ or $j \geq X_q$. In the latter case, $X_q \leq j < X_{q+1}$, $(\alpha \setminus \alpha^n)(l)$ is set to 0 in step $q$.
In the first case, $(\alpha \setminus \alpha^n)(l) = \alpha^{i+1}(j)$. With a short induction, we can see that $\overline{\alpha^{i+1}} X_q = \overline{\underline{0}} X_q$: $\forall s \in \mathbb{N} \forall t \in \mathbb{N}[t \geq s \rightarrow \overline{\alpha^t} X_s = \overline{\underline{0}} X_s]$, bij induction on $s$.
Base step: For $s = 0$, this is trivial as $X_0 = 0$.
Induction hypothesis: $\forall t \in \mathbb{N}[t \geq s' \rightarrow \overline{\alpha^t} X_{s'} = \overline{\underline{0}} X_{s'}]$.
Induction step: Let $s = s' + 1$ and let $t \geq s$. By the induction hypothesis: $\overline{\alpha^t} X_{s'} = \overline{\underline{0}} X_{s'}$. Since $t \neq s'_1$, in step $s'$ $\alpha^t(X_{s'}), \ldots, \alpha^t(X_s - 1)$ are all set to 0, so $\overline{\alpha^t} X_s = \overline{\underline{0}} X_s$. □

So if $\langle\!\langle q, X_q \rangle\!\rangle \leq l$, then always $(\alpha \setminus \alpha^n)(l) = 0$, which completes our proof! □

## 5.3 Ordered multi-dart functions

When translating from countable sets to infinite sequences in the previous section, we allowed all kinds of functions, dropping the requirement that the function, when presented with two sequences that correspond to equal sets, will give equivalent results. If anything, this makes the result stronger than what we originally asked for.

But we can make it yet stronger! In the sequence-translations of the theorems, if the function we use does respect equality, the corresponding set will be such that each member is unequal to the image of the rest - no matter how we translate the rest. More precisely, if, in 5.2, $F$ respects set equality, then we will find some $\alpha$ such that for any $k$, $\alpha^k$ is constructively different from any $F(\beta)^n$, where $\beta$ is any element of $\mathcal{C}$ such that $\{\beta^m | m \in \mathbb{N}\} = \{\alpha^m | m \in \mathbb{N} | m \neq k\}$.
To get a new result, we could ask ourselves: can we always find such an $\alpha$, even if $F$ does not always respect set equality (or when we don't know)?

To elaborate on this question, define the notion of a permutation:

**Definition 3.** Permutation
An element $\alpha$ of $\mathcal{N}$ is called a *permutation of* $n$ if for each $m < n$ there is some $k < n$ such that $\alpha(k) = m$. So the first $n$ places of $\alpha$ contain the number $0 \ldots n-1$ in some arbitrary order.

An element $\beta$ of $\mathcal{C}$ is called a *permutation of* $\mathbb{N}$ if for each $m$ there is a unique $k$ such that $\alpha(k) = m$. So $\beta$ would have all the elements of $\mathbb{N}$, in some arbitrary order.

For any permutation $\pi$ and $\alpha \in \mathcal{N}$, define $\alpha^\pi$ as the sequence with $(\alpha^\pi)^n = \alpha^{\pi(n)}$ for all $n$. Now we can find, with very little difficulty, that the extension of theorem 10 holds:

**Theorem 12.** *$A^n$ extended*
*For all $F : \mathcal{C}_n \to \mathcal{C}$ there exists some $\alpha$ such that for all permutations $\pi$ of $n$: $\forall k \leq n \forall i \exists m[\alpha^k(m) \neq F((\alpha \setminus \alpha^k)^\pi)^i(m)]$.*

I will not prove the theorem here, but it is easy - we simply enumerate all possible permutations (there are only finitely many!), and continue, in step $\langle\!\langle k, m \rangle\!\rangle$ (if $k < n$, otherwise we skip the step) $\alpha$ in such a way that $\alpha^k$ will be unequal to $F(\beta)^m$ for every permutation $\beta$ of $\alpha \setminus \alpha^k$.

We can not apply the same principle to the infinite case: the permutations of any given countable sequence are not finite - not even countable, which would also lead to a solution.

I have not yet been able to prove or reject this extension of the theorem. Formally, it becomes:

**Conjecture 2.** *$A^\omega$ extended*
*For all $F : \mathcal{C} \to \mathcal{C}$ there exists some $\alpha$ such that for all permutations $\pi$ of the natural numbers: $\forall k \forall i \exists m[\alpha^k(m) \neq F((\alpha \setminus \alpha^k)^\pi)^i(m)]$.*

I will leave this question open for now - it might be an interesting assignment, or more trivial than it seems at first.

# Chapter 6

# The axiom of symmetry and measurability

## 6.1 Introduction

When defending his axiom of symmetry, Freiling formulated a stronger version of the axiom where a *null set* is associated to every point, rather than just a countable set. This variation implies the standard axiom, as every countable set of reals is a null set.

In this chapter, I will investigate this question, and the standard multi-dart extensions. I will use the fan theorem, in the strong form, as explained in chapter 2.

## 6.2 Definitions

First we will need to know what a measurable subset of $\mathbb{R}$ is; or, at least, what a null set is.

So define the notion of a finite *areal*: this is a finite sequence of rational intervals, say an element of $S^*$.

For any finite areal $a \in S^*$, define $\mu(a)$ as "the total size of the area covered by $a$". That is, define $\mu$ with a recursion on the length of $a$: for $s \in S$, $a$ a finite areal, $\mu(\langle\rangle) = 0,\ \mu(a * \langle s \rangle) = \mu(a) + |s| - \mu(\langle a(0) \cap s, a(1) \cap s, \ldots, a(\mathrm{len}(a) - 1) \cap s \rangle)$.

A measurable areal, then, is a sequence $\alpha \in \mathcal{N}$ with the property that $\forall n[\alpha(n) \in S]$ and that the sequence $\mu(\overline{\alpha}0),\ \mu(\overline{\alpha}1),\ \mu(\overline{\alpha}2),\ \ldots$ converges.
Define $\mu(\alpha) = \lim_{n\to\infty} \mu(\overline{\alpha}n)$.
A measurable areal can be thought of as a constructive form of an *open set*.

For a finite or measurable areal, write $x \in \alpha$ to denote: $\exists m, n[\alpha(n)' < x(m)' < x(m)'' < \alpha(n)'']$. Then $x \notin \alpha := \neg(x \in \alpha)$. For a stronger version of non-inclusion, say $x \notin \alpha := \forall m \exists n[x(n)'' < \alpha(m)' \vee \alpha(m)'' < x(n)']$, so $x$ is really outside (and not even on an edge) of every interval in $\alpha$.

Now we could define measurability on subsets of the real line $\mathbb{R}$ as "a set is measurable if it can be approached by closed subsets and open supersets" [1]. From such a definition, we can derive that all measurable areals are, indeed, measurable.
However, we don't particularly need this. For the results in this chapter, I will only need null sets, and measurable areals themselves. As such, we can leave the exact definition of a measurable set in the middle - it would only complicate things needlessly.
We do need to define a null set: this is a subset of $\mathbb{R}$ with the property that for each $k$, it is covered by a measurable areal $\alpha$ such that $\mu(\alpha) < 2^{-k}$.

For ease of writing, I also define the following abbreviations: let $a, b \in S,\ \alpha, \beta$ measurable or finite areals:

**$a \cup b$:** the finite areal $\langle a, b \rangle$

**$\alpha \subseteq b$:** $\forall m[\alpha(m) \subseteq b]$

**$\alpha \cap \beta$:** the finite or measurable areal that covers exactly those intervals that are covered both by $\alpha$ and $\beta$. If $\alpha$ and $\beta$ are both measurable areals, for example, this will be the $\gamma$ such that $\gamma(\langle\!\langle n, m \rangle\!\rangle) = \alpha(n) \cap \beta(m)$. If, for example, the first one is finite, use a variation of $\langle\!\langle , \rangle\!\rangle$ that induces a bijection $\{0, \ldots, n-1\} \times \mathbb{N}$ to $\mathbb{N}$ instead.

**$\alpha \cup \beta$:** the finite or measurable areal that contains all the intervals either in $\alpha$ or $\beta$.

Write $\mathcal{A}$ for the class of measurable areals.
We define $\widetilde{\mathcal{N}}$ to be the class of those elements $\alpha$ of $\mathcal{N}$ such that $\forall n[\alpha^n \in \mathcal{A} \wedge \mu(\alpha^n) < 2^{-n}]$.
Using this definition, we can almost identify $\widetilde{\mathcal{N}}$ with the class of null-sets: for each $A \subseteq \mathbb{R}$ we know that $A$ has measure 0 if and only if, for some $\alpha \in \widetilde{\mathcal{N}}$, every $x \in A$ is contained in every $\alpha^n$ (or, brief, $A \subseteq \bigcap_{n\in\mathbb{N}} \alpha^n$).

For $A \in \widetilde{\mathcal{N}}$ we define: $x \in A := \forall n[x \in A^n]$, $x \notin A := \exists n[x \notin A^n]$.

## 6.3 Ponderings

How would you constructively prove a theorem like $\mathrm{A}_{\mathrm{null}}$ that states that for each function that maps the reals to the nullsets, there are two points not belonging to each other's image? When a nullset, to us, is nothing more than a shrinking sequence of measurable areals, should we point out an element of this sequence that does not cover a given point $x$ to see that $x$ is outside the nullset? It sounds reasonable - but if we want to interpret "not belonging to each other's image" in a strong way like that, doesn't the axiom suddenly become a lot stronger?

I would say it does. Note that every nullset can be represented by a shrinking sequence of measurable areals with the additional property that each of these areals can be chosen strictly inside the previous one. So if we do this, and if we can then find some $x$, $y$, $n$ and $m$ such that $x \notin F(y)^n$ and $y \notin F(x)^m$, then taking $k = \max(n, m)$ we have that $x \notin F(y)^k \wedge y \notin F(x)^k$. So, for any $F$ that maps the reals to the nullsets, we can find some function $G$ that maps to measurable areals, such that each $F(x)$ is covered by $G(x)$, and $G$ has a symmetric pair.

[1] For example, a subset $X$ of $[0, 1]$ is measurable with measure $\nu$ if, for each $\epsilon > 0$, we can find measurable areals $\alpha, \beta$ such that $1 - \mu(\alpha) > \nu - \epsilon$, $\mu(\beta) < \nu + \epsilon$ and $\{x | x \in \mathbb{R} | x \notin \alpha\} \subseteq X \subseteq \{x | x \in \mathbb{R} | x \in \beta\}$.

From this, one may wonder whether there is some $k$ such that *any* function $G$ that maps to the finite areals of size $2^{-k}$ has a symmetric pair.

We will find that this very constructive interpretation of $A_{null}$ holds: we can find a number $k$ such that each function that maps the reals to measurable areals of measure less than $2^{-k}$ has a symmetric pair. In fact, we will see that it holds for $k = 1$. It is easy to see that a smaller $k$ is not possible!

## 6.4 Finite version

In order to make the reader familiar with the idea of the proof, I will first show a version for finite areals:

**Theorem 13.** *Let $F$ be a function that maps $[0, 1]$ to $\mathbb{N}^*$ such that $\forall x[F(x)$ is a finite areal with $\mu(F(x)) \leq \frac{1}{2}]$. $F$ does not have to respect $\equiv$* [2]*.*
*Then there are $x, y \in [0, 1]$ such that $x \notin F(y)$ and $y \notin F(x)$.*

*Proof.* First of all, since $S^* \subseteq \mathbb{N}$, we can apply the continuity principle to obtain that $\forall x \in [0, 1]\exists n\forall y \in [0, 1][\overline{x}n = \overline{y}n \to F(x) = F(y)]$. Limiting our function to $[0, 1]^*$, we can apply the (extended) fan theorem to this formula and conclude that there exists some $N \in \mathbb{N}$ such that $\forall x, y \in [0, 1]^*[\overline{x}N = \overline{y}N \to F(x) = F(y)]$.

Now look at all standardintervals $a$ of order $N$ and define, for all $s \in S^*$, $Fa = s$ iff for some $x \in a$, $F(x) = s$. By the choice of $N$, we can test this by taking *any* $x \in a$.
Each $Fa$ has measure at most $\frac{1}{2}$, and thus covers at most $2^{N-1}$ intervals $b \in S_N$ completely. With a simple counting argument we see that there must be an interval $b \in S_N$ such that $\#\{a|a \in S_N|b \subseteq Fa\} \leq 2^{N-1}$ (we can just enumerate all the elements $b$ of $S_N$ and determine whether they fulfill the condition). $Fb$ will also cover at most half the standard order $N$ intervals.

Let $A = \{a|a \in S_N|b \subseteq Fa\}$ and $B = \{a|a \in S_N|a \subseteq Fb\}$. Then either $b \in B$ or $b \notin B$.

Suppose $b \in B$. Then, trivially, $b \in A$. So $\#(A \cup B) \leq 1 + \#(A \setminus \{b\}) + \#(B \setminus \{b\}) \leq 1 + (2^{N-1} - 1) + (2^{N-1} - 1) = 2^N - 1$, so there is some $c \in S_N$ with $c \notin A \cup B$. $c$ is not completely covered by $Fb$, so there is some $y \in c$ such that $y \notin Fb$. $b$ is not completely covered by $Fc$, so there is some $x \in b$ with $x \notin Fc$. These $x$ and $y$ satisfy the theorem.

Now suppose $b \notin B$. Then there is a subinterval $c \subseteq b$ such that $c$ is completely uncovered by $Fb$. Take $x$ inside $c$. Then $x \notin F(x)$ and we're done. ☐

Note that, unlike in the solution to the standard axiom (and extensions), we do not use a symmetric argument to build up $x$ and $y$ here. We just fix a special interval that is covered only by the images of "few" of the other intervals and take $x$ there. For $y$, we then only have to pick it in some interval that both isn't covered by the image of $x$ (there are few of those) and whose image doesn't cover $x$ (also few, as that's how we chose $x$). The case split $b \in B \vee b \notin B$ is only necessary for the situation that always $\mu(F(x)) = \frac{1}{2}$.

[2] This clause is necessary to make the problem in any way interesting: if $F$ does respect $\equiv$, it is constant by the continuity principle!

As a bonus, this is a sharp theorem; if we allow $F$ to map to larger sets, even by any $\epsilon > 0$, it is possible to construct such functions that do not have a symmetric pair.

*Proof of this claim:*

Give me any $\epsilon$. Determine $N \in \mathbb{N}$ such that $2^{-N+1} < \epsilon$. I will create a function $F$ such that for any two standardintervals $a, b \in S_N$, either $\forall x \in [a', a''][x \in F(b)]$ or $\forall x \in [b', b''][x \in F(a)]$.

*Idea of the proof:*

The image of each standardinterval of order $N$ may cover $2^{N-1} + 1$ standard intervals of the same order completely. We always make the image of such an interval cover the interval itself; for the rest we just choose the images of the intervals in such a way that for every pair of standard intervals $a, b$, either $a \in F(b)$ or $b \in F(a)$ (but never both).

*Construction:*

First create a finite areal $\alpha$ of size $2^{-N}$ such that all points $\frac{a}{2^N}$ ($a \in \{0, \ldots, 2^N\}$) are covered by $\alpha$; say $\alpha(n) = \langle n \cdot 2^{-N} - \frac{2^{-N}}{2^{-N}+1}, 2^{-N} + \frac{2^{-N}}{2^{-N}+1} \rangle$.
Now define, for $n \in \{0, \ldots, 2^N - 1\}$, a finite areal $An$ as follows:

$$An = \begin{cases} \langle X(0), \ldots, X(n) \rangle * \langle X(2^{N-1}), \ldots, X(2^N - n - 1) \rangle & \text{if } n < 2^{N-1} \\ \langle X(n), \ldots, X(2^N - 1) \rangle * X(2^N - n), \ldots, X(2^{N-1} - 1) \rangle & \text{if } n \geq 2^{N-1} \end{cases}$$

Here, $X(a)$ is the standardinterval $\langle a * 2^{-N}, (a+1) * 2^{-N} \rangle$.

Now define a function $F : \mathbb{R} \to \mathcal{A}$ as follows: given $x \in \mathbb{R}$, determine the first $n$ such that either $x(n)$ falls strictly inside one of the intervals in $\alpha$, or $x(n)$ falls strictly inside some standardinterval of order $N$. Then define $F(x)$ to be $\alpha$ if $x(n)$ proved that $x$ falls inside $\alpha$, otherwise $F(x) = \alpha \cup An$.

*Correctness proof:*

Some observations about the definition of $An$:

- For all $n$: $X(n)$ is one of the intervals in $An$.
- If $n < m < 2^{N-1}$: $X(n)$ is one of the intervals in $Am$.
- If $2^{N-1} \leq n < m$: $X(m)$ is one of the intervals in $An$.
- If $n < 2^{N-1} \leq m$: if $m < 2^N - n$, then $X(m)$ is one of the intervals in $An$, otherwise $m - 2^N \geq -n$, so $n \geq 2^N - m$ and $X(n)$ is one of the intervals in $Am$.
- For $n < 2^{N-1}$: $An$ covers $2^{N-1} + 1$ standardintervals of order $N$ completely. For $n \geq 2^{N-1}$: $An$ covers $2^{N-1}$ of these intervals.

Also note that $F$ is defined without any free choice. For all $x$, $\mu(F(x)) \leq \mu(\alpha) + (2^{N-1} + 1) \cdot 2^{-N} = 2^{-N} + 2^{-1} + 2^{-N} = 2^{-N+1} + 2^{-1} < 2^{-1} + \epsilon$.

Give me any $x, y \in [0, 1]$. If we first find that $x \in \alpha$ then we know that $x \in F(y)$ and are done. So assume $x$ is inside some standardinterval $a$. Similarly, assume $y$ is inside some standardinterval $b$.
Now we have seen that always either $a$ is covered by $F(y)$ or $b$ is covered by $F(x)$. $\square$

## 6.5 The axiom of symmetry for sets with small measure

The infinite case is a lot more difficult, because we can't point out intervals whose members all have exactly the same image. Fortunately, we can still apply a similar reasoning, where we find a single real that isn't covered "too much" by the images of the rest. To keep a bit of space in the choice of intervals, the function in this theorem will only map to sets of measure strictly smaller than $\frac{1}{2}$.

We intend to prove the following:

**Theorem 14.** $A_{<\frac{1}{2}}$
*Let $F$ be a function that maps $[0,1]$ to $\mathcal{A}$ such that $\forall x[\mu(F(x)) < \frac{1}{2}]$.*
*Then there exist $x, y \in [0,1]$ such that $x \notin F(y)$ and $y \notin F(x)$.*

Having this theorem $\mathrm{A}_{\mathrm{null}}$ naturally becomes trivial.

The theorem itself is not easy to prove. We will do it in two steps:

- Find some real number $x$ and a measurable areal $\alpha$ such that $x$ is only covered by the images of points in $\alpha$. We should choose $x$ in such a way that $\alpha$ has small measure, say no more than $\frac{1}{2}$.
- Find a real number $y$ that is both outside $\alpha$ and $F(x)$. This should be possible because their joint measure is smaller than 1.

We will need the (extended) fan theorem again. To make the proof easier to follow, I will use separate lemmas for both steps.

First we need the well-known fact that if a measurable subclass of $[0,1]$ has measure less than 1, there will be some real in $[0,1]$ outside of it. We will need this lemma for the second step, which is the easier of the two - although the method used in this proof is very similar to what we will use for the first step.

**Lemma 1.**
*Let $\beta \in \mathcal{A}$ be such that $\mu(\beta) < 1$.*
*Then there exists $x \in [0,1]$ such that $x \notin \beta$.*

This lemma will follow without any additional principles.

*Idea of the proof:*

As a very rough guide to what we are going to do, essentially the idea is as follows: let $\beta \in \mathcal{A}$ be such that $\mu(\beta \cap [0,1]) < 1$. Then we can show that $\mu(\beta \cap [0, \frac{1}{2}]) < \frac{1}{2} \vee \mu(\beta \cap [\frac{1}{2}, 1]) < \frac{1}{2}$. Suppose $\mu(\beta \cap [0, \frac{1}{2}]) < \frac{1}{2}$. Then $\mu(\beta \cap [0, \frac{1}{4}]) < \frac{1}{4} \vee \mu(\beta \cap [\frac{1}{4}, \frac{1}{2}]) < \frac{1}{4}$. And so on. By choosing a smaller interval every time, eventually we fix a value $x$ that suffices!

Now in more detail.
We will build $x$ inductively, while learning more about $\beta$. In each step $n$, we start by determining the first $M$ places of $\beta$, where $M$ is chosen so large that the remainder of $\beta$ is relatively small ($\mu(\beta) - \mu(\overline{\beta}M) < 2^{-t_n}$ for some cleverly chosen $t_n$). Assuming we have already defined $x(n)$, we can then define $x(n+1)$ to be the half of $x(n)$ that is least covered by $\overline{\beta}N$.

Of course, for this construction to be useful, we would always have to be able to choose a half that is not completely covered by the start of $\beta$. We will also have to take precautions to avoid that the resulting $x$ lies on the edge of any of the intervals in $\beta$, as we wanted strict separation ($\notin$) of $x$ from $\beta$.

So we will create, in step $n$, a set $A_n$ of standardintervals that are strictly separated from $\overline{\beta}M$ and inside $x(n+1)$. Using the clever definition of $t_n$, we will be able to see that $A_n$ will always cover a reasonable part of $[0,1]$ - more than what the remainder of $\beta$ can spoil. By the construction, $x$ will end up in each of the $A_n$, securing it is strictly outside $\beta$.

*Observations:*

To avoid making exceptions, it would be nice if none of the intervals in $\beta$ overlapped (so there would never be $n, m$ such that $n \neq m \wedge \beta(m)' < \beta(n)'' \wedge \beta(n)' < \beta(m)''$).
But we can modify $\beta$ to force this. Simply cut each interval $\beta(n)$ up in subintervals, so that each subinterval has either been completely covered by the intervals $\beta(0), \ldots, \beta(n-1)$, or not at all. Then make a new areal $\beta'$ by enumerating all the parts that have not been covered before. There are some points $x$ with $x \in \beta$ and not $x \in \beta'$ (endpoints of intervals), but $x \notin \beta$ if and only if $x \notin \beta'$.

To conclude, in the proof of this theorem we can safely assume that $\beta$ does not have any overlaps.

*Construction:*

$\mu(\beta) < 1$, so there must be $t$ such that $\mu(\beta) < 1 - 2^{-t}$. Now define $t_0 := t$ and for all $n > 0$: $t_n := t + 2(n+1)$.

I will define, recursively, for all $n$: $M_n, R_n \in \mathbb{N}$ as well as a set $A_n \subseteq S_{R_n}$ and $x(n+1) \in S_{n+1}$, $x(n+1) \subseteq x(n)$ (so $x$ will be an element of $[0,1]^*$). The following conditions will hold for each $n$:

1. $\#A_n \geq 2^{R_n} \cdot 2^{-t} \cdot 4^{-(n+1)}$
2. $\forall a \in A_n [a \subseteq x(n+1)]$
3. $\forall a \in A_{n+1} \exists b \in A_n [a \subseteq b]$
4. $\mu(\beta) - \mu(\overline{\beta}M_n) < 2^{-t_n}$ if $n > 0$
5. $R_{n+1} \geq R_n$

To this end, we first define $M_0 = 0$. $\beta$ is a measurable areal, so $\mu(\overline{\beta}k)$ converges to $\mu(\beta)$ and we can find, for all $n > 0$, $M_n$ such that $|\mu(\overline{\beta}M_n) - \mu(\beta)| < 2^{-t_n}$. Since $\mu(\overline{\beta}k)$ is increasing for $k$ increasing this gives us point 4 for all $n$.
We also define $x(0) =\langle\!\langle 0,1 \rangle\!\rangle$ and, for writing ease, $A_{-1} = (\langle\!\langle 0,1 \rangle\!\rangle)$, $R_{-1} = 0$. Note that $A_{-1} \subseteq S_{R_{-1}}$ and $x(-1+1) \in S_{-1+1}$.

Now we define, inductively, for all $n$, $A_n$, $R_n$ and $x(n+1)$ as follows:

Determine $R_n$ such that either $M_{n+1} \cdot 2 \cdot 2^{-R_n} < 2^{-t_n-1}$ if $n = 0$, or $M_{n+1} \cdot 2 \cdot 2^{-R_n} < 2^{-t_n}$ if $n > 0$. Choose $R_n$ large enough, such that $R_n \geq R_{n-1}$, to secure point 5 for the inductionhypothesis. We automatically have that $R_n \geq t + n + 1$, so $2^{R_n} \cdot 2^{-(t+n+1)}$ will always be an integer.

Let $B$ be the set of all standardintervals $a \in S_{R_n}$ such that $a$ is not strictly outside $\beta(M_n), \ldots, \beta(M_{n+1}-1)$ (so $a$ has some overlap, at least an endpoint, with these parts of $\beta$). Since we assumed $\beta$ wouldn't overlap itself, $\mu(\langle\beta(M_n), \ldots, \beta(M_{n+1}-1)\rangle)$ will at most be $\mu(\beta) - \mu(\overline{\beta}M_n)$.
So $\#B$ = (number of intervals of order $R_n$ that are covered by $\beta$) + (number of other intervals of order $R_n$ that share points with $\beta$) $\leq 2^{R_n} \cdot \mu(\langle\beta(M_n), \ldots, \beta(M_{n+1}-1)\rangle) + (M_{n+1} - M_n) \cdot 2 \leq 2^{R_n} \cdot (\mu(\beta) - \mu(\overline{\beta}M_n)) + M_{n+1} \cdot 2 \leq$
$\begin{cases} 2^{R_n} \cdot (1 - 2^{-t_0}) + 2^{R_n} \cdot (2^{-t_0-1}) = 2^{R_n} \cdot (1 - 2^{-t_0-1}) & \text{if } n = 0 \\ 2^{R_n} \cdot 2^{-t_n} + 2^{R_n} 2^{-t_n} = 2^{R_n} \cdot 2^{-t_n+1} & \text{if } n > 0 \end{cases}$

Now let $C$ be the set consisting of all standardintervals of order $R_n$ that are not in $B$, but are covered by some element of $A_{n-1}$ (so $C$ contains all standardintervals of order $R_n$ that we would like $x$ to be in). Since $R_n \geq R_{n-1}$ we know that an interval $a \in S_{R_n}$ is either covered completely by an interval $b \in S_{R_{n-1}}$ or not at all.
If $n = 0$, then $C$ contains exactly the $a \in S_{R_n}$ that are not in $B$, so $\#C \geq 2^{R_n} \cdot 2^{-(t_0+1)} = 2^{R_n} \cdot 2^{-(t+2n+1)}$.
If $n > 0$, then by the inductionhypothesis $\#A_{n-1} \geq 2^{R_{n-1}} \cdot 2^{-t} \cdot 4^{-n}$, so since $A_{n-1}$ contains standardintervals of order $R_{n-1}$, at least $2^{R_n} \cdot 2^{-(t+2n)}$ standardintervals of order $R_n$ are covered by some element of $A_{n-1}$. At most $2^{R_n} \cdot 2^{-t_n+1} = 2^{R_n} \cdot 2^{-(t+2n+1)}$ of these are covered by $B$. So $\#C \geq 2^{R_n} \cdot 2^{-(t+2n+1)}$.

Now look at the rational intervals $a := \langle\!\langle x(n)', \frac{x(n)''-x(n)'}{2} \rangle\!\rangle$ and $b := \langle\!\langle \frac{x(n)''-x(n)'}{2}, x(n)'' \rangle\!\rangle$. Since all the intervals in $C$ are covered by elements of $A_{n-1}$, and those again are covered by $x(n)$, and because $R_n \geq n+1$, at least one of the halves $a, b$ of $x(n)$ will contain at least half the elements of $C$. Take $x(n+1)$ to be this half of $x(n)$. Take $A_n$ to be the subset of $C$ that contains only those intervals that are covered by $x(n+1)$. Then $\#A_n \geq 2^{R_n} \cdot 2^{-(t+2n+1)} \cdot \frac{1}{2} = 2^{R_n} \cdot 2^{-t} \cdot 4^{-(n+1)}$. This secures point 1 for the inductionstep. From the rest of the construction we see directly that the other conditions are filled as well.

*Correctness proof:*

First, we note that $\forall n[x(R_n) \in A_n]$.
After all, for any $b \in A_{R_n}$, both $b \subseteq x(R_n)$ (by 2) and for some $a \in A_n$: $b \subseteq a$ (by 3 and a trivial induction).
We can determine whether $x(R_n)$ and $a$ are the same or not; but if they are not, they would at most have a single point in common (both being standard intervals of order $R_n$), and $b$ couldn't be covered by both. So $a = x(R_n)$, and thus $x(R_n) \in A_n$.

Now, to conclude that $x \# \beta$, we must see that $\forall m \exists n[x(n)'' < \beta(m)' \vee \beta(m)'' < x(n)']$.
So let $m \in \mathbb{N}$ and find $k$ such that $M_k \leq m < M_{k+1}$. In step $k$ we choose $A_k$ in such a way that $\forall a \in A_k[x$ is strictly outside $\beta(M_k), \ldots, \beta(M_{k+1}-1)]$, so we see that $\forall a \in A_k[a'' < \beta(m)' \vee \beta(m)'' < a']$.
So we can choose $n = R_k$ to get the wanted conclusion. $\square$

To prove theorem 14, we also need to work on the first of the two steps we discussed before: finding a number $x$ that is only covered by the images of points in a relatively small areal. Here, too, I will use counting arguments and reasoning with formulas to get the idea of the proof exact.
In this lemma, I will use the fan theorem.

**Lemma 2.**
*Let $F$ be a function that maps $[0,1]$ to $\mathcal{A}$ such that $\forall x \in [0,1][\mu(F(x)) < \frac{1}{2}]$ ($F$ does not have to respect $\equiv$).*
*Then there exist $x \in [0,1]$ and $\beta \in \mathcal{A}$ with the properties that $\mu(\beta) \leq \frac{1}{2}$ and $\forall y \in [0,1]^*[y \notin \beta \to x \notin F(y)]$.*

*Idea of the proof:*

We will build $x$ recursively, while slowly learning more about $F(y)$ for all standard reals $y$ (simultaneously).
During the procedure, we will keep a list of standard intervals that are still "okay" for $x$ to be in. In each step $n$, the list will only consists of subintervals of the okay intervals of the previous step.

First of all, note that we can determine most of $F(y)$ (for any given $y$) while only a beginning of $y$ is known (by the continuity principle). Using the fan theorem, we can find an $N$ such that any standard real $y$ inside the same standardinterval of order $N$ will have the same value for $\overline{F(y)}M$. Here we can choose $M$ however we want - we choose it in such a way that $\mu(\overline{F(y)}M) > \mu(F(y)) - 2^{-t_n}$ for a clever choice of $t_n$.

So in step $n$ we can consider all standardintervals of the corresponding order $N$ and determine the parts of our okay intervals that are covered only by relatively few of these. We will then choose some standard subintervals of the previous okay intervals that aren't covered "too much" (where too much is given by some cleverly chosen equation), and call these *good*.
Pick the half of $x(n)$ that covers most of these intervals; call the good intervals inside $x(n)$ okay.

This way, $x$ will end up inside one of the chosen okay intervals in each step. For each step $N$, take the standardintervals whose image covers $x$ in the areal $\beta$; $\beta$ will not be larger than $\frac{1}{2}$.

*Observations:*

Since $\forall x \in [0,1][\mu(F(x)) < \frac{1}{2}]$, we can see that $\forall x \exists t \in \mathbb{N}[\mu(F(x)) < \frac{1}{2} - 2^{-t}]$. Applying the fan theorem to that statement (choosing a fan-representation of $[0,1]$), we find a $T \in \mathbb{N}$ such that $\forall x \exists t \leq T[\mu(F(x)) < \frac{1}{2} - 2^{-t}]$, so $\forall x[\mu(F(x)) < \frac{1}{2} - 2^{-T}]$.

Since $F$ maps to $\mathcal{A}$ and by the definition of measurable areal, we also know that $\forall m \forall x \exists n[\mu(\overline{F(x)}n) \geq \mu(F(x)) - 2^{-m}]$. Applying the fan theorem to this, too, we find: $\forall m \exists N \in \mathbb{N} \forall x[\mu(\overline{F(x)}N) \geq \mu(F(x)) - 2^{-m}]$.

We can not draw these conclusions without the fan theorem.

As before, we will also assume that $F$ maps to measurable areals without overlap. Since we suggested a well-defined procedure to get an areal without overlaps from a general measurable areal, we can assume this without needing additional choice axioms.

*Construction:*

First we determine $T$ such that $\forall x[\mu(F(x)) < \frac{1}{2} - 2^{-T}]$.
Define, for $n > 1$: $t_n := 2T + 3n + 4$.
For each $n > 0$ we can determine $M_n$ such that $\forall x \in [0,1][\Sigma_{m \geq M_n} \mu(F(x)(m)) < 2^{-t_n}]$. Choose also $M_0 := 0$. Applying the fan theorem again for every $n$, we can find $N_n$ such that $\forall x, y \in [0,1][\overline{x}(N_n + 1) = \overline{y}(N_n + 1) \to \overline{F(x)}M_{n+1} = \overline{F(y)}M_{n+1}]$.

I will now define, recursively, $x \in [0,1]^*$ and $\beta \in \mathcal{A}$. Like in lemma 1, step 0 is slightly different from the other steps, but here I will handle it separately as it is too different from the other steps; it might, however, demonstrate the global idea for the rest of the proof.

**Step** 0**:**

Determine $R_0$ such that $M_1 \cdot 2 \cdot 2^{-R_0} < 2^{-(T+1)}$.

Let, for $x \in [0,1]^*$ (we will mainly concern ourselves with standard $x$, because this is the easiest form of reals to work with and it's enough), $B_x$ be the set of all standardintervals $a \in S_{R_0}$ that are not strictly outside $F(x)(0), \ldots, F(x)(M_1-1)$ (so these $a$ have some overlap, or share an endpoint with $\overline{F(x)}M_1$). Then $\#B_x \leq 2^{R_0} \cdot \mu(\overline{F(x)}M_1) + 2 \cdot M_1 \leq 2^{R_0} \cdot (\frac{1}{2} - 2^{-T} + 2^{-(T+1)}) = 2^{R_0} \cdot (\frac{1}{2} - 2^{-(T+1)})$.

Now let $X$ be the set of pairs $(a,b) \in S_{N_0} \times S_{R_0}$ such that $\forall x \in a[b \in B_x]$ - by definition of $N_0$, $\overline{F(x)}M_1$ and therefore $B_x$ only depend on $\overline{x}(N_0+1)$, and thus on the standardinterval of order $N_0$ that it's part of. So $X$ is decidable.
For each $a \in S_{N_0}$ there can be at most $2^{R_0} \cdot (\frac{1}{2} - 2^{-(T+1)})$ $b \in S_{R_0}$ for which $(a,b) \in X$. So $\#X \leq 2^{N_0} \cdot 2^{R_0} \cdot (\frac{1}{2} - 2^{-(T+1)})$.

Define $Y$ as the set of those $b \in S_{R_0}$ for which $\#\{a | a \in S_{N_0} | (a,b) \in X\} \leq 2^{N_0} \cdot (\frac{1}{2} - 2^{-(T+2)})$.
We can determine the number of elements of $Y$.
Now, suppose $\#Y < 2^{R_0} \cdot 2^{-(T+1)}$.
Then $2^{R_0} \cdot (1 - 2^{-(T+1)}) \cdot 2^{N_0} \cdot (\frac{1}{2} - 2^{-(T+2)}) < \#X \leq 2^{N_0} * 2^{R_0} \cdot (\frac{1}{2} - 2^{-(T+1)})$
So $(1 - 2^{-(T+1)}) \cdot (\frac{1}{2} - 2^{-(T+2)}) < \frac{1}{2} - 2^{-(T+1)}$
So $(1 - 2^{-(T+1)}) \cdot (1 - 2^{-(T+1)}) < 1 - 2^{-T}$
So $1 - 2^{-T} + 2^{-(2T+2)} < 1 - 2^{-T}$, and $2^{-(2T+2)} < 0$. $\lightning$
So we may conclude: $\#Y \geq 2^{R_0} \cdot 2^{-(T+1)}$.

We want to create $x \in [0,1]^*$, so first define $x(0) = \langle 0, 1 \rangle$. All the elements of $Y$ are contained in $x(0)$, so either $\langle 0, \frac{1}{2} \rangle$ or $\langle \frac{1}{2}, 1 \rangle$ contains at least half the elements of $Y$ (we had taken $R_0 > 0$). Take $x(1)$ this half. Choose $A_0 = \{b | b \in Y | b \subseteq x(1)\}$. Then $\#A_0 \geq \frac{1}{2} \cdot \#Y \geq 2^{R_0} \cdot 2^{-(T+1)} \cdot 2^{-1} = 2^{R_0} \cdot 2^{-T} \cdot 4^{-1}$.

Define, for $x \in [0,1]^*$: $C_{x,0} = \{a | a \in S_{N_0} | (a, x(R_0)) \in X\}$. If $x(R_0) \in A_0$, then $x(R_0) \in Y$, so $\#C_{x,0} \leq 2^{N_0} \cdot (\frac{1}{2} - 2^{-(T+2)})$. If, for $y \in [0,1]^*$, $y(N_0) \notin C_{x,0}$, then $(y(N_0), x(R_0)) \notin X$, so $x(R_0) \notin B_y$ and therefore $x(R_0)$ is strictly outside $F(y)(0), \ldots, F(y)(M_1-1)$.

In the rest of the recursion I will define, in each step $n$: $R_n \in \mathbb{N}$, $A_n \subseteq S_{R_n}$, $x(n+1)$ and, for all $y \in [0,1]^*$, $C_{y,n} \subseteq S_{N_n}$ that satisfy the following conditions:

- $\forall a \in A_n[a \subseteq x(n+1)]$
- $\forall a \in A_{n+1} \exists b \in A_n[a \subseteq b]$
- $\#A_n \geq 2^{R_n} \cdot 2^{-T} \cdot 4^{-(n+1)}$
- $\forall x, y \in [0,1]^*[y(N_n) \notin C_{x,n} \to \forall t \in \{M_n, \ldots, M_{n+1}-1\}[x(R_n)'' < F(y)(t)' \vee F(y)(t)'' < x(R_n)']]$
- for $n > 0$: $\forall x \in [0,1]^*[x(R_n) \in A_n \to \#C_{x,n} \leq 2^{N_n} \cdot 2^{-(T+n+2)}]$
- $R_n \leq R_{n+1}$

For $n = 0$, all the conditions have been satisfied by step 0. Using the above condition as induction hypothesis, we will define the rest.

**Step** $n = k + 1$**:**

Determine some $R_n$ such that $M_{n+1} \cdot 2 \cdot 2^{-R_n} < 2^{-t_n}$. Choose $R_n \geq R_k$.

Let, for $x \in [0,1]^*$, $B_x$ be the set of all standardintervals $a \in S_{R_n}$ such that $a$ is not strictly outside $F(x)(M_n), \ldots, F(x)(M_n - 1)$.
Then $\#B_x \leq 2^{R_n} \cdot \mu(\langle F(x)(M_n), \ldots, F(x)(M_{n+1} - 1)\rangle) + 2 \cdot (M_{n+1} - M_n) \leq 2^{R_n} \cdot (\mu(F(x)) - \mu(\overline{F(x)}M_n)) + 2 \cdot M_{n+1} \leq 2^{R_n} \cdot (2^{-t_n} + 2^{-t_n}) = 2^{R_n} * 2^{-t_n+1}$.

Now let $X$ be the set of pairs $(a, b) \in S_{N_n} \times S_{R_n}$ such that $\forall x \in a[b \in B_x]$. By definition of $N_n$, $\overline{F(x)}M_{n+1}$ and therefore $B_x$ only depend on $\overline{x}(N_n + 1)$, and thus on the standardinterval of order $N_n$ that it's part of. For each $a \in S_{N_n}$ there can be at most $2^{R_n} \cdot 2^{-t_n+1}$ $b \in S_{R_n}$ for which $(a, b) \in X$. So $\#X \leq 2^{N_n} \cdot 2^{R_n} \cdot 2^{-t_n+1}$.

Define $Y$ as the set of those $b \in S_{R_n}$ for which $\#\{a | a \in S_{N_n} | (a, b) \in X\} \leq 2^{N_n} \cdot 2^{-(T+n+2)}$. We can determine the number of elements of $Y$.
Now suppose $\#Y < 2^{R_n} \cdot (1 - 2^{-(T+1)} \cdot 4^{-n})$.
Then $2^{R_n} \cdot 2^{-(T+1)} \cdot 4^{-n} \cdot 2^{N_n} \cdot 2^{-(T+n+2)} < \#X \leq 2^{N_n} \cdot 2^{R_n} \cdot 2^{-t_n+1}$.
So $2^{-(T+1+2n+T+n+2)} < 2^{-t_n+1}$.
So $2^{-(2T+3n+3)} < 2^{-t_n+1} = 2^{-(2T+3n+3)}$. $\lightning$
So we may conclude: $\#Y \geq 2^{R_n} \cdot (1 - 2^{-(T+1)} \cdot 4^{-n})$.

Define $Z := \{b | b \in Y | \exists a \in A_k[b \subseteq a]\}$. Since an element $a$ of $A_k$ is a standardinterval of order $R_k \leq R_n$, it will always either cover a given element $b$ of $Y$ completely or have no overlap with it (although they may touch in a single point). Therefore, using the induction hypothesis on the size of $A_k$, $A_k$ will cover at least $2^{R_n} \cdot 2^{-T} \cdot 4^{-n}$ standardintervals of order $R_n$. At most $2^{R_n} \cdot 2^{-(T+1)} \cdot 4^{-n}$ of these do not occur in $Y$. So $\#Z \geq 2^{R_n} \cdot 2^{-T} \cdot 4^{-n} - 2^{R_n} \cdot 2^{-(T+1)} \cdot 4^{-n} = 2^{R_n} \cdot 2^{-(T+1)} \cdot 4^{-n}$.

As $A_k \subseteq x(n)$ by induction hypothesis, $\forall b \in Z[b \subseteq x(n)]$, and from the construction of $R_n$ we see that $R_n \geq n + 1$. Now let $s := x(n)' + \frac{x(n)'' - x(n)'}{2}$. Then either $\langle x(n)', s\rangle$ covers half the intervals in $Z$, or $\langle s, x(n)''\rangle$ does. Choose for $x(n + 1)$ this half (or the first, if both halves satisfy the condition). Choose $A_n = \{b | b \in Z | b \subseteq x(n + 1)\}$. Then $\#A_n \geq \frac{1}{2} \cdot \#Z \geq 2^{R_n} \cdot 2^{-(T+1)} \cdot 4^{-n} \cdot 2^{-1} = 2^{R_n} \cdot 2^{-T} \cdot 4^{-(n+1)}$.

To conclude, define for $x \in [0,1]^*$, $C_{x,n} = \{a | a \in S_{N_n} | (a, x(R_n)) \in X\}$. If $x(R_n) \in A_n$, then $x(R_n) \in Y$, so $\#C_{x,n} \leq 2^{N_n} \cdot 2^{-(T+n+1)}$. If, for $y \in [0,1]^*$, $y(N_n) \notin C_{x,n}$, then $(y(N_n), x(R_n)) \notin X$, so $x(R_n) \notin B_y$ and therefore $x(R_n)$ is strictly outside $F(y)(M_n), \ldots, F(y)(M_{n+1} - 1)$.

It is easy to see that the variables created in this construction satisfy all the conditions of the induction hypothesis.

*Correctness proof:*

First, note that for all $n$, $x(R_n) \in A_n$: we can see this in the same way as in the proof of lemma 1: $x(R_n)$ is either in $A_n$ or has no overlap with its elements, but the elements of $A_{R_n}$ are both covered by $x(R_n)$ and by some element of $A_n$; so $x(R_n)$ must be an element of $A_n$. $\square$

The inductive construction above gives us $x \in [0,1]^*$. Using $x$, we can define $\beta$ as follows: $\beta(0), \ldots, \beta(\#C_{x,0} - 1)$ are the intervals in $C_{x,0}$, $\beta(\#C_{x,0}), \ldots, \beta(\#C_{x,0} + \#C_{x,1} - 1)$ are the intervals in $C_{x,1}$ and so on.

$\beta$ is a measurable areal: for each $n > 0$, $\#C_{x,n} \leq 2^{N_n} \cdot 2^{-(T+n+2)}$ and consists of intervals of length $2^{-N_n}$; as such, $\mu(\langle\beta(\Sigma_{0\leq k<n}\#C_{x,k}), \ldots, \beta(\Sigma_{0\leq k\leq n}\#C_{x,k} - 1)\rangle \leq 2^{-(T+n+2)})$. We can easily define, step by step, a real number $y$ such that $y = \lim_{k\to\infty} \mu(\overline{\beta}k) \leq (\frac{1}{2} - 2^{-(T+2)}) + \Sigma_{n>0}2^{-(T+n+2)} = (\frac{1}{2} - 2^{-(T+2)}) + 2^{-(T+2)} = \frac{1}{2}$.

To prove the lemma, we must see that for all $y \in [0,1]^*$ such that $y \notin \beta$: $x \notin F(y)$.
So let $y \notin \beta$ and let $n \in \mathbb{N}$. We must see that there exists some $m$ such that $x(m)$ is strictly outside $F(y)(n)$. To this end, determine $k$ such that $M_k \leq n < M_{k+1}$. Since $y \notin \beta$, $y(N_k)$ can not be contained in $C_{x,k}$.
But $C_{x,k}$ contains (among others) all the intervals $a$ such that for $z \in [0,1]^*$ with $z(N_k) = a$, $x(R_k)$ is not strictly outside $F(z)(n)$. Thus, we can choose $m = R_k$. Then $x(m)$ is strictly outside $F(y)(n)$, which is what we had to prove. □

***Proof of theorem 14***

With these two lemmas, the theorem has become trivial:
Let $F$ be a function that maps $[0,1]$ to $\mathcal{A}$ such that $\forall x[\mu(F(x)) < \frac{1}{2}]$. Using lemma 2 we find $x \in [0,1]^*$ and $\beta \in \mathcal{A}$ such that for $y \in [0,1]^*$: $y \notin \beta \to x \notin F(y)$. Choose $\alpha \in \mathcal{N}$ such that for all $n$: $\alpha(2n) = \beta(n)$, $\alpha(2n+1) = F(x)(n)$. It is easy to see that $\alpha \in \mathcal{A}$, and that $\mu(\alpha) \leq \mu(F(x)) + \mu(\beta) < 1$.
Now we can apply lemma 1 to find $y \in [0,1]^*$ with $y \notin \alpha$. So $y \notin F(x)$, and $y \notin \beta$, so $x \notin F(y)$. Since $[0,1]^* \subset [0,1]$ we have found $x$ and $y$ that satisfy the theorem. □

## 6.6 Extensions of $\mathrm{A}_{\mathrm{null}}$

### 6.6.1 $\mathrm{A}_{\mathrm{null}}$

The standard axiom of symmetry for null sets follows trivially from $\mathrm{A}_{<\frac{1}{2}}$:

**Theorem 15.** *$A_{\mathrm{null}}$*
*Let $F$ be a function that maps $\mathbb{R}$ to $\widetilde{\mathcal{N}}$.*
*Then there exist $x, y$ such that $x \notin F(y) \wedge y \notin F(x)$.*

*Proof.* Apply theorem 14 to the function $x \to F(x)^1$. This gives us $x, y$ such that $x \notin F(y)^1$ and $y \notin F(x)^1$ and thus, in a very strong way, what we have to prove. □

But now we could ask ourselves: do we still have the multidart-extensions as we saw in chapter 5?
The answer is: we do. This time, I will only go into the infinite dart extensions of $\mathrm{A}_{\mathrm{null}}$ itself; as the statements hold, there is no real need to consider the finite alternatives as well. There are other interesting extensions we could think about, particularly for extending not $\mathrm{A}_{\mathrm{null}}$ but $\mathrm{A}_\epsilon$ (where each point is mapped to a measurable areal of at most measure $\epsilon$), but I will not treat those in detail here.

The extensions here will be a bit trickier than in chapter 5, because the basic theorem $\mathrm{A}_{\mathrm{null}}$ was not derived in a symmetric way. So we can't just extend the proof method by working a bit on each of the $x_i$ in turns - at least, not without taking some extra precautions.

### 6.6.2 $\mathbf{A}^{*\omega}_{\text{null}}$

First, let us consider simply the throwing of infinitely many darts.

**Theorem 16.** *$A^{*\omega}_{null}$*
*Let $F$ be a function that maps $[0,1]$ to $\widetilde{\mathcal{N}}$.*
*Then there exist $x_0, x_1, x_2, \ldots \in [0,1]$ such that $\forall n, m[n \neq m \to x_n \notin F(x_m)]$.*

*Idea of the proof:*

The construction in the proof of lemma 2 is not specific to functions that map points to areals with measure $\frac{1}{2}$. We could replace $\frac{1}{2}$ in this lemma by $2^{-n}$ and use almost the same proof to derive this result.

So find, say, some $x_0$ that isn't covered too much by the $F^{37}$-image of other reals, and a measurable areal $\alpha_0$ of measure at most $2^{-37}$ such that whenever $y \notin \alpha_0$, $x_0 \notin F(y)^{37}$. It shouldn't be hard for the points made in the rest of the construction to avoid landing in $F(x_0)^{37}$ or $\alpha_0$, as their joined measure is no more than $2^{-36}$. If we avoid this, then for all $n > 0$ we will know that $x_n \notin F(x_0)$ and $x_0 \notin F(x_n)$.

Then find another real, $x_1$, and a measurable areal $\alpha_1$ of measure at most $2^{-42}$, such that whenever $y \notin \alpha_1$, $x_1 \notin F(y)^{42}$. Choose $x_1$ in a clever way, that ensures it will both be outside $F(x_0)^{37}$ and $\alpha_0$.
Continue like this. Every step $n$, find some real $x_n$ and a measurable areal $\alpha_n$ such that whenever $y$ is outside $\alpha_n$, $x_n$ will be outside $F(y)^{n+\text{something}}$. Pick $x_n$ outside all the previous $\alpha_n$, and outside the image of the previous $x_k$s.

Of course, 37, 42 and *something* in this reasoning are arbitrary; we just have to choose these values in such a way that we always have space to find suitable $x_n$.

Unfortunately, it doesn't follow from our previous lemmas (or even directly from their proof) that we can choose the $x_n$ that aren't covered "too much" outside a given areal $\alpha_k$. So before we can see the full proof of the theorem, we need another lemma.

**Lemma 3.**
*Let $T \in \mathbb{N}$, $T \geq 4$.*
*Let $\alpha \in \mathcal{A}$ have the property that $\mu(\alpha) \leq \frac{1}{2}$.*
*Let $F$ be a function that maps $[0,1]$ to $\mathcal{A}$ such that $\forall x \in [0,1][\mu(F(x)) \leq 2^{-T}]$.*
*Then there exist $x \in [0,1]^*$ and $\beta \in \mathcal{A}$ such that $x \notin \alpha$, $\mu(\beta) < 2^{-T+4}$ and $\forall y \in [0,1]^*[y \notin \beta \to x \notin F(Y)]$.*

I have deliberately formulated this lemma in a relatively weak way; it would be quite possible to allow for larger areals, and create smaller $\beta$. However, we don't need this for theorem 16 - $F$ can map to measurable areals as small as we want! So in this case, we might as well go for an easy proof - the larger choice of $T$ allows a bit more space to avoid exceptions.

*Construction:*

The proof of this lemma will be rather similar to that of lemma 2, but cater for $\alpha$.

Like in lemma 2, we first define for $n > 0$: $t_n = 2T + 3n + 4$. As before, I will define $x \in [0,1]^*$ and $\beta \in \mathcal{A}$ recursively.

**Step** $0$:

First of all, determine $M_1$ such that $\forall x \in [0,1][\mu(\overline{F(x)}M_1) \geq \mu(F(x)) - 2^{-t_1-1}]$ and, using the fan theorem, $N_0$ such that $\forall x, y \in [0,1][\overline{x}(N_0+1) = \overline{y}(N_0+1) \rightarrow \overline{F(x)}M_1 = \overline{F(y)}M_1]$.
We can also find $U \in \mathbb{N}$ such that $\mu(\overline{\alpha}U) \geq \mu(\alpha) - 2^{-t_1-1}$.
Knowing these values, determine $R_0$ such that $M_1 \cdot 2 \cdot 2^{-R_0} < 2^{-T}$ and $U \cdot 2 \cdot 2^{-R_0} < 2^{-T}$.

Let, for $x \in [0,1]^*$, $B_x$ be the set of all standardintervals $a \in S_{R_0}$ such that $a$ is not strictly outside $F(x)(0), \ldots, F(x)(M_1-1)$. Then $\#B_x \leq 2^{R_0} \cdot \mu(\overline{F(x)}M_1) + 2 \cdot M_1 \leq 2^{R_0} \cdot \mu(F(x)) + 2^{R_0} \cdot 2^{-T} \leq 2^{R_0} \cdot 2^{-T+1}$.

Now let $X$ be the set of pairs $(a,b) \in S_{N_0} \times S_{R_0}$ such that $\forall x \in a[b \in B_x]$ - by definition of $N_0$, $\overline{F(x)}M_1$ and therefore $B_x$ only depend on $\overline{x}(N_0+1)$, and thus on the standardinterval of order $N_0$ that it's part of. For each $a \in S_{N_0}$ there can be at most $2^{R_0} \cdot 2^{-T+1}$ $b \in S_{R_0}$ for which $(a,b) \in X$. So $\#X \leq 2^{N_0} \cdot 2^{R_0} \cdot 2^{-T+1}$.

Define $Y$ as the set of those $b \in S_{R_0}$ for which $\#\{a | a \in S_{N_0} | (a,b) \in X\} \leq 2^{N_0} \cdot 2^{3-T}$. We can determine the number of elements of $Y$.
Now, suppose $\#Y < (\frac{1}{2} + 2^{-T+1}) \cdot 2^{R_0}$.
Then $2^{R_0} \cdot (\frac{1}{2} - 2^{-T+1}) \cdot 2^{N_0} \cdot 2^{3-T} < 2^{N_0} \cdot 2^{R_0} \cdot 2^{-T+1}$.
So $(2^{-1} - 2^{-T+1}) \cdot 2^{3-T} < 2^{-T+1}$.
So $(2^{T-2} - 2^0) \cdot 2^{3-T} < 1$.
So $2 - 2^{3-T} < 1$, and $2^{3-T} > 1$, which suggests $T < 3$. ↯
So we may conclude: $\#Y \geq (\frac{1}{2} + 2^{-T+1}) \cdot 2^{R_0}$.

We have chosen $R_0$ large enough that there are at most $2^{R_0} \cdot \mu(\overline{\alpha}U) + 2^{R_0} \cdot 2^{-T} \leq 2^{R_0} \cdot (\mu(\alpha) + 2^{-T}) \leq 2^{R_0} \cdot (\frac{1}{2} + 2^{-T})$ standardintervals of order $R_0$ that are not strictly outside $\overline{\alpha}U$. If we remove these intervals from $Y$ and call the remaining set $Z$, that leaves $\#Z \geq 2^{R_0} \cdot 2^{-T}$.

We want to create $x \in [0,1]^*$, so first define $x(0) = \langle 0, 1 \rangle$. All the elements of $Z$ are contained in $x(0)$, so either $\langle 0, \frac{1}{2} \rangle$ or $\langle \frac{1}{2}, 1 \rangle$ contains at least half the elements of $Z$ (it is easy to see that we have taken $R_0 > 0$). Take $x(1)$ this half. Choose $A_0 = \{b | b \in Z | b \subseteq x(1)\}$. Then $\#A_0 \geq \frac{1}{2} \cdot \#Z \geq 2^{-1} \cdot 2^{R_0} \cdot 2^{-T} \geq 2^{R_0} \cdot 2^{-T} \cdot 4^{-1}$.

Define, for $x \in [0,1]^*$: $C_{x,0} = \{a | a \in S_{N_0} | (a, x(R_0)) \in X\}$. If $x(R_0) \in A_0$, then $x(R_0) \in Y$, so $\#C_{x,0} \leq 2^{N_0} * 2^{3-T}$. If, for $y \in [0,1]^*$, $y(N_0) \notin C_{x,0}$, then $(y(N_0), x(R_0)) \notin X$, so $x(R_0) \notin B_y$ and therefore $x(R_0)$ is strictly outside $F(y)(0), \ldots, F(y)(M_1-1)$.
It is also easy to see that $A_0$ has the property that if $x(R_0) \in A_0$, then $x \notin \overline{\alpha}U$.

Now define a function $G$ that maps $[0,1]$ to $\mathcal{A}$ as follows: $\forall x \forall n \forall m[(n < M_1 \rightarrow G(x)(n) = F(x)(n)) \wedge (n = M_1 + 2m \rightarrow G(x) = F(x)(M_1+m)) \wedge (n = M_1 + 2m + 1 \rightarrow G(x)(n) = \alpha(U+m))]$. So $G$ maps each $y$ to the union of $F(y)$ and the part of $\alpha$ we hadn't excluded yet. Then $\forall x \in [0,1][\Sigma_{k \geq M_1} \mu(G(x)(k)) \leq 2^{-t_1}]$.
For $n > 0$, we can also choose $M_{n+1}$ such that $\forall x[\Sigma_{k \geq M_{n+1}} |G(x)(k)| \leq 2^{-t_{n+1}}]$, and matching $N_n$ such that $\forall x, y \in [0,1][\overline{x}(N_n+1) = \overline{y}(N_n+1) \rightarrow \overline{G(x)}M_{n+1} = \overline{G(y)}M_{n+1}]$.

Now $G, A_0, R_0$ and for all $x$, $B_{x,0}$ satisfy the induction conditions from the proof of lemma 2, and the definitions of $t_n, M_n$ and $N_n$ correspond too. Using the induction step given there, we have, for all $n, x$: $A_n, R_n$ and $C_{x,n}$.

Using this construction we define $x$, and can take $\beta$ as the measurable areal that consists of all intervals in $C_{x,n}$ for some $n$. We can see that $\beta$ is indeed a measurable areal, and $\mu(\beta) \leq \Sigma_n \#C_{x,n} \cdot 2^{-N_n} = \#C_{x,0} * 2^{-N_0} + \Sigma_{n>0} \#C_{x,n} \cdot 2^{-N_n} \leq 2^{3-T} + 2^{-T-1} \cdot \Sigma_{n>0} 2^{-n} = 2^{-T+3} + 2^{-T-1} = 2^{-T} \cdot (2^3 + 2^{-1}) < 2^{-T+4}$.

If we have some $y \in [0,1]^*$ such that $y \# \beta$, then $x \# G(y)$. Thus, for all $n \geq U$, $x$ is strictly outside $\alpha(n)$. Since $x(R_0)$ must be an element of $A_0$, we know that for $n < U$, $x$ is also strictly outside $\alpha(n)$, and we may conclude that $x \# \alpha$. From $x \# G(y)$ we also see that $x \# F(y)$, which completes the proof for the lemma.

***Proof of theorem 16***

I will define the $x_n$ inductively; in step $n$, $x_n$ will be defined, along with a measurable areal $\alpha_n$ such that $\mu(\alpha_n) < 2^{-(n+2)}$ and $\forall y \in [0,1]^*[y \# \alpha_n \to (x_n \# F(y)^{n+7} \wedge y \# F(x_n)^{n+7})]$. For writing ease, say $T_n = n + 7$.

**Step** $0$**:**

Let $G$ be the function $x \to F(x)^{T_0}$, so $\mu(G(x)) < 2^{-T_0}$ for all $x \in [0,1]$.
Apply lemma 3 to $G$ (take for $\alpha$ an areal consisting of empty intervals). This gives us $x_0 \in [0,1]^*$ and $\beta_0 \in \mathcal{A}$ such that $\mu(\beta_0) \leq 2^{-T_0+4}$ and for $y \in [0,1]^*$: $y \# \beta_0 \to x_0 \# F(y)^{T_0}$.

Let $\alpha_0 = \beta_0 \cup F(x_0)^{T_0}$. Then $\mu(\alpha_0) \leq 2^{-T_0+4} + 2^{-T_0} < 2^{-T_0+5} = 2^{-2}$ and $\forall y \in [0,1]^*$: $y \# \alpha_0 \to (x_0 \# F(y)^{T_0} \wedge y \# F(x_0)^{T_0})$.

**Step** $n > 0$**:**

Let $\alpha = \alpha_0 \cup \alpha_1 \cup \ldots \cup \alpha_{n-1}$. Since each $\alpha_m$ has $\mu(\alpha_m) < 2^{-(m+2)}$, we see that $\mu(\alpha) < \Sigma_{m<n} 2^{-(m+2)} < \Sigma_m 2^{-(m+2)} = \frac{1}{2}$.
Let $G$ be the function $x \to F(x)^{T_n}$. We can apply lemma 3 to $G$ and $\alpha$ to get $x_n \in [0,1]^*$ and $\beta_n \in \mathcal{A}$ such that $\forall m < n[x_n \# \alpha_m]$ and $\mu(\beta) \leq 2^{-T_n+4}$.

Define $\alpha_n = \beta_n \cup G(x_n)$. Then $\mu(\alpha_n) \leq 2^{-T_n+4} + 2^{-T_n} < 2^{-T_n+5} = 2^{n-2}$ and $\forall y \in [0,1]^*$: $y \# \alpha_n \to (x_n \# F(y)^{T_n} \wedge y \# F(x_n)^{T_n})$.

This shows all we need to prove: for let $n, m$ be given, $n \neq m$. One of the two must be the smaller, say $n < m$. Then $x_m \# \alpha_n$, so $x_n \# F(x_m)^{T_n}$ and $x_m \# F(x_N)^{T_n}$. □

### 6.6.3 $\mathbf{A}^{\omega}_{\text{null}}$

The other extension I will consider is using a function in countably many variables, instead of just one.

#### Original statement and reformulation

**Theorem 17.** $A^{\omega}_{null}$ *(naively phrased)*
*Let $F$ be a sequence that maps hsets of elements of $[0,1]$ to $\widetilde{\mathcal{N}}$.*
*Then there exists a set of reals $X = \{x_n | n \in \mathbb{N}\}$ such that $\forall n[x_n \# F(X \setminus \{x_n\}]$.*

To make it easier to reason about this statement, it will help to replace the "hsets of reals" by sequences in $\mathcal{C}$. This will make the statement somewhat stronger, but since it will still prove to be true, this won't be a problem!
To get this adaptation of the theorem formulated properly, I will first need a few definitions.

I will use a very specific pairing function, with some nice properties that will make the calculations a bit easier. Take $\langle\!\langle , \rangle\!\rangle : \mathbb{N}^2 \hookrightarrow \mathbb{N}$ as $\langle\!\langle a, b \rangle\!\rangle = 2^a(1 + 2b) - 1$. This function is bijective, because all positive numbers can be uniquely written as the product of a power of 2 and an

odd number. The function is also increasing in both parameters.
As we will see, this pairing function has the nice property that, if the start of an infinite sequence $\alpha$ is known up to $2^n - 1$ places, there are exactly $n$ subsequences $\alpha^k$ for which a number of places are known.

Since finite initial segments of elements of $\mathcal{C}$ will be used repeatedly in the proof, I would like to index subsequences of finite sequences as well. This can be done in a similar way as we have defined $\alpha^n$ for $\alpha \in \mathcal{C}$.
So for $a \in \{0,1\}^*, n \in \mathbb{N}$, write $a_n$ for the finite sequence in $\{0,1\}^*$ such that $\forall k[k < \text{len}(a_n) \to a_n(k) = a(\langle\!\langle n, k \rangle\!\rangle)) \wedge (k < \text{len}(a_n) \leftrightarrow \langle\!\langle n, k \rangle\!\rangle < \text{len}(a))]$.
Using the definition of our pairing function $\langle\!\langle, \rangle\!\rangle$, we see that for $b \in \{0,1\}^{2^n-1}$ we have finite sequences $b_0, b_1, \ldots, b_{n-1}$ with $b_k \in \{0,1\}^{2^{n-(k+1)}}$ (for $k < n$), but no $b_n$ or higher (these are all empty sequences).
These lengths can easily be verified with some easy calculations: $\langle\!\langle k, 2^{n-(k+1)} - 1 \rangle\!\rangle = 2^k(1 + 2^{n-k} - 2) - 1 = 2^n - 2^k - 1 < 2^n - 1 = \text{len}(a)$ and $\langle\!\langle k, 2^{n-(k+1)} \rangle\!\rangle = 2^k(1 + 2^{n-k}) - 1 = 2^n + 2^k - 1 \geq 2^n > \text{len}(a)$.
It is easy to see that if a finite sequence $a$ is the initial segment of some $\alpha \in \mathcal{C}$, then each $a_n$ is that initial segment of $\alpha^n$.

Recall from chapter 5, the definition of $\alpha \setminus \alpha^n$ (for $\alpha \in \mathcal{C},\ n \in \mathbb{N}$): this is the element of $\mathcal{C}$ such that $\alpha \setminus \alpha^n(\langle\!\langle i, j \rangle\!\rangle) = \begin{cases} \alpha(\langle\!\langle i, j \rangle\!\rangle) & \text{if } i < n \\ \alpha(\langle\!\langle i+1, j \rangle\!\rangle) & \text{otherwise} \end{cases}$
This will be used as a replacement for taking an element out of a set.

For a finite variation, define for given $n,\ k < n,\ a \in \{0,1\}^{2^n-1}$, $a \setminus a_k$ as the element $b \in \{0,1\}^{2^{n-1}-1}$ such that, for $s < k$, $b_s$ is an initial segment of $a_s$, and for $k \leq s < n-1$, $b_s = a_{s+1}$.
With a minor bit of numberfiddling we can see that always $\overline{\alpha \setminus \alpha^k}(2^{n-1} - 1) = \overline{\alpha}(2^n - 1) \setminus (\overline{\alpha}(2^n - 1))_k$.

Looking at the definition closely, it is clear that we throw away quite a lot of information from $a$ by taking $a \setminus a_n$: some subsequences of $a$ will only partly be taken into $a \setminus a_n$.
To see exactly how much we lose, suppose that, for some $n, k$, we are given $b \in \{0,1\}^{2^{n-1}-1}$ and $z \in \{0,1\}^{2^{n-(k+1)}}$, and we must find $a$ such that $a_k = z$ and $a \setminus a_k = b$. Then $a_k$ and $a_{k+1}, \ldots, a_{n-1}$ are fixed by $z$ and $b$, but for $s \in \{0, \ldots, k-1\}$, only the first $\text{len}(b_s) = 2^{(n-1)-(s+1)}$ places of $a_s$ are fixed, so $2^{n-s-2}$ places of $a_s$ can be chosen randomly. This leaves in total $\Sigma_{0 \leq s < k} 2^{n-s-2} = 2^{n-1} \cdot \Sigma_{1 \leq s \leq k} 2^{-s} = 2^{n-1} \cdot (1 - 2^{-k})$ unfixed places in $a$. Each place can be filled with two different digits, so this gives us $2^{2^{n-1} \cdot (1-2^{-k})}$ choices for $a$.

It is possible to interpret some $a \in \{0,1\}^*$ as a standardinterval of order $\text{len}(a)$: start at $\langle 0, 1 \rangle$, and interpret a 0 as taking the left subinterval, 1 as going to the right; the sequence 001, for example, would encode the interval $[\frac{1}{8}, \frac{2}{8}]$. Write $Ia$ for the standardinterval that corresponds to $a \in \{0,1\}^*$.
Extending this to infinite sequences, we can interpret any $\alpha \in \mathcal{C}$ as an element $x$ of $[0,1]^*$; just let $x(n) = I(\overline{\alpha}n)$. Write $I\alpha$ for the real associated with $\alpha$.

As done before in chapters 3 and 5, we can interpret a set as a countable sequence, while only making the theorem stronger. Limiting the reals I want to consider for the solution to standardreals will not hurt the generality of the problem either. Therefore, we can interpret $F$ as a function that assigns a nullset (element of $\widetilde{\mathcal{N}}$) to each element of $\mathcal{C}^{\mathbb{N}}$.

Since there is a one-on-one correspondence between $\mathcal{C}$ and $\mathcal{C}^{\mathbb{N}}$ (using $\langle\!\langle , \rangle\!\rangle$), this leads to the following reformulation of theorem 17:

**Theorem 18.** $A^{\omega}_{null}$ *(reformulation)*
*Let $F$ be a function that maps $\mathcal{C}$ to $\widetilde{\mathcal{N}}$.*
*Then there exists $x \in \mathcal{C}$ such that the real associated with each $x^n$ is strictly outside $F(x \setminus x^n)$.*
*In formula: $\forall n \exists m \forall i \exists j [I(\overline{x^n}j)'' < F(x \setminus x^n)^m(i)' \vee F(x \setminus x^n)^m(i)'' < I(\overline{x^n}j)']$.*

**proof of the theorem**

*Idea of the proof:*

In the following, we will construct a sequence $x \in \mathcal{C}$ inductively. During the proof, we will keep track of sets $A_k$ of finite sequences: we only consider choices for $x$ that start with an element of $A_k$. These $A_k$ will have the nice property that for any $y \in \mathcal{C}$ that starts with an element of $A_k$, $Iy^n$ is strictly outside most of $F(y \setminus y^n)^m$ for certain fixed choices of $n$, $m$ and "most". Then we will see that for each $x$ and $n$, there will be some $m$ such that $Ix^n$ is outside $F(x \setminus x^n)^m$.

To get this right, we must, of course, choose $x$ and the sets $A_k$ in a very specific way.

So in each step $k = \langle\!\langle n, m \rangle\!\rangle$, we first decide how much of $\alpha \in \mathcal{C}$ must be known to ensure that all but a "small" remainder of, say, $F(\alpha)^{n+37}$ is known (exact figures are given in the construction phase).
Suppose $\alpha$ must be known up to $N$ places. We then consider all finite sequences in $\{0,1\}^N$ and the part of their $F^{n+37}$ image we know (and haven't treated in a previous step). Cutting up $[0,1]$ into sufficiently small standardintervals $b \in S_R$, we select the intervals $b$ that are not covered "too much" by elements of $\{0,1\}^N$.

Representing standardintervals of order $R$ as finite 0/1-sequences of length $R$, we can make a reasonably large set $\subseteq \{0,1\}^N \times \{0,1\}^R$ of pairs, where $(a,b)$ is in the set if $Ib$ is strictly outside $F(a)^{n+37}$.
We can then combine such $a$ and $b$ into a single sequence: if $N$ and $R$ are chosen cleverly, we can find a sequence $c$ such that $c^n = b$ and $c \setminus c^n = a$. Call the set that consists of all such $c$, where $b$ is one of the sequences that isn't covered "too much", $B$.

The idea now is to choose $A_k$ as a subset of $B$. We already know that for $\alpha \in \mathcal{C}$ that start with an element of $B$, $\alpha^n$ is outside some part of $F(\alpha \setminus \alpha^n)^{n+37}$: say it is outside $F(\alpha \setminus \alpha^n)^{n+37}(X), \ldots, F(\alpha \setminus \alpha^n)^{n+37}(Y)$, where $F(\alpha \setminus \alpha^n)^{n+37}(0), \ldots, F(\alpha \setminus \alpha^n)^{n+37}(X-1)$ have been discussed in a previous step.

Now, it would be really nice if we would just know that such $\alpha^n$ would be outside $F(\alpha \setminus \alpha^n)^{n+37}(i)$ for *all* $i \leq Y$. For then we could, inductively, conclude that $x^n$ is outside $F(x \setminus x^n)^{n+37}$ completely (since $x$ would have initial segments in each $A_n$).
We can get this conclusion if we trim down $B$: take only those elements of $B$ that extend some sequence in $A_{k-1}$. Inductively, this will do the trick. Some numberfiddling will show that there will always be some elements left. For writing ease, group these elements together in a set $D$.

To finish the step, we should also extend the known part of $x$. We do this in a way that ensures that all the elements of $A_k$ start with the known part of $x$ (we need this to ensure

that $A_k$ will always contain an initial segment of $x$).
Just adding a single place, we can either add a 0 or a 1. Choose the extension such that the number of elements of $D$ that starts with the known part of $x$ is maximal. Let $A_k$ be this part of $D$.

This was just a sketch, although admittedly rather detailed. In the construction, I will use several calculations to show correctness of the reasoning layed out above.
$n + 37$ was just a simple example function to indicate "a small areal". In fact, we will use a function in $\langle\!\langle\, n, 0 \,\rangle\!\rangle$, which is exponentially bigger! However, for the idea of the proof this doesn't make a difference.

*Construction:*

The construction will be somewhat similar to the one in lemma 2.

I will define, in each step $k$, some $T_k \in \mathbb{N}$ and $A_k \subseteq \{0,1\}^{2^{T_k}-1}$ such that $\#A_k \geq 2^{2^{T_k}-1} \cdot 4^{-(k+1)}$. This $A_k$ will contain the initial segments of all choices of $x$ we might consider.
The $A_k$ will be such that if $i < j$, then for $a \in A_j$ there is some $b \in A_i$ such that $b$ is the initial segment of $a$. So the set of sequences that start with some element of $A_{k+1}$ will always be a subset of the sequences that start with an element of $A_k$.
In step $k$ I will also choose $x(k)$, such that if $k = \langle\!\langle\, a, b \,\rangle\!\rangle$, then for $c \in A_k$, $c_a$ extends $\overline{x^a}(b+1)$ (so $Ic_a \subseteq I(\overline{x^a}(b+1))$). This will guarantee that $x$ always extends some element of each $A_i$.

To start, we define for all $n, m$: $t_{n,m} = 4 \cdot \langle\!\langle\, n, m \,\rangle\!\rangle + 6$. Further, we choose $M_{n,m}$ to be such that $\forall x \in \mathcal{C}[\mu(\overline{F(x)^{t_{n,0}}}M_{n,m}) \geq \mu(F(x)^{t_{n,0}}) - 2^{-t_{n,m}}]$, and $M_{n,m+1} > M_{n,m}$. For $M_{n,0}$ we always take 0; since $\mu(F(x)^v) < 2^{-v}$ for all $v$ (by definition of $\widetilde{\mathcal{N}}$), this can be done.
To avoid having to make exceptions, we will also need $T_{-1} = 0, A_{-1} = \{\langle\rangle\}$.

**Step $k := \langle\!\langle\, n, m \,\rangle\!\rangle$:**

Using the fan theorem, we can determine $N'$ such that $\forall x, y \in \mathcal{C}[\overline{x}N' = \overline{y}N' \rightarrow \overline{F(x)^{t_{n,0}}}M_{n,m+1} = \overline{F(y)^{t_{n,0}}}M_{n,m+1}]$. Also determine $R'$ such that $M_{n,m+1} \cdot 2 \cdot 2^{-R'} < 2^{-t_{n,m}}$. It will prove easier to work with values of a standardform, so choose $T_k$ such that $T_k > T_{k-1}$, $2^{T_k - 1} - 1 \geq N'$ and $2^{T_k-(n+1)} \geq R'$. Then define $N = 2^{T_k-1} - 1$, $R = 2^{T_k-(n+1)}$.

Let $X \subseteq \{0,1\}^N \times \{0,1\}^R$ be the set of pairs $(a, b)$ for which $I_b$ is not strictly outside $\langle F(a{*}\underline{0})^{t_{n,0}}(M_{n,m}), \ldots, F(a{*}\underline{0})^{t_{n,0}}(M_{n,m+1}-1)\rangle$. Since this finite areal can only cover intervals with united measure of less then $2^{-t_{n,m}}$, $\#X \leq 2^N \cdot (2^R \cdot 2^{-t_{n,m}} + 2^{R'} \cdot 2^{-t_{n,m}}) = 2^N \cdot 2^R \cdot 2^{-t_{n,m}+1}$.

Let $Y$ be the set of those $b \in \{0,1\}^R$ such that $\#\{a | a \in \{0,1\}^N | (a, b) \in X\} \leq 2^N \cdot 2^{-(2k+2)}$.
Suppose $\#Y < 2^R \cdot (1 - 2^{-(2k+2)})$.
Then $2^R \cdot 2^{-(2k+2)} \cdot 2^N * 2^{-(2k+2)} < \#X \leq 2^R \cdot 2^N \cdot 2^{-t_{n,m}+1}$.
Thus $2^{-(4k+4)} < 2^{-t_{n,m}+1}$, and it follows that $t_{n,m} \leq 4k + 5$. Contradiction with the definition of $t_{n,m}$. $\lightning$
So we may conclude that $\#Y \geq 2^R \cdot (1 - 2^{-(2k+2)})$.

Now define for $y \in Y$: $Z_y = \{a | a \in \{0,1\}^N | (a, y) \notin X\}$. By definition of $Y$ we know that $\#Z_y \geq 2^N \cdot (1 - 2^{-(2k+2)})$.

Let $B = \{a | a \in \{0,1\}^{2^{T_k}-1} | a_n \in Y \wedge a \setminus a_n \in Z_{a_n}\}$. Since, for $a \in \{0,1\}^{2^{T_k}-1}$, $a_n$ has length $2^{T_k-(n+1)} = R$ and $a \setminus a_n$ has length $2^{T_k-1} - 1 = N$, we see that

$$
\begin{aligned}
\#B = \;& \Sigma_{y\in Y} \#Z_y \cdot 2^{2^{T_k-1}\cdot(1-2^{-n})} \\
\geq \;& \Sigma_{y\in Y} 2^{2^{T_k-1}-1} \cdot (1-2^{-(2k+2)}) \cdot 2^{2^{T_k-1}\cdot(1-2^{-n})} \\
= \;& \Sigma_{y\in Y} 2^{2^{T_k-1}\cdot 2-2^{T_k-n-1}-1} \cdot (1-2^{-(2k+2)}) \\
\geq \;& 2^{2^{T_k-(n+1)}} \cdot (1-2^{-(2k+2)}) \cdot 2^{2^{T_k}-2^{T_k-(n+1)}-1} \cdot (1-2^{-(2k+2)}) \\
= \;& 2^{2^{T_k}-1} \cdot (1-2^{-(2k+2)})^2 \\
= \;& 2^{2^{T_k}-1} \cdot (1-2^{-(2k+1)}+2^{-(4k+4)}) \\
\geq \;& 2^{2^{T_k}-1} \cdot (1-2^{-(2k+1)})
\end{aligned}
$$

Further, let $C = \{a|a \in \{0,1\}^{2^{T_k}-1}|\overline{a}(2^{T_{k-1}}-1) \in A_{k-1}\}$. Then $\#C = \#A_{k-1} \cdot 2^{2^{T_k}-1-(2^{T_{k-1}}-1)} \geq 2^{2^{T_{k-1}}-1} \cdot 4^{-k} \cdot 2^{2^{T_k}-1} \cdot 2^{-(2^{T_{k-1}}-1)} = 2^{2^{T_k}-1} \cdot 4^{-k}$.
Now choose $D$ to be $B \cap C$. There are at most $2^{2^{T_k}-1} \cdot 2^{-(2k+1)}$ elements of $C$ that aren't in $B$, so $\#D \geq 2^{2^{T_k}-1} \cdot 2^{-(2k+1)}$.

By induction, $\overline{x}k$ has already been defined. If $m > 0$, then for $c \in A_{\langle\langle n,m-1\rangle\rangle}$, $c_n$ is a direct extension of $\overline{x^n}m$. Since for all $l < k$, the elements of $A_l$ extend some element of $A_{l-1}$, in each element $c \in A_{k-1}$ $c_n$ will also be an extension of $\overline{x^n}m$ - this of course still holds when $m = 0$. So for all $c \in D$, too, $c_n$ will always be an extension of $\overline{x^n}m$.

Now consider $\overline{x^n}m \cdot \langle 0\rangle$ and $\overline{x^n}m \cdot \langle 1\rangle$. For each $c \in D$ $c_n$ will extend either of these (as $D$ contains longer elements than $A_{k-1}$, so by induction the elements of $D$ will at least have length $k$). Since $D$ is a finite set, we can determine one $i \in \{0,1\}$ such that $\overline{x^n}m * \langle i\rangle$ is the initial segment of the $n^{\text{th}}$ subsequence of at least half the elements of $D$. Let $x(k) = x(\langle\langle n,m\rangle\rangle) = i$. Choose $A_k = \{c|c \in D|\overline{c_n}(m+1) = \overline{x^n}(m+1)\}$. Then $\#A_k \geq \frac{1}{2} \cdot \#D \geq 2^{2^{T_k}-1} \cdot 2^{-(2k+1)} \cdot 2^{-1} = 2^{2^{T_k}-1} \cdot 4^{-(k+1)}$.
Now it is easy to see that all induction conditions are satisfied.

*Correctness proof:*

First, we will see that for all $k$: $\overline{x}(2^{T_k}-1) \in A_k$.
To prove this, I will first show that every $c \in A_{\langle\langle T_k-1,2^{T_k}-1\rangle\rangle}$ extends $\overline{x}(2^{T_k}-1)$.
So let $l < T_k$. Consider $c \in A_{\langle\langle l,2^{T_k}-2\rangle\rangle}$. By the construction, $c_l$ will extend $\overline{x^l}(2^{T_k}-1)$, so also $(\overline{x}(2^{T_k}-1))_l$ (which is an initial segment of $\overline{x^l}(2^{T_k}-1)$). Since always $\langle\langle l, 2^{T_k}-2\rangle\rangle < \langle\langle l+1, 2^{T_k}-2\rangle\rangle$, we have that for $0 \leq s < T_k$: for every $c \in A_{\langle\langle T_{k-1},2^{T_k}-2\rangle\rangle}$, $c_s$ extends $(\overline{x}(2^{T_k}-1))_s$. Thus we can conclude: every $c \in A_{\langle\langle T_k-1,2^{T_k}-1\rangle\rangle}$ extends $\overline{x}(2^{T_k}-1)$.

But $\langle\langle T_k-1, 2^{T_k}-1\rangle\rangle > k$, so for $c \in A_{\langle\langle T_k-1,2^{T_k}-1\rangle\rangle}$ there must be some $b \in A_k$ such that $\overline{x}(2^{T_k}-1) = \overline{c}(2^{T_k}-1) = b$. And thus we may conclude that $\overline{x}(2^{T_k}-1) \in A_k$. $\square$

To prove the theorem, we must see that $\forall n \exists m \forall i \exists j [I_{\overline{x^n}j}$ is strictly outside $F(x \setminus x^n)^m(i)]$. So let $n \in \mathbb{N}$ and take $m = t_{n,0}$. Let $i \in \mathbb{N}$. Determine $z \in \mathbb{N}$ for which $M_{n,z} \leq i < M_{n,z+1}$. By the definition of the $M$s it is clear that there must always be such a $z$. For writing ease, say $k = \langle\langle n,z\rangle\rangle$.
As we just saw, $\overline{x}(2^{T_k}-1) \in A_k$, so in step $k$, the set $X$ does not contain a pair $((\overline{x}(2^{T_k}-1)) \setminus (\overline{x}(2^{T_k}-1))_n, (\overline{x}(2^{T_k}-1))_n)$. This means that $I_{\overline{x^n}(2^{T_{\langle\langle n,z\rangle\rangle}-(n+1)})}$ is strictly outside $\bigcup_{s=M_{n,m}}^{M_{n,m+1}-1} F(x \setminus x^n)^{t_{n,0}}(s)$, so certainly outside $F(x \setminus x^n)^{t_{n,0}}(i)$.
If we choose $j = 2^{T_{\langle\langle n,z\rangle\rangle}-(n+1)}$, we are done. $\square$

## 6.7 Further research

In this chapter, we saw that, accepting the fan theorem, we can derive a much stronger statement than the axiom of symmetry: $\mathrm{A}_{<\frac{1}{2}}$, which claims that any function that associates each point in $[0,1]$ to a measurable areal with measure smaller than $\frac{1}{2}$ will have a symmetric pair.

One could wonder whether classically, without an assumption of continuity, we have a similar statement, or perhaps a slightly weaker statement where each point is associated to a set of some measure $\epsilon$.
Of course, if we accept the continuum hypothesis and axiom of choice, such a statement would not be true, as it trivially implies the standard axiom of symmetry. But could it hold in combination with the axiom of choice? Or, if not, with the normal axioms of set theory?
I haven't given this question too much thought myself - but it might, perhaps, be an interesting question for classical research.

Staying in an intuitionistic setup, a question I haven't found the time to research properly is the validity of a statement $\mathrm{A}_{=\frac{1}{2}}$: what if each point is associated to a measurable areal of exactly measure $\frac{1}{2}$? The method used to prove theorem 14 doesn't work anymore, since it made a measurable areal $\beta$ of slightly larger size than the areals mapped to by the function used - but we don't have that space now. In the finite variation (theorem 13) we could circumvent this problem to get a sharp result, but the infinite version is trickier.

Another point of research could be extending the theorems $\mathrm{A}_{2^{-n}}$ (where each point is associated to a measurable areal of measure at most $2^{-n}$) to multiple darts. Several extensions come to mind - for example: if the areals all have measure $2^{-n}$, how many points can you choose such that each of them is (strictly or not strictly) outside the areal associated with any of the others? What variation of $\mathrm{A}^{*m}_{2^{-\mathrm{n}}}$ is the best we can get?

Additionally, for those who reject the fan theorem the question whether $\mathrm{A}_{\mathrm{null}}$ can be derived *without* the fan theorem would be very interesting. We used the fan theorem for the proof of $\mathrm{A}_{<\frac{1}{2}}$ (and even there we could question whether it could, perhaps, be avoided), but $\mathrm{A}_{\mathrm{null}}$ is a lot weaker than that!

I haven't looked (much) into these questions myself, or even defined most of them properly. Some of these might be interesting, others are probably trivial.

# Chapter 7

# Other small sets

In the previous chapters, we have discussed the validity of various versions of the axiom of symmetry. In each case, the sets associated to the points were, in some clear way, *small* - countable sets of ordinals are negligible compared to the class of all ordinals, the "chance" of a point being in a null set is 0, and so on.
A last question Freiling considered was whether the axiom leads to a true statement if you use a very different kind of smallness, like the notion of a meagre set. However, since meagre subsets of $[0, 1]$ might well have measure 1, the corresponding statements can not be "justified" in the same probabilistic way as the other ones! But even though Freiling couldn't defend it, he did not have a proof that this $\mathrm{A}_{\mathrm{meagre}}$ axiom was false, even when additionally assuming $\mathrm{A}_{\mathrm{null}}$.

We shall see that under our constructive interpretation, this "axiom" becomes true. To see this, we must first give a constructively precise notion of "meagre set".

## 7.1 Definitions

Classically, a meagre set is the countable union of nowhere dense sets. A set $A$ is nowhere dense if, given any interval $a$, there is some subinterval $b \subseteq a$ such that $b$ has no overlap with $A$ (so there is no interval where $A$ is dense).
In a constructive setup, there is no reason not to take exactly the same approach! However, to avoid having to make ambiguous choices uncountably many times, I will define these "sets" as spreads.

**Definition 4.** nowhere dense sets
$S$ is countable, so we may define an enumeration $\chi$ of all the rational intervals in $[0, 1]$.
Using this, let $\mathrm{D}$ be the spread such that for all $\alpha \in \mathcal{N}$: $\alpha \in \mathrm{D} \leftrightarrow \forall n[\alpha(n) \in S \wedge \chi_n' \leq \alpha(n)' < \alpha(n)'' \leq \chi_n'']$.

Classically, a subset of $[0, 1]$ is nowhere dense if, and only if, its complement covers the union of the intervals of some $\alpha \in \mathrm{D}$. As such, we can identify the nowhere dense sets with the elements of $\mathrm{D}$. Note that this is different from the way spreads are normally treated: when speaking about the set given by some $\alpha \in \mathrm{D}$, this is *not* the set $\{x|\exists n[x \in \alpha(n)]\}$.
For some $\alpha \in \mathrm{D}$ and some real $x$, we say that $x$ is covered by $\alpha$ if, for each $n$, $x \notin \alpha(n)$.

**Definition 5.** meagre sets
Let M be the spread such that for each $\alpha \in \mathcal{N}$: $\alpha \in \mathrm{M} \leftrightarrow \forall n, m[\alpha(\langle\!\langle n, m \rangle\!\rangle) \in S \wedge \chi'_m \leq \alpha(\langle\!\langle n, m \rangle\!\rangle)' < \alpha(\langle\!\langle n, m \rangle\!\rangle) \leq \chi''_m]$.
So $\alpha \in \mathrm{M} \leftrightarrow \forall n[\alpha^n \in \mathrm{D}]$.

Sticking with the classical idea of meagreness, every meagre set should be the countable union of nowhere dense sets; so we can represent these as a sequence of elements of D, which is exactly what the elements of M are.
For $\beta \in \mathrm{M}$, $x$ is covered by $\beta$ if there is some $n$ such that $x$ is covered by $\beta^n$.

## 7.2 $\mathbf{A}_{\text{nowhere dense}}$

To give an idea of the reasoning of the proof, I will first show that the straightforward interpretation of the axiom holds for functions to nowhere dense sets rather than meagre ones. I will only use the weak continuity principle.

**Theorem 19.** $A_{\text{nowhere dense}}$
*Let $F$ be a function that maps $[0,1]$ to D.*
*Then there exist $x, y \in [0,1]$ such that $\exists n[x \in F(y)(n)] \wedge \exists n[y \in F(x)(n)]$.*

Since the union of the intervals in an $\alpha \in \mathrm{D}$ gives the complement of what we think of as a nowhere dense set, this theorem is similar to the other symmetry axioms we have seen so far.

*Proof.* Since $x \to F(x)(0)$ is a function that maps $[0,1]$ to a subset of $\mathbb{N}$, we can apply the weak continuity principle. This gives us:

$$\forall x \in [0,1] \exists n \forall y \in [0,1][\overline{x}n = \overline{y}n \to F(x)(0) = F(y)(0)]$$

Determine such $n$ for $x$ the standard real representing 0 ($x(m) = \langle 0, 2^{-m} \rangle$). Also find $s = F(x)(0)$.
Then for all $y \in [0,1]^*$ with $y(n) = \langle 0, 2^{-n} \rangle$, $F(y)(0) = s$.
$s$ is some interval. Determine any $y \in s$. Also determine $m$ such that $\chi_m = \langle 0, 2^{-n} \rangle$. Let $t \in S$ be $F(y)(m)$. Then $t \subseteq [0, 2^{-n}]$ by definition of D.
Choose any standard $x \in t$ with $x(n) = \langle 0, 2^{-n} \rangle$. Then $x \in F(y)(m)$ and $y \in F(x)(0)$. $\square$

## 7.3 $\mathbf{A}_{\text{meagre}}$

With this proof in our pocket, we can turn our attention to the main problem of this section.

**Theorem 20.** $A_{\text{meagre}}$
*Let $F$ be a function that maps $[0,1]$ to M.*
*Then there exist $x, y \in [0,1]$ such that $\forall n[\exists m[x \in F(y)^n(m)] \wedge \exists m[y \in F(x)^n(m)]]$.*

*Idea of the proof:*

The idea of the proof is to first find intervals that $x$ and $y$ must be in to assume that $x \in F(y)^0$ and $y \in F(x)^0$ (loose notation - I mean to say that there is some $m$ such that $x \in F(y)^0(m)$, and similar for $F(x)^0$). Finding these intervals is done in the same way as the previous proof. Then, inside these intervals, we find subintervals, such that for $x$ and $y$ in these intervals, also $x \in F(y)^1$ and $y \in F(x)^1$. We continue doing this infinitely long, which ultimately defines $x$ and $y$ completely.

*Construction:*

Applying the weak continuity principle once more, we have that we can find, for all $z \in [0,1]$ and $n \in \mathbb{N}$, some $m \in \mathbb{N}$ such that every real $w$ that corresponds on $m$ places with $z$ will have $F(z)(m) = F(w)(m)$.
We will use this in an inductive definition of $x$ and $y$. To make things easy, I will limit my choices of $x$ and $y$ to standard reals.

For the formal construction, first define $x(0) = y(0) = \langle 0, 1\rangle$. Take $n_0 = m_0 = 0$. In each step $k$, we will define $n_{k+1},\ m_{k+1},\ i_k,\ j_k$ and $\overline{x}(n_{k+1}+1),\ \overline{y}(n_{k+1}+1)$.

**Step $k$:**

Let $u \in \mathbb{R}$, equal to $x(n_k)'$. Find $i \in \mathbb{N}$ such that $\chi_i = y(m_k)$. For reasoning in the correctness proof later, call this $i$ $i_k$.
Determine $p$ such that for any $c$ in $[0,1]^*$ that agrees with $u$ on the first $p+1$ places, $F(u)^k(i) = F(c)^k(i)]$ (such $p$ exists due to the weak continuity principle). We can safely assume that $p \geq n_k$.
Choose, for $n_k < q \leq p$: $x(q) = \langle x(n_k)', x(n_k)' + 2^{-q}\rangle$. Then we can be sure that $\overline{x}(p+1) = \overline{u}(p+1)$, so whatever the further choices for $x$, $F(x)^k(i) = F(u)^k(i)$.

$F(u)^k(i)$ is an interval, so there must be some $r \in \mathbb{N}$ and a standardinterval $a$ of order $r$ such that $a$ is strictly inside $F(u)^k(i)$. Let $v$ be any standard real with $v(r) = a$.
Now find $j \in \mathbb{N}$ such that $\chi_j = x(p)$. We will later refer to this $j$ as $j_k$.
Using WCP, determine $s \in \mathbb{N}$ such that $\forall d \in [0,1]^*[\overline{v}(s+1) = \overline{d}(s+1) \to F(v)^k(j) = F(d)^k(j)]$. We can always choose $s \geq r$. Define $m_{k+1} = s$.
Since $v$ is contained in $F(u)^k(i)$ which is covered by $\chi_i = y(m_k)$ (since $F$ maps to M), we can choose $y(m_k+1) = v(m_k+1),\ \ldots,\ y(s) = v(s)$. Now we can be sure that $F(y)^k(j)$ will be equal to $F(v)^k(j)$.

There must be $l \in \mathbb{N}$ and a standardinterval $b$ of order $l$ such that $b$ is strictly inside $F(v)^k(j)$. Let $w \in [0,1]^*$ be some standard real with $w(l) = b$.
By our choice of $j$, $b$ is strictly inside $x(p)$, so we may define, for $p < q \leq l$: $x(q) = w(q)$. Let $n_{k+1} = l$.

*Correctness proof:*

Since we take $a$ and $b$ *strictly* inside the previous intervals in the procedure, we will have, for each $k$, that $n_{k+1} > n_k$ and $m_{k+1} > n_k$. Therefore, this procedure defines $x$ and $y$ completely (over time). We have already seen that they are welldefined (always $x(n+1) \subseteq x(n),\ y(n+1) \subseteq y(n)$).

To prove the theorem, it will suffice to show that for each $k$, $x \in F(y)^k(j_k)$ and $y \in F(x)^k(i_k)$. But this follows immediately from the construction! $\square$

# Chapter 8

# Conclusion

In this essay, we have discussed the axiom of symmetry and its natural extensions from an intuitionistic point of view.

We have seen a straightforward way to interpret the axiom of symmetry constructively, by simply reading the quantifiers in a strong way. If we also assume that the function used in the basic theorem is given in a constructive way such that the strong continuity principle applies to it, we have found that the "axiom" holds.

Most of the extensions and variations Freiling discusses turn out to be true as well under this interpretation, usually by the mere assumption of the weak continuity principle:

- the various natural extensions with multiple darts (as long as we stick to countably many) follow in a similar way to the standard theorem of symmetry;
- the variation that assigns to each real a null set rather than necessarily a countable set turns out to be okay as well - but the method used to prove this requires that we accept the fan theorem in addition to the continuity principle;
- by the same means, we have extended the variation with null sets to multiple darts (in different forms);
- again, by the same means, we have gone beyond Freiling's claims, and have derived a variation of the symmetrytheorem where real numbers are mapped not just to null sets, but to sets with any measure smaller than $\frac{1}{2}$;
- even the variation where the reals are mapped to meagre sets becomes a true statement - which is quite a surprising result, since it seems impossible to defend this statement from Freiling's philosophical point of view.

Apart from these results, we have examined what happens if we use countable ordinals and sets of countable ordinals instead of reals and sets of reals in the axiom of symmetry. In classical mathematics, this variation of the symmetry axiom fails; in an intuitionistic setup, it seems to fail only if we assume Kuroda's conjecture of double negation shift.
It does not, however, fail in a very constructive way (as discussed in the chapter dealing with this question).

To conclude, from an intuitionistic point of view axioms of symmetry seem to be acceptable; if we assume some of Brouwer's principles, we can prove these "axioms" to be true.

However, intuitionistic research of the symmetry axiom doesn't have to end here. In the previous sections, I have outlined several problems that I haven't been able or haven't found the time to solve - mostly going on on the main extensions of the symmetry axiom, and some of those a full research subject in themselves. But we could go further! In this essay I mostly answered the questions Freiling asked - but there is no reason to limit ourselves to those. We could try to map the reals to subsets of reals that are in some other way small. And maybe there are interesting variations with multiple functions instead of, or combined with, multiple darts?

Additionally, one might be interested in the classical consequences of the theorems derived in this essay. To classical mathematicians, these theorems are meaningful in a straightforward way ("if we assume that the continuity principle holds for this particular function, then ..."), but we might make the statements more useful by adapting the proofs a little. In chapter 3 we have seen how this works out for the standard axiom of symmetry.
Other than the classical consequences of the theorems seen here, one might consider the classical question by itself: would it hold if we assume the negation of the axiom of choice? Are there other conditions a function could fulfill to be guaranteed a symmetric pair? Many of these questions have already been treated in other work.

As a closing remark, I believe we may be content. The axiom of symmetry has revealed itself to be interesting in intuitionistic as well as classical mathematics. And while many questions relating to it have been answered, there is still plenty of opportunity for additional research.

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