COMPLEXITY OF CONDITIONAL TERM REWRITING

CYNTHIA KOP, AART MIDDELDORP, AND THOMAS STERNAGEL

Department of Computer Science
University of Innsbruck, Austria*
e-mail address: kop@di.ku.dk

Department of Computer Science
University of Innsbruck, Austria
e-mail address: aart.middeldorp@uibk.ac.at

Department of Computer Science
University of Innsbruck, Austria
e-mail address: thomas.sternagel@uibk.ac.at

ABSTRACT. We propose a notion of complexity for oriented conditional rewrite systems satisfying certain restrictions. This notion is realistic in the sense that it measures not only successful computations, but also partial computations that result in a failed rule application. A transformation to unconditional context-sensitive rewrite systems is presented which reflects this complexity notion, as well as a technique to derive runtime and derivational complexity bounds for the result of this transformation.

1. INTRODUCTION

Conditional term rewriting [31, Chapter 7] is a well-known computational paradigm. First studied in the eighties and early nineties of the previous century, in more recent years transformation techniques have received a lot of attention. Various automatic tools for (operational) termination [12, 22, 32] as well as confluence [34] have been developed.

In this paper we consider the following question: What is the greatest number of steps that can be done when evaluating terms, for starting terms of a given size? For unconditional rewrite systems this question has been investigated extensively and numerous techniques have been developed that give an upper bound on the resulting notions of derivational and runtime complexity (e.g. [6, 15, 16, 25, 26]). Tools that support complexity methods ([4, 29, 39]) are under active development and compete annually in the complexity competition.[1]

[1]http://cbr.uibk.ac.at/competition/

We are not aware of any techniques or tools for conditional (derivational and runtime) complexity—or indeed, even of a *definition* for conditional complexity. This may be for a good reason, as it is not obvious what such a definition should be. Of course, simply counting steps without taking the conditions into account will not do. Counting successful rewrite steps both in the reduction and in the evaluation of conditions is a natural idea. This two-dimensional view is seen for instance in studies of (operational) termination [21, 22] and certain transformations from conditional rewrite systems to unconditional ones (e.g., unravelings [23, 31]). However, we will argue that this approach—considering only the *successful* evaluation steps—still gives rise to an unrealistic notion of complexity. Modern rewrite engines like Maude [7] that support conditional rewriting can spend significant resources on evaluating conditions that in the end prove to be useless for rewriting the term at hand. This should be taken into account when defining complexity.

Contribution. We propose a new notion of conditional complexity for a relatively large class of reasonably well-behaved conditional rewrite systems. This notion aims to capture the maximal number of rewrite steps that can be performed when reducing a term to normal form, including the steps that were computed but turned out to be ultimately not useful. In order to reuse existing methodology for deriving complexity bounds, we present a transformation into unconditional rewrite systems that can be used to estimate the conditional complexity, building on the ideas of *structure-preserving* transformations [1, 8] but including several new ideas. The transformed system is context-sensitive (Lucas [19, 20]), which is not yet supported by current complexity tools; however, ignoring the corresponding restrictions, we still obtain an upper bound on the conditional complexity.

Organization. The remainder of the paper is organized as follows. In the next section we recall some preliminaries. Based on the analysis of conditional complexity in Section 3, we introduce our new notion formally in Section 4. Section 5 presents a transformation to context-sensitive rewrite systems, and in Section 6–8 we present an interpretation-based method targeting the resulting systems, as well as two optimizations of the technique to demonstrate that we can obtain tight bounds on realistic systems. Section 9 concludes with initial experiments, related work, and suggestions for future work.

2. Preliminaries

We assume familiarity with (conditional) term rewriting and all that (e.g., [5, 31, 36]) and only shortly recall important notions that are used in the following.

In this paper we consider *oriented* conditional (term) rewrite systems (CTRSs for short). *Conditional rewrite rules* have the form $\ell \to r \Leftarrow c$, where c is a sequence $a_1 \approx b_1, \ldots, a_k \approx b_k$ of equations. An oriented CTRS is a set $\mathcal{R}$ of conditional rules. The rewrite relation $\to_{\mathcal{R}}$ associated with $\mathcal{R}$ is formally defined as the union of a series of approximations $\to_{\mathcal{R}_i}$, where

- $\mathcal{R}_0 = \varnothing$,
- $\mathcal{R}_{i+1} = \{\ell\sigma \to r\sigma \mid \ell \to r \Leftarrow c \in \mathcal{R} \text{ and } a\sigma \to_{\mathcal{R}_i}^* b\sigma \text{ for all } a \approx b \in c\}$.

In the sequel we will primarily use the observation that $s \to_{\mathcal{R}} t$ if and only if there exist a position p in s, a rule $\ell \to r \Leftarrow c$ in $\mathcal{R}$, and a substitution σ such that $s|_p = \ell\sigma$, $t = s[r\sigma]_p$, and $\mathcal{R} \vdash c\sigma$, where the latter denotes that $a\sigma \to_{\mathcal{R}}^* b\sigma$ for all $a \approx b \in c$. We may write $s \xrightarrow{\epsilon} t$ for a rewrite step at the root position and $s \xrightarrow{>\epsilon} t$ for a non-root step.

Given a (C)TRS $\mathcal{R}$ over a signature $\mathcal{F}$, the root symbols of left-hand sides of rules in $\mathcal{R}$ are called *defined symbols* and every other symbol in $\mathcal{F}$ is a *constructor symbol*. These sets are denoted by $\mathcal{F}_\mathcal{D}$ and $\mathcal{F}_\mathcal{C}$, respectively. For a given symbol f, we write $\mathcal{R}{\restriction}f$ for the set of rules in $\mathcal{R}$ whose left-hand sides have root symbol f. A *constructor term* consists of constructor symbols and variables. A *basic term* is a term $f(t_1, \ldots, t_n)$ where $f \in \mathcal{F}_\mathcal{D}$ and $t_1, \ldots, t_n$ are constructor terms. We call $\mathcal{R}$ *semi-finite* if $\mathcal{R}{\restriction}f$ is finite for every $f \in \mathcal{F}_\mathcal{D}$. Let $\Sigma(\mathcal{F}, \mathcal{V})$ be the set of substitutions mapping to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. For substitutions σ and τ we write $\sigma \to_\mathcal{R}^* \tau$ to denote $\sigma(x) \to_\mathcal{R}^* \tau(x)$ for all variables $x \in \mathcal{V}$. A term s is *terminating* if there is no infinite reduction $s \to_\mathcal{R} s_1 \to_\mathcal{R} s_2 \to_\mathcal{R} \cdots$. A *normal form* is a term s such that there is no term t with $s \to_\mathcal{R} t$. We say that t is a normal form of s if $s \to_\mathcal{R}^* t$ and t is a normal form. Note that it is possible for a normal form to instantiate the left-hand side of a rule, which is not true for TRSs.

A (C)TRS is *finitely branching* if there are only finitely many distinct terms reachable in one rewrite step from any given term. All semi-finite (C)TRSs are finitely branching, but they may have an infinite signature. Given a terminating and finitely branching TRS $\mathcal{R}$ over a signature $\mathcal{F}$, the *derivation height* of a term t is defined as $\mathsf{dh}(t) = \max\{n \mid t \to^n u$ for some term $u\}$. This leads to the notion of *derivational complexity* $\mathsf{dc}_\mathcal{R}(n) = \max\{\mathsf{dh}(t) \mid |t| \leqslant n\}$, where $|t|$ is the number of symbols occurring in t. If we restrict the definition to basic terms t we get the notion of *runtime complexity* $\mathsf{rc}_\mathcal{R}(n)$ [14].

Rewrite rules $\ell \to r \Leftarrow c$ of CTRSs are classified according to the distribution of variables among ℓ, r, and c. In this paper we consider 3-CTRSs, where the rules satisfy $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell, c)$. A CTRS $\mathcal{R}$ is *deterministic* if for every rule $\ell \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ in $\mathcal{R}$ we have $\mathcal{V}\mathsf{ar}(a_i) \subseteq \mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_{i-1})$ for $1 \leqslant i \leqslant k$.

We write $s \xrightarrow{\triangleright} t$ if there exist a position p in s, a rule $\ell \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$, a substitution σ, and an index $1 \leqslant i \leqslant k$ such that $s|_p = \ell\sigma$, $a_j\sigma \to^* b_j\sigma$ for all $1 \leqslant j < i$, and $t = a_i\sigma$. A CTRS is *quasi-decreasing* if there exists a well-founded order $>$ with the subterm property (i.e., $\rhd \subseteq >$ where $s \rhd t$ if t is a proper subterm of s) such that both $\to$ and $\xrightarrow{\triangleright}$ are included in $>$ [9]. We additionally define here that a term s is quasi-decreasing if there is no infinite sequence $s = u_0 \ (\to \cup \xrightarrow{\triangleright}) \ u_1 \ (\to \cup \xrightarrow{\triangleright}) \cdots$. Clearly, a CTRS is quasi-decreasing if and only if all its terms are, but individual terms may be quasi-decreasing even if the CTRS is not. Quasi-decreasingness ensures termination and, for finite CTRSs, computability of the rewrite relation. Quasi-decreasingness coincides with *operational termination* [21]. We call a CTRS *constructor-based* if the right-hand sides of conditions as well as proper subterms of the left-hand sides of rules are constructor terms.

Limitations. We restrict ourselves to constructor-based deterministic 3-CTRSs, where the right-hand sides of conditions use only variables not occurring in the left-hand side or in earlier conditions. That is, for every rule $f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k \in \mathcal{R}$:

- $\ell_1, \ldots, \ell_n, b_1, \ldots, b_k$ are constructor terms without common variables,
- $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_k)$ and $\mathcal{V}\mathsf{ar}(a_i) \subseteq \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_{i-1})$ for $1 \leqslant i \leqslant k$.

We will call such systems *CCTRSs*. Furthermore, we will focus on *strong* CCTRSs: semi-finite CCTRSs such that, for every rule $f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k \in \mathcal{R}$,

- $f(\ell_1, \ldots, \ell_n)$ and $b_1, \ldots, b_k$ are linear terms: no variable occurs more than once in them.

Note that, even in strong CCTRSs, the left-hand sides of conditions are *not* required to be linear. We will develop a complexity notion for the general case of CCTRSs, but limit the

work on transformations (in Section 5 and beyond) to strong CCTRSs. We will particularly consider *confluent* CCTRSs. While confluence is not needed for the formal development in this paper, without it the complexity notion we define is not meaningful, as discussed below.

To appreciate the limitations, note that in CTRSs which are not deterministic 3-CTRSs, the rewrite relation is undecidable in general, which makes it hard to define what complexity means. The restrictions with regards to variables and constructors in strong CCTRSs are the natural extension of the common restriction to left-linear constructor TRSs in unconditional rewriting. They closely correspond to *pattern guards* [10], a language extension of Haskell. Semi-finiteness actually *weakens* the standard restriction that $\mathcal{R}$ must be finite.

The limitation to CCTRSs is important because, in confluent CCTRSs, the approach to computation is unambiguous: To evaluate whether a term $\ell\sigma$ reduces with a rule $\ell \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ of a CCTRS, we start by reducing $a_1\sigma$ and, finding an instance of b_1, extend σ to the new variables in b_1 resulting in σ', continue with $a_2\sigma'$, and so on. Assuming confluence, if there is an extension of σ which satisfies all conditions then, no matter how we reduce, this procedure will either find it or—if $\ell\sigma$ is not quasi-decreasing— enter into an infinite reduction, a possibility which is also interesting from a complexity standpoint. However, if confluence or any of the restrictions on the conditions were dropped, this would no longer be the case and we might be unable to verify the applicability of a rule without enumerating all possible reducts of its conditions. The restrictions are needed to obtain Lemma 3.4, which will be essential to justify the way we handle failure.

We do not limit interest to quasi-decreasing CCTRSs—which would correspond to the usual approach of limiting interest to terminating TRSs in the unconditional setting—but will rather define the complexity of non-quasi-decreasing terms to be infinite. This is done in order to unify proof efforts, especially for Theorem 5.12.

Example 2.1. The CTRS $\mathcal{R}_{\mathsf{fib}}$ consisting of the rewrite rules

$$0 + y \to y \qquad (2.1) \qquad \mathsf{fib}(0) \to \langle 0, \mathsf{s}(0)\rangle \qquad (2.3)$$

$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \quad (2.2) \qquad \mathsf{fib}(\mathsf{s}(x)) \to \langle z, w\rangle \;\Leftarrow\; \mathsf{fib}(x) \approx \langle y, z\rangle, \; y + z \approx w \qquad (2.4)$$

is a quasi-decreasing and confluent strong CCTRS. The requirements for quasi-decreasingness are satisfied (e.g.) by the lexicographic path order with precedence $\mathsf{fib} > \langle \cdot, \cdot \rangle > + > \mathsf{s}$. Because the 3-CTRS $\mathcal{R}_{\mathsf{fib}}$ is orthogonal, right-stable, and properly oriented, confluence follows from the result of [35].

Notation. To simplify the notation and shorten proofs, we will use the following convention throughout the paper. Given a rule $\rho\colon \ell \to r \Leftarrow c$,

- the conditional part c consists of the conditions $a_1 \approx b_1$, …, $a_k \approx b_k$ for some $k \geqslant 0$ (which depends on ρ),
- for all $0 \leqslant j \leqslant k$, $c_j^{\leqslant}$ denotes the sequence $a_1 \approx b_1$, …, $a_j \approx b_j$.

In addition, we will sometimes refer to ℓ as b_0 and to r as a_{k+1}.

With these conventions, the limitations on rules can be reformulated as follows. For every rule $\ell \to r \Leftarrow c$:

- $b_1, \ldots, b_k$ and the proper subterms of b_0 are constructor terms,
- $\mathcal{V}\mathsf{ar}(b_i) \cap \mathcal{V}\mathsf{ar}(b_j) = \varnothing$ for all $0 \leqslant i, j \leqslant k$ with $i \neq j$ and, in a strong CCTRS, the terms $b_0, \ldots, b_k$ are linear,
- $\mathcal{V}\mathsf{ar}(a_i) \subseteq \mathcal{V}\mathsf{ar}(b_0, \ldots, b_{i-1})$ for all $1 \leqslant i \leqslant k + 1$.

3. ANALYSIS

Before we can define a notion of complexity, we must consider a model of computation. Unlike unconditional term rewriting, it is not obvious how a term in a CTRS is reduced to normal form. Even taking the approach for confluent CCTRSs sketched in Section 2 as a basis, some unresolved questions remain. In this section, we will study both computation and complexity by an appeal to intuition. In the next section we will formalize the results.

We start our analysis with a deceivingly simple CCTRS to illustrate that the notion of complexity for conditional systems is not obvious.

Example 3.1. The CCTRS $\mathcal{R}_{\mathsf{even}}$ consists of the following six rewrite rules:

$$\mathsf{even}(0) \to \mathsf{true} \qquad (3.1) \qquad \mathsf{odd}(0) \to \mathsf{false} \qquad (3.4)$$

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \approx \mathsf{true} \quad (3.2) \qquad \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{even}(x) \approx \mathsf{true} \quad (3.5)$$

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{even}(x) \approx \mathsf{true} \quad (3.3) \qquad \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{odd}(x) \approx \mathsf{true} \quad (3.6)$$

If, like in the unconditional case, we count the number of steps needed to normalize a term, then a term $t_n = \mathsf{even}(\mathsf{s}^n(0))$ has derivation height 1, since $t_n \to \mathsf{false}$ or $t_n \to \mathsf{true}$ in a single step. To reflect actual computation, the rewrite steps to verify the condition should be taken into account. Viewed like this, normalizing t_n takes $n + 1$ rewrite steps.

However, this still seems unrealistic as a rewriting engine cannot know in advance which rule to attempt first. For example, when rewriting t_9, rule (3.2) may be tried first, which requires normalizing $\mathsf{odd}(\mathsf{s}^8(0))$ to verify the condition. After finding that the condition fails, rule (3.3) is attempted. Thus, for $\mathcal{R}_{\mathsf{even}}$, a tool implementing conditional term rewriting with a random rule selection strategy would select a rule with a failing condition about half the time. If we assume a worst possible selection and count all rewrite steps performed during the computation, we need $2^{n+1} - 1$ steps to normalize t_n.

Although this exponential upper bound may come as a surprise, a powerful rewrite engine like Maude [7] does not perform much better, as can be seen from the data in Table 1. Unlike term rewriting (which is non-deterministic by nature), Maude employs a top-down rule selection strategy, so the order in which the rules are presented makes a difference in the outcome—although, as it turns out, not a substantial one for Example 3.1 or other examples in this paper. For rows three and four we presented the rules to Maude in the order given in Example 3.1. If we change the order to (3.4), (3.6), (3.5), (3.1), (3.3), (3.2) we obtain the last two rows, showing an exponential number of steps in all cases. Regardless of the order on the rules, we never obtain the optimal linear bound for all tested terms.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$2^{n+1} - 1$	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191
$\mathsf{even}(\mathsf{s}^n(0))$	1	3	3	11	5	37	7	135	9	521	11	2059	13
$\mathsf{odd}(\mathsf{s}^n(0))$	1	2	6	4	20	6	70	8	264	10	1034	12	4108
$\mathsf{even}(\mathsf{s}^n(0))$	1	2	7	8	31	32	127	128	511	512	2047	2048	8191
$\mathsf{odd}(\mathsf{s}^n(0))$	1	3	4	15	16	63	64	255	256	1023	1024	4095	4096

Table 1: Number of steps required to normalize $\mathsf{even}(\mathsf{s}^n(0))$ and $\mathsf{odd}(\mathsf{s}^n(0))$ in Maude.

From the above we conclude that *a realistic definition of conditional complexity should take failed computations into account.* This conclusion opens new questions, however; most pertinently, the question of how to handle repeated failed attempts. It is obvious that we cannot allow repeatedly trying (and failing) the same rule at the same position. For instance, it would be foolish to attempt to reduce $\mathsf{even}(\mathsf{s}(0))$ with rule (3.2), fail, then try the same rule *again* ten more times before turning to (3.3) and count the steps for all the failed attempts in the reduction cost. Thus, we must impose some restrictions on duplicated attempts. To this end, let us consider what constitutes a duplicated attempt.

Example 3.2. The CCTRS $\mathcal{R}_{\mathsf{fg}}$ consists of the following two rewrite rules:

$$\mathsf{f}(x) \to x \qquad\qquad (3.7) \qquad\qquad \mathsf{g}(x) \to \mathsf{a} \Leftarrow x \approx \mathsf{b} \qquad (3.8)$$

Consider $t_{n,m} = \mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a})))$. As we have not imposed an evaluation strategy, one approach to evaluate this term could be as follows. We try using (3.8) on the subterm $\mathsf{g}(\mathsf{f}^m(\mathsf{a}))$. This fails in m steps. With (3.7) at the root position we obtain $t_{n-1,m}$. We again attempt (3.8), failing in m steps. Repeating this scenario results in $n \cdot m$ rewrite steps before we reach $t_{0,m}$.

In the above example we keep attempting—and failing—to rewrite an *unmodified copy* of a subterm we tried before, with the same rule. Even though the position of the subterm $\mathsf{g}(\mathsf{f}^m(\mathsf{a}))$ changes, we already know that this reduction will fail. Hence, once we fail a conditional rule on given subterms, it is reasonable not to try the same rule again on (copies of) the same subterms, even after a successful step. In our model of computation we should therefore keep track of previous failed attempts. This will be made formal in Section 4.

Example 3.3. Continuing with $t_{0,m}$ from the preceding example, we could try to use (3.8), which fails in m steps. Next, (3.7) is applied on a subterm, and we obtain $t_{0,m-1}$. Again we try (3.8), failing after executing $m - 1$ steps. Repeating this alternation results eventually in the normal form $t_{0,0}$, but not before computing $\frac{1}{2}(m^2 + 3m)$ rewrite steps in total.

Like in Example 3.2, we keep coming back to a subterm which we have already tried before in an unsuccessful attempt. The difference is that the subterm has been rewritten between successive attempts. According to the following general result, we need not reconsider a failed attempt to apply a conditional rewrite rule if only the arguments were changed.

Lemma 3.4. *Given a CCTRS $\mathcal{R}$, suppose $s \xrightarrow{>\epsilon}{}^* t$ and let $\rho \colon \ell \to r \Leftarrow c$ be a rule such that s is an instance of ℓ. If $t \xrightarrow{\epsilon}_\rho u$ then there exists a term v such that $s \xrightarrow{\epsilon}_\rho v$ and $v \to^* u$.*

So if we can rewrite a term at the root position eventually, and the term already matches the left-hand side of the rule with which we can do so, then we can rewrite the term with this rule immediately and obtain the same result. Note that this lemma does *not* assume confluence, quasi-decreasingness or left-linearity, and so is broadly applicable.

Proof. Suppose $s = \ell\sigma$ with $\mathsf{dom}(\sigma) \subseteq \mathcal{V}\mathsf{ar}(\ell)$, and let τ be a substitution such that $t = \ell\tau$, $u = r\tau$, and $\mathcal{R} \vdash c\tau$, which exists since ρ applies to t at the root position. Because ℓ is a basic term, all steps in $s \xrightarrow{>\epsilon}{}^* t$ take place in the substitution part σ of $\ell\sigma$ and thus $\sigma(x) \to^* \tau(x)$ for all $x \in \mathcal{V}\mathsf{ar}(\ell)$. Defining the substitution σ' as follows, we have $s = \ell\sigma = \ell\sigma'$ and $\sigma' \to^* \tau$:

$$\sigma'(x) = \begin{cases} \sigma(x) & \text{if } x \in \mathcal{V}\mathsf{ar}(\ell) \\ \tau(x) & \text{if } x \notin \mathcal{V}\mathsf{ar}(\ell) \end{cases}$$

Let $a \approx b$ be a condition in c. From $\mathcal{V}\mathsf{ar}(b) \cap \mathcal{V}\mathsf{ar}(\ell) = \varnothing$ we infer $a\sigma' \to^* a\tau \to^* b\tau = b\sigma'$. It follows that $\mathcal{R} \vdash c\sigma'$ and thus $s \xrightarrow{\epsilon}_\rho r\sigma'$. Hence we can take $v = r\sigma'$ as $r\sigma' \to^* r\tau = u$. $\qquad\square$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
m	0	1	2	3	4	5	6	7	8	9	10	11	12
$n \cdot m$	0	1	4	9	16	25	36	49	64	81	100	121	144
$\frac{1}{2}(m^2 + 3m)$	0	2	5	9	14	20	27	35	44	54	65	77	90
$\mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a})))$	0	3	8	16	28	45	68	98	136	183	240	308	388
$\mathsf{g}(\mathsf{f}^m(\mathsf{a}))$	0	2	6	13	24	40	62	91	128	174	230	297	376

Table 2: Number of steps required to normalize $\mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a})))$ and $\mathsf{g}(\mathsf{f}^m(\mathsf{a}))$ in Maude.

From the above observations we conclude that, to avoid unnecessary repetitions, we can simply mark occurrences of defined symbols with the rules we have already tried without success—or, symmetrically, with the rules we have yet to try, as we will do in Section 4.

Table 2 compares these theoretical considerations to actual computations of $\mathcal{R}_{\mathsf{fg}}$ in Maude. Interestingly, Maude seems to perform worse on evaluating $\mathsf{g}(\mathsf{f}^m(\mathsf{a}))$ than the realistic $m + 1$ bound. Thus, it seems that Maude could benefit from incorporating the implications of Lemma 3.4. However, it should be remarked that when presenting $\mathcal{R}_{\mathsf{fg}}$ as a *functional module* [7, Chapter 6], Maude will switch to an innermost evaluation strategy and compute the normal form $\mathsf{g}(\mathsf{a})$ of $\mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a})))$ in $m + n$ steps.

In this paper, we will assume that rewriting takes Lemma 3.4 into account, and thus avoids repeatedly reevaluating the same term. Also unlike Maude, we will not impose an evaluation order on the rules, nor a strategy for the position in a term that must be rewritten first, but allow free choice as is common in term rewriting.

Another important aspect to consider is how to define a "failed" reduction. Intuitively, a rule $\ell \to r \Leftarrow c$ should be considered not applicable on a term $\ell\sigma$ if there is no extension σ' of σ such that $\mathcal{R} \vdash c\sigma'$. Yet in Example 3.1 we already concluded that the second rule was not applicable to t_9 simply after reducing $\mathsf{odd}(\mathsf{s}^8(0))$ to its normal form false, because false does not match the right-hand side true of the condition. As remarked in Section 2, this is possible due to our restrictions. The following lemma makes this observation formal.

Lemma 3.5. *Let $\rho\colon \ell \to r \Leftarrow c$ be a rule in a confluent CCTRS $\mathcal{R}$ and σ a substitution such that $\mathsf{dom}(\sigma) \subseteq \mathcal{V}\mathrm{ar}(\ell)$ and $\ell\sigma$ is quasi-decreasing. Then ρ is not applicable to $\ell\sigma$ if and only if there is an extension σ' of σ, and some $1 \leqslant i \leqslant k$ such that $\mathcal{R} \vdash c_{i-1}^{\leqslant}\sigma$ and $a_i\sigma' \to^* u$ for some normal form u which is not an instance of b_i.*

Proof. (Recall that c is $a_1 \approx b_1$, ..., $a_k \approx b_k$ and $c_{i-1}^{\leqslant}$ denotes $a_1 \approx b_1$, ..., $a_{i-1} \approx b_{i-1}$.) We first prove the "only if" direction. So suppose that ρ is not applicable to $\ell\sigma$. We define extensions $\sigma_0, \ldots, \sigma_{i-1}$ of σ such that $\sigma_j(x) = \sigma(x)$ for all $x \in \mathcal{V}\mathrm{ar}(\ell)$, $\mathsf{dom}(\sigma_j) \subseteq \mathcal{V}\mathrm{ar}(\ell, b_1, \ldots, b_j)$ for all $0 \leqslant j < i$, $\mathcal{R} \vdash c_j^{\leqslant}\sigma_j$, and $a_i\sigma_{i-1} \to^* u$ for some normal form u which is not an instance of b_i. Then $\sigma' = \sigma_{i-1}$ satisfies the requirements of the lemma. Let $\sigma_0 = \sigma$ and suppose $\sigma_1, \ldots, \sigma_{j-1}$ have been defined. We have $\ell\sigma = \ell\sigma_{j-1} \xrightarrow{\rhd} a_j\sigma_{j-1}$ and hence $a_j\sigma_{j-1}$ is terminating by quasi-decreasingness. Let u be a normal form of $a_j\sigma_{j-1}$. If u is an instance of b_j, say $u = b_j\tau$ with $\mathsf{dom}(\tau) \subseteq \mathcal{V}\mathrm{ar}(b_j)$, then we let $\sigma_j = \sigma_{j-1} \cup \tau$. Note that σ_j is well-defined as $\mathsf{dom}(\sigma_{j-1}) \cap \mathcal{V}\mathrm{ar}(b_j) = \varnothing$. In this case σ_j clearly satisfies the above conditions. If u is not an instance of b_j then we are done by letting $i = j$. Note that the latter must happen for some j since we assumed that ρ is not applicable.

Next we prove the "if" direction. Suppose σ', i, and u exist with the stated properties. For a proof by contradiction, also suppose that the rule is applicable, so there is an extension τ of σ such that $\mathcal{R} \vdash c\tau$. Define the substitution $\tau{\downarrow}$ as $\{x \mapsto \tau(x){\downarrow}_{\mathcal{R}} \mid x \in \mathsf{dom}(\tau)\}$, where $\tau(x){\downarrow}_{\mathcal{R}}$ denotes the unique normal form of $\tau(x)$. (This is well-defined because $\ell\tau = \ell\sigma$ is quasi-decreasing and thus $a_j\tau$ and $b_j\tau$ are quasi-decreasing for all $1 \leqslant j \leqslant k$. Therefore, all subterms of $\ell\tau$, $b_1\tau$, …, $b_k\tau$ are terminating. Since we may assume $\mathsf{dom}(\tau) \subseteq \mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_k)$—as each a_j uses only variables in $\mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_{j-1})$—confluence ensures that $\tau(x)$ has a unique normal form for every $x \in \mathsf{dom}(\tau)$.) Fix $1 \leqslant j \leqslant k$. We have $a_j\tau \to^* b_j\tau \to^* b_j(\tau{\downarrow})$ and $a_j\tau \to^* a_j(\tau{\downarrow})$. Since b_j is a constructor term, $b_j(\tau{\downarrow})$ is a normal form and thus $a_j(\tau{\downarrow}) \to^* b_j(\tau{\downarrow})$ by confluence. We claim that $\sigma'(x) \to^* \tau{\downarrow}(x)$ for all $x \in \mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_{i-1})$. If $x \in \mathcal{V}\mathsf{ar}(\ell)$ then $\sigma'(x) = \sigma(x) = \tau(x)$. Hence also $\sigma'(x) \to^* \tau{\downarrow}(x)$. Suppose the claim holds for $x \in \mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_{j-1})$ with $1 \leqslant j < i$. From $\mathcal{V}\mathsf{ar}(a_j) \subseteq \mathcal{V}\mathsf{ar}(\ell, b_1, \ldots, b_{j-1})$ we infer $a_j\sigma' \to^* a_j(\tau{\downarrow}) \to^* b_j(\tau{\downarrow})$. Also $a_j\sigma' \to^* b_j\sigma'$ and thus $b_j\sigma' \to^* b_j(\tau{\downarrow})$ by confluence. As b_j is a constructor term, $\sigma'(x) \to^* \tau{\downarrow}(x)$ for all $x \in \mathcal{V}\mathsf{ar}(b_j)$. This completes the proof of the claim. From the claim we find $a_i\sigma' \to^* a_i(\tau{\downarrow}) \to^* b_i(\tau{\downarrow})$. Using $a_i\sigma' \to^* u$ and confluence, we obtain $b_i(\tau{\downarrow}) = u$, contradicting the assumption that u is not an instance of b_i. $\qquad\square$

Thus, if we reduce the conditions of a rule and find a normal form that does not instantiate the required right-hand side, we can safely conclude that the rule does not apply.

A final aspect to consider is when to stop reducing a condition. Should we stop once we obtain the right shape? Or should we allow—or even enforce—reductions to normal form?

Example 3.6. Consider the following CCTRS implementing addition:

$$\mathsf{plus}(x, y) \to y \Leftarrow x \approx 0 \qquad\qquad \mathsf{plus}(x, y) \to \mathsf{s}(\mathsf{plus}(z, y)) \Leftarrow x \approx \mathsf{s}(z)$$

Let $t = \mathsf{plus}(\mathsf{plus}(\mathsf{s}^9(0), 0), \mathsf{s}(0))$. To reduce t at the root with the second rule, we must evaluate the condition $\mathsf{plus}(\mathsf{s}^9(0), 0) \to^* \mathsf{s}(z)$. This is satisfied in a single step, reducing to $\mathsf{s}(z)\{z \mapsto \mathsf{plus}(\mathsf{s}^8(0), 0)\}$. Should we therefore reduce immediately, to $\mathsf{s}(\mathsf{plus}(\mathsf{plus}(\mathsf{s}^8(0), 0), \mathsf{s}(0)))$? Or should we continue reducing the condition until we obtain a normal form $\mathsf{s}^8(0)$ and then reduce to $\mathsf{s}(\mathsf{plus}(\mathsf{s}^8(0), \mathsf{s}(0)))$? Similarly, if we try to reduce t at the root with the first rule, we obtain in one step an instance of $\mathsf{s}(z)$, which does not unify with 0. Since every reduct of $\mathsf{s}(z)\sigma'$ is still an instance of $\mathsf{s}(z)$, we could immediately conclude that the condition will fail.

Both questions are a matter of *strategy*, and different approaches might adopt different choices. One could argue that it makes little sense to continue reducing a term for a condition when we already know that it is satisfied, much like we said it makes no sense to keep reevaluating the same failing condition. However, since we aim for a general definition, we have decided not to pursue this. That is, in Example 3.6 we may choose to stop evaluating the conditions and reduce with the rule (resp. conclude failure) once we obtain an instance of the desired pattern (resp. a term for which we can easily see that it will never reduce to such an instance), but this is not compulsory. Specific evaluation strategies can easily be added to the corresponding definitions and transformations later.

Although a large part of our complexity notion deals with failed reductions, there are many CCTRSs where this is not relevant. Consider for instance Example 2.1 in which the conditions of the one conditional rule are not expected to fail; they merely evaluate the result of a smaller term to a normal form (or at least a constructor instance), and use its subterms. Correspondingly, as can be seen in Table 3, the time needed to normalize terms in the Fibonacci CCTRS grows roughly as fast as the Fibonacci sequence itself, with no additional exponential growth for failed attempts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathsf{fib}(\mathsf{s}^n(0))$	1	3	7	13	23	40	69	119	205	353	607	1042	1785

Table 3: Number of steps required to normalize $\mathsf{fib}(\mathsf{s}^n(0))$ in Maude.

4. Conditional Complexity

In Section 3 we have come to an intuitive understanding of how a term s in a (confluent) CTRS can be reduced, and what the corresponding complexity should be:

- In every step we select a position p and a rule $\ell \to r \Leftarrow c$ matching the corresponding subterm (i.e., $s|_p = \ell\sigma$ for some σ).
- We then start evaluating the conditions in c from left to right, extending σ as we go, until we have either confirmed all conditions or obtain a failing condition.
- In the former case, we reduce $s|_p$ by this rule (obtaining $s[r\sigma']_p$ for the extension σ' of σ found by evaluating the conditions in c). In the latter case, we mark the subterm $s|_p$ to indicate that we should not try the rule $\ell \to r \Leftarrow c$ on this subterm again.
- The complexity of a conditional reduction is then obtained by counting all rewrite steps, including those in successful and failed condition evaluations.

In this section, we will formalize this intuition. A key aspect is the ability to mark terms, so as to avoid continuously repeating the same reduction attempt. To achieve this, we will label defined function symbols by subsets of the rules used to define them. Then, we define a variation $\rightharpoonup$ of the rewrite relation $\to$ which explicitly includes failed computations. This relation is used as the basis to define a complexity measure in a natural way.

This section is structured as follows. First we define labeled terms and the labeled rewrite relation $\rightharpoonup$ (Section 4.1). Then we analyze how $\rightharpoonup$ relates to the unlabeled conditional rewrite relation $\to$ (Section 4.2) and define derivation height and complexity (Section 4.3).

4.1. **Labeled Terms and Reduction.**

Definition 4.1. Let $\mathcal{R}$ be a CCTRS over a signature $\mathcal{F}$. The labeled signature $\mathcal{G}$ is defined as $\mathcal{F}_{\mathcal{C}} \cup \{f_R \mid f \in \mathcal{F}_{\mathcal{D}} \text{ and } R \subseteq \mathcal{R}{\restriction}f\}$. A labeled term is a term in $\mathcal{T}(\mathcal{G}, \mathcal{V})$.

Intuitively, the label R in f_R records the defining rules for f which have not yet been tried. In examples we will generally conflate the rules in R with labels identifying them.

Definition 4.2. Let $\mathcal{R}$ be a CCTRS over a signature $\mathcal{F}$. The mapping $\mathsf{label}\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{G}, \mathcal{V})$ labels every defined symbol f with $\mathcal{R}{\restriction}f$. The mapping $\mathsf{erase}\colon \mathcal{T}(\mathcal{G}, \mathcal{V}) \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ removes the labels from defined symbols.

We obviously have $\mathsf{erase}(\mathsf{label}(t)) = t$ for every $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\mathsf{erase}(t) = \mathsf{label}(t) = t$ for constructor terms t. The identity $\mathsf{label}(\mathsf{erase}(t)) = t$ does not hold for arbitrary $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$.

Definition 4.3. A *labeled normal form* is a term in $\mathcal{T}(\mathcal{F}_{\mathcal{C}} \cup \{f_{\varnothing} \mid f \in \mathcal{F}_{\mathcal{D}}\}, \mathcal{V})$.

Example 4.4. In $\mathcal{R}_{\mathsf{fib}}$ from Example 2.1, the labeled signature $\mathcal{G}$ consists of 0, s, $\langle \cdot \rangle$, $+_R$ for every subset R of $\{(2.1), (2.2)\}$, and fib_R for every subset R of $\{(2.3), (2.4)\}$. We have

$$\mathsf{label}(\mathsf{fib}(\mathsf{s}(0) + 0)) = \mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0) +_{\{(2.1),(2.2)\}} 0)$$

Examples of labeled normal forms are $\mathsf{s}(0)$ and $\mathsf{fib}_{\varnothing}(0 +_{\varnothing} \mathsf{s}(\mathsf{s}(0)))$.

The relation $\rightharpoonup$ will be designed so that a ground labeled term can be reduced if and only if it is not a labeled normal form (see Lemma 4.11). First, with Definition 4.5 we can remove rules from a label if they will never apply due to an impossible matching problem.

Definition 4.5. Let $\mathcal{R}$ be a CCTRS. For labeled terms s and t we write $s \xrightarrow{\perp} t$ if there exist a position $p \in \mathcal{P}os(s)$ and a rewrite rule $\rho \colon \ell \to r \Leftarrow c$ in $\mathcal{R}$ such that

(1) $s|_p = f_R(s_1, \ldots, s_n)$ with $\rho \in R$,

(2) $t = s[f_{R \setminus \{\rho\}}(s_1, \ldots, s_n)]_p$, and

(3) there exist linear labeled normal forms $u_1, \ldots, u_n$ with fresh variables and a substitution σ such that $s|_p = f_R(u_1, \ldots, u_n)\sigma$ and $f(u_1, \ldots, u_n)$ does not unify with ℓ.

The last item ensures that rewriting (using $\rightharpoonup$) strictly below position p cannot give a reduct that matches ℓ, since all such reducts will still be instances of $f_R(u_1, \ldots, u_n)$. Furthermore, if s is ground and $s|_p = f_R(s_1, \ldots, s_n)$ where R is non-empty and all $s_1, \ldots, s_n$ are labeled normal forms, then either $f(s_1, \ldots, s_n)$ is an instance of ℓ, or $\xrightarrow{\perp}$ applies to $s|_p$.

Example 4.6. In Example 4.4 we have

$$\mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0) +_{\{(2.1),(2.2)\}} 0) \xrightarrow{\perp} \mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0) +_{\{(2.2)\}} 0)$$

because $\mathsf{s}(0) + 0$ does not unify with $0 + y$, and both $\mathsf{s}(0)$ and 0 are linear labeled normal forms. Also

$$\mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0 +_{\{(2.1)\}} 0) +_{\{(2.1),(2.2)\}} 0) \xrightarrow{\perp} \mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0 +_{\{(2.1)\}} 0) +_{\{(2.2)\}} 0)$$

since $\mathsf{s}(x)$ and 0 are linear labeled normal forms and $\mathsf{s}(x) + 0$ does not unify with $0 + y$.

Second, Definition 4.7 describes how to "reduce" labeled terms in general. This definition is designed to reduce *ground* terms in the way roughly described in Section 3. Labeled terms are reduced without any strategy, but subterms keep track of which rules have not yet been attempted, thus implicitly avoiding duplication.

Definition 4.7. A *labeled* reduction is a sequence $t_1 \rightharpoonup t_2 \rightharpoonup \cdots \rightharpoonup t_m$ of labeled terms where $s \rightharpoonup t$ if either $s \xrightarrow{\perp} t$ or there exist a position $p \in \mathcal{P}os(s)$, rewrite rule $\rho \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$, substitution σ, and index $0 \leqslant j \leqslant k$ such that

(1) $s|_p = f_R(s_1, \ldots, s_n)$ with $\rho \in R$ and $s_i = \ell_i\sigma$ for all $1 \leqslant i \leqslant n$,

(2) $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$ for all $1 \leqslant i \leqslant j$,

and either

(3) $j = k$ and $t = s[\mathsf{label}(r)\sigma]_p$

in which case we speak of a *successful* step, or

(4) $j < k$ and there exist a linear labeled normal form u and a substitution τ such that
   (a) $\mathsf{label}(a_{j+1})\sigma \rightharpoonup^* u\tau$ and u does not unify with b_{j+1}, and
   (b) $t = s[f_{R \setminus \{\rho\}}(s_1, \ldots, s_n)]_p$,

which is a *failed* step.

A *complexity-conscious reduction* is a labeled reduction complete with proofs of the sub-requirements, i.e., a sequence $(t_1 \rightharpoonup t_2), \ldots, (t_{m-1} \rightharpoonup t_m)$ of *complexity-conscious steps*, where each complexity-conscious step $s \rightharpoonup t$ is a tuple combining s, t, p, ρ, j and the complexity-conscious reductions $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$ for $1 \leqslant i \leqslant j$ and possibly $\mathsf{label}(a_{j+1})\sigma \rightharpoonup^* u\tau$. We will denote complexity-conscious reductions as labeled reductions, and simply assume the underlying condition evaluations given.

It is easy to see that for all ground labeled terms s which are not labeled normal forms, either $s \overset{\perp}{\rightharpoonup} t$ for some term t or there are p, ρ, σ such that $s|_p$ "matches" ρ in the sense that the first requirement in Definition 4.7 is satisfied. In the latter case, the conditions are evaluated left-to-right; as all b_j are linear constructor terms on fresh variables, $\mathsf{label}(a_j)\sigma \rightharpoonup^* b_j\sigma$ simply indicates that $a_j\sigma$—with labels added to allow reducing defined symbols in a_j—reduces to an instance of b_j. A successful reduction occurs when we manage to reduce each $\mathsf{label}(a_i)\sigma$ to $b_i\sigma$. A failed reduction occurs when we start reducing $\mathsf{label}(a_i)\sigma$ and obtain a term that will never reduce to an instance of b_i.

Example 4.8. Continuing Example 4.4, we have the following complexity-conscious reduction:

$$\mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0) +_{\{(2.1),(2.2)\}} 0) \overset{\perp}{\rightharpoonup} \mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0) +_{\{(2.2)\}} 0) \qquad \text{(Example 4.6)}$$
$$\rightharpoonup \mathsf{fib}_{\{(2.3),(2.4)\}}(\mathsf{s}(0 +_{\{(2.1),(2.2)\}} 0)) \qquad \text{(successful step)}$$
$$\rightharpoonup \langle \mathsf{s}(0), \mathsf{s}(0) \rangle \qquad \text{(successful step)}$$

The first successful step uses the unconditional rule (2.2). The second successful step uses rule (2.4) and the complexity-conscious reductions

$$\mathsf{fib}_{\{(2.3),(2.4)\}}(0 +_{\{(2.1),(2.2)\}} 0) \rightharpoonup \mathsf{fib}_{\{(2.3),(2.4)\}}(0) \rightharpoonup \langle 0, \mathsf{s}(0) \rangle$$

and

$$0 +_{\{(2.1),(2.2)\}} \mathsf{s}(0) \rightharpoonup \mathsf{s}(0)$$

for the evaluation of the conditions, all by successful steps without conditions.

Example 4.9. In the CCTRS of Example 3.1 we have the following complexity-conscious reduction:

$$\mathsf{even}_{\{(3.1),(3.2),(3.3)\}}(\mathsf{s}(0)) \rightharpoonup \mathsf{even}_{\{(3.1),(3.3)\}}(\mathsf{s}(0)) \qquad \text{(failed step)}$$
$$\overset{\perp}{\rightharpoonup} \mathsf{even}_{\{(3.3)\}}(\mathsf{s}(0)) \qquad \text{(matching failure)}$$
$$\rightharpoonup \mathsf{false} \qquad \text{(successful step)}$$

The first step fails with $j = 0$ because

$$\mathsf{label}(\mathsf{odd}(0)) = \mathsf{odd}_{\{(3.4)),(3.5),(3.6)\}}(0) \overset{\perp}{\rightharpoonup} \mathsf{odd}_{\{(3.4)),(3.5)\}}(0) \rightharpoonup \mathsf{false}$$

and false is a linear labeled normal form which does not unify with true. The third step succeeds because $\mathsf{label}(\mathsf{even}(0)) = \mathsf{even}_{\{(3.1)),(3.2),(3.3)\}}(0) \rightharpoonup \mathsf{true}$.

There is one possibility remaining which is not covered by Definition 4.7; in a non-quasi-decreasing setting, a condition may give rise to an infinite reduction, neither failing nor succeeding. To handle this case we introduce a third definition.

Definition 4.10. We write $s \overset{\triangleright}{\rightharpoonup} t$ if there exist a position $p \in \mathcal{P}\mathsf{os}(s)$, rewrite rule $\rho \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$, substitution σ and $1 \leqslant j < k$ such that

(1) $s|_p = f_R(s_1, \ldots, s_n)$ with $\rho \in R$ and $s_i = \ell_i\sigma$ for all $1 \leqslant i \leqslant n$,
(2) $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$ for all $1 \leqslant i \leqslant j$, and
(3) $t = \mathsf{label}(a_{j+1})\sigma$

We write $s \overset{\infty}{\rightharpoonup}$ if there is an infinite sequence $s = s_0 \ (\rightharpoonup \cup \overset{\triangleright}{\rightharpoonup}) \ s_1 \ (\rightharpoonup \cup \overset{\triangleright}{\rightharpoonup}) \cdots$

Definition 4.10 completes labeled reduction; all ground labeled terms s are either labeled normal forms, or can be reduced using $\rightharpoonup$ or $\overset{\triangleright}{\rightharpoonup}$. This is verified in the following lemma.

Lemma 4.11. *For every ground labeled term s one of the following alternatives holds:*

(1) $s \overset{\infty}{\rightharpoonup}$,

(2) $s \rightharpoonup t$ *for some term t, or*

(3) s *is a labeled normal form.*

Proof. We non-deterministically construct a (finite or infinite) sequence $s = s_0, s_1, s_2, \ldots$ of ground terms as follows. Assuming s_i has been defined, if there is some u such that $s_i \rightharpoonup u$ then we take any such u as s_{i+1}. Otherwise, if there is some v with $s_i \overset{\triangleright}{\rightharpoonup} v$ then we take $s_{i+1} = v$. If there are multiple such v, we choose one with the largest possible number j (cf. (2) in Definition 4.10) of successful conditions with respect to a rule ρ satisfying (1) in Definition 4.10. If no s_{i+1} has been defined, we terminate the construction and let $N = i$.

If the constructed sequence is infinite then $s \overset{\infty}{\rightharpoonup}$ and thus statement (1) holds. So suppose the sequence is finite.

We claim that s_N is a labeled normal form. For a proof by contradiction, assume that s_N is not a labeled normal form, so it has a subterm whose root symbol has a non-empty label. Choosing a minimal such subterm, we find a position $p \in \mathcal{P}\mathrm{os}(s_N)$ such that $s_N|_p = f_R(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ are labeled normal forms and $R \neq \varnothing$, say $\rho\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c \in R$. Now, if $f(u_1, \ldots, u_n)$ does not instantiate $f(\ell_1, \ldots, \ell_n)$, then $s_N \overset{\perp}{\rightharpoonup} s_N[f_{R \setminus \{\rho\}}(u_1, \ldots, u_n)]_p$ because $u_1, \ldots, u_n$ are ground labeled normal forms, contradicting the fact that s_N is the last element of the sequence. It follows that a substitution σ exists such that $u_i = \ell_i \sigma$ for $1 \leqslant i \leqslant n$. If $k = 0$ then $s_N \rightharpoonup s_N[r\sigma]_p$, otherwise $s_N \overset{\triangleright}{\rightharpoonup} \mathsf{label}(a_1)\sigma$. In both cases we obtain a contradiction to the choice of N.

Next we prove $s_i \rightharpoonup s_{i+1}$ for $0 \leqslant i < N$. Aiming for a contradiction, consider the largest i such that $s_i \rightharpoonup s_{i+1}$ does not hold. Then we have $s_i \overset{\triangleright}{\rightharpoonup} s_{i+1}$, so there exist a position p, a rule $\rho\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c \in R$, an index $1 \leqslant j < k$, and a substitution σ such that $s_i|_p = f_R(u_1, \ldots, u_n)$ with $f(u_1, \ldots, u_n) = f(\ell_1, \ldots, \ell_n)\sigma$, $\mathsf{label}(a_l)\sigma \rightharpoonup^* b_l\sigma$ for all $1 \leqslant l \leqslant j$, and $s_{i+1} = \mathsf{label}(a_{j+1})\sigma$. We have $s_{i+1} \rightharpoonup^* s_N$ by the choice of i. If s_N is not an instance of b_{j+1} then, since s_N is a ground labeled normal form, the condition has failed and we have $s_i \rightharpoonup s_i[f_{R \setminus \{\rho\}}(u_1, \ldots, u_n)]_p$, contradicting the choice for s_{i+1}. So s_N does instantiate b_{j+1}. But then, if $j + 1 < k$, we should have chosen $s_{i+1} = \mathsf{label}(a_{j+2})\sigma$ according to the construction of s_{i+1}, and if $j + 1 = k$ then $s_{i+1} = s_i[r\sigma]_p$ should have been chosen instead.

Now, if $N = 0$ then $s = s_N$ is a labeled normal form and thus statement (3) holds. If $N > 0$ statement (2) holds as $s = s_0 \rightharpoonup s_1$. $\quad\square$

Note that the three alternatives in Lemma 4.11 are not exclusive: It is possible to have $s \rightharpoonup t$ as well as $s \overset{\infty}{\rightharpoonup}$ for a term s.

4.2. Labeled versus Unlabeled Reduction.

The relation $\rightharpoonup$ provides an alternative approach to evaluation which keeps track of failed rule application attempts, whereas $\overset{\infty}{\rightharpoonup}$ is the counterpart of non-quasi-decreasingness. As may be expected, there is a strong connection between the relations $\to$ and $\rightharpoonup$. This connection is made formal in Theorem 4.13 and the subsequent lemmata.

Definition 4.12. Let $\mathcal{R}$ be semi-finite and $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$. We write $\|t\|$ for the total number of rules occurring in all labels in t.

Semi-finiteness ensures that $\|t\|$ is a well-defined natural number.

Theorem 4.13. *Let $\mathcal{R}$ be a CCTRS.*

    (1) Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$.
        (a) If $s \to t$ then $\mathsf{label}(s) \rightharpoonup \mathsf{label}(t)$.
        (b) If $s \to^* t$ then $\mathsf{label}(s) \rightharpoonup^* \mathsf{label}(t)$.
    (2) Let $s, t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$.
        (a) If $s \rightharpoonup t$ then either $\mathsf{erase}(s) \to \mathsf{erase}(t)$ or both $\mathsf{erase}(s) = \mathsf{erase}(t)$ and, if $\mathcal{R}$ is semi-finite, $\|s\| > \|t\|$.
        (b) If $s \rightharpoonup^* t$ then $\mathsf{erase}(s) \to^* \mathsf{erase}(t)$.

Proof. We use induction on the total number of *rewrite steps* of $\to$ and $\rightharpoonup$, respectively. This is the number of steps used both directly in the reduction, and those needed to verify the conditions $a_i\sigma \to^* b_i\sigma$ or $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$.

    (1) We derive cases (1a) and (1b) by simultaneous induction on the total number of rewrite steps needed to derive $s \to t$ and $s \to^* t$.
        (a) There exist a position $p \in \mathcal{P}\mathsf{os}(s)$, a rule $\rho\colon \ell \to r \Leftarrow c$, and a substitution σ such that $s|_p = \ell\sigma$, $t = s[r\sigma]_p$, and $\mathcal{R} \vdash c\sigma$. Let σ' be the (labeled) substitution $\mathsf{label} \circ \sigma$. Fix $1 \leqslant i \leqslant k$. We have $\mathsf{label}(a_i\sigma) = \mathsf{label}(a_i)\sigma'$ and $\mathsf{label}(b_i\sigma) = b_i\sigma'$ (as b_i is a constructor term). Because $a_i\sigma \to^* b_i\sigma$ is used in the derivation of $s \to t$ we can apply the induction hypothesis for part (b), resulting in $\mathsf{label}(a_i\sigma) \rightharpoonup^* \mathsf{label}(b_i\sigma)$. Furthermore, writing $\ell = f(\ell_1, \ldots, \ell_n)$, we obtain $\mathsf{label}(\ell) = f_{\mathcal{R}\restriction f}(\ell_1, \ldots, \ell_n)$. Hence $\mathsf{label}(s) = \mathsf{label}(s)[\mathsf{label}(\ell)\sigma']_p \rightharpoonup \mathsf{label}(s)[\mathsf{label}(r)\sigma']_p = \mathsf{label}(t)$ because conditions (1)–(3) in Definition 4.7 are satisfied.
        (b) If $s = t$ then the result is obvious. If $s \to u \to^* t$ then $\mathsf{label}(s) \rightharpoonup \mathsf{label}(u)$ follows by case (1a), and the induction hypothesis yields $\mathsf{label}(u) \rightharpoonup \mathsf{label}(t)$.
    (2) We prove both statements by simultaneous induction on the total number of steps required to derive $s \rightharpoonup t$ and $s \rightharpoonup^* t$. For part (a) we distinguish two cases.
        • Suppose $s \xrightarrow{\perp} t$ or $s \rightharpoonup t$ by a failed step. In either case we have $\mathsf{erase}(s) = \mathsf{erase}(t)$. Moreover, if all labels have finite size, also $\|s\| = \|t\| + 1$.
        • Suppose $s \rightharpoonup t$ by a successful step. So there exist a position $p \in \mathcal{P}\mathsf{os}(s)$, a rule $\rho\colon \ell \to r \Leftarrow c$ in $\mathcal{R}$, a substitution σ, and terms $\ell', a_1', \ldots, a_k'$ such that $s|_p = \ell'\sigma$ with $\mathsf{erase}(\ell') = \ell$, $a_i'\sigma \rightharpoonup^* b_i\sigma$ with $\mathsf{erase}(a_i') = a_i$ for all $1 \leqslant i \leqslant k$, and $t = s[\mathsf{label}(r)\sigma]_p$. Let σ' be the (unlabeled) substitution $\mathsf{erase} \circ \sigma$. We have $\mathsf{erase}(s) = \mathsf{erase}(s)[\ell\sigma']_p$ and $\mathsf{erase}(u) = \mathsf{erase}(s)[r\sigma']_p$. Since the sequence $a_i'\sigma \rightharpoonup^* b_i\sigma$ is used as a strict subpart of the derivation of $s \rightharpoonup t$, we obtain $a_i\sigma' = \mathsf{erase}(a_i'\sigma) \to^* \mathsf{erase}(b_i\sigma) = b_i\sigma'$ from the induction hypothesis, for all $1 \leqslant i \leqslant k$. Hence $\mathcal{R} \vdash c\sigma'$, so indeed $\mathsf{erase}(s) \to \mathsf{erase}(t)$.
    Again, part (b) easily follows from part (a). □

Example 4.14. In Examples 3.2 and 3.3 we encountered the reduction

$$t_{n,m} = \mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a}))) \to^* \mathsf{g}(\mathsf{a}) = t_{0,0}$$

By Theorem 4.13, we immediately obtain a labeled reduction

$$\mathsf{label}(t_{n,m}) = \mathsf{f}^n_{\{(3.7)\}}(\mathsf{g}_{\{(3.8)\}}(\mathsf{f}^m_{\{(3.7)\}}(\mathsf{a}))) \rightharpoonup^* \mathsf{g}_{\{(3.8)\}}(\mathsf{a}) = \mathsf{label}(t_{0,0})$$

Note that we can reduce this term further to $\mathsf{g}_\varnothing(\mathsf{a})$ and obtain a labeled normal form.

Lemma 4.15. *Let $\mathcal{R}$ be a CCTRS.*

(1) *If $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $s \xrightarrow{\rhd} t$ then $\mathsf{label}(s) \xrightarrow{\rhd} \mathsf{label}(t)$.*

(2) *If $s, t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ and $s \xrightarrow{\rhd} t$ then $\mathsf{erase}(s) \xrightarrow{\rhd} \mathsf{erase}(t)$.*

Proof. (1) There exist a position $p \in \mathcal{P}\mathsf{os}(s)$, a rule $\rho \colon \ell \to r \Leftarrow c$, a substitution σ, and an index $1 \leqslant i \leqslant k$ such that $s|_p = \ell\sigma$, $t = a_i\sigma$ and $a_j\sigma \to^* b_j\sigma$ for all $1 \leqslant j < i$. Write $\sigma' = \mathsf{label} \circ \sigma$. By Theorem 4.13(1), $\mathsf{label}(a_j)\sigma' = \mathsf{label}(a_j\sigma) \rightharpoonup^* \mathsf{label}(b_j\sigma) = b_j\sigma'$ for all $1 \leqslant j < i$. Let $\ell = f(\ell_1, \dots, \ell_n)$ and $R = \mathcal{R}{\restriction}f$. Clearly, $\rho \in R$ and therefore $\mathsf{label}(s)|_p = f_R(\mathsf{label}(s_1), \dots, \mathsf{label}(s_n)) = f_R(\ell_1\sigma', \dots, \ell_n\sigma') \xrightarrow{\rhd} \mathsf{label}(a_i)\sigma' = \mathsf{label}(t)$ as required.

(2) There exist a position $p \in \mathcal{P}\mathsf{os}(s)$, a rule $\rho \colon f(\ell_1, \dots, \ell_n) \to r \Leftarrow c$, a substitution σ, and an index $1 \leqslant i \leqslant k$ such that $s|_p = f_R(s_1, \dots, s_n)$ with $\rho \in R$ and $s_j = \ell_j\sigma$ for all $1 \leqslant j \leqslant n$, $\mathsf{label}(a_j)\sigma \to^* b_j\sigma$ for all $1 \leqslant j < i$, and $t = \mathsf{label}(a_i)\sigma$. Write $\sigma' = \mathsf{erase} \circ \sigma$. By Theorem 4.13(2), $\mathsf{erase}(\mathsf{label}(a_j)\sigma) = a_j\sigma' \to^* b_j\sigma' = \mathsf{erase}(b_j\sigma)$ for $1 \leqslant j < i$. We obtain $\mathsf{erase}(s) = \mathsf{erase}(s)[f(\ell_1, \dots, \ell_n)\sigma']_p \xrightarrow{\rhd} a_i\sigma' = \mathsf{erase}(t)$. $\square$

Lemma 4.16. *A term $s \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ in a semi-finite CCTRS $\mathcal{R}$ is non-quasi-decreasing if and only if $\mathsf{label}(s) \xrightarrow{\infty}$.*

Proof. If s is not quasi-decreasing then there exists an infinite sequence $s = u_0 \ (\to \cup \xrightarrow{\rhd}) \ u_1 \ (\to \cup \xrightarrow{\rhd}) \ \cdots$. We obtain $\mathsf{label}(u_0) \ (\rightharpoonup \cup \xrightarrow{\rhd}) \ \mathsf{label}(u_1) \ (\rightharpoonup \cup \xrightarrow{\rhd}) \ \cdots$ from Theorem 4.13(1) and Lemma 4.15(1). Thus $\mathsf{label}(s) = \mathsf{label}(u_0) \xrightarrow{\infty}$. Conversely, if $\mathsf{label}(s) \xrightarrow{\infty}$ then there is an infinite sequence $\mathsf{label}(s) = u_0 \ (\rightharpoonup \cup \xrightarrow{\rhd}) \ u_1 \ (\rightharpoonup \cup \xrightarrow{\rhd}) \ \cdots$. From Theorem 4.13(2) and Lemma 4.15(2) we obtain $\mathsf{erase}(u_i) \to^= \mathsf{erase}(u_{i+1})$ or $\mathsf{erase}(u_i) \xrightarrow{\rhd} \mathsf{erase}(u_{i+1})$ for every $i \geqslant 0$. Since $\mathsf{erase}(u_i) = \mathsf{erase}(u_{i+1})$ implies $\|u_i\| > \|u_{i+1}\|$, this gives an infinite sequence of $\to$ and $\xrightarrow{\rhd}$ steps starting from $\mathsf{erase}(u_0) = s$. $\square$

Example 4.17. Consider the CCTRS consisting of the single rule $\rho \colon \mathsf{a} \to \mathsf{b} \Leftarrow \mathsf{a} \approx \mathsf{b}$. We have $\mathsf{label}(\mathsf{a}) = \mathsf{a}_{\{\rho\}} \xrightarrow{\rhd} \mathsf{label}(\mathsf{a})$. Hence $\mathsf{label}(\mathsf{a}) \xrightarrow{\infty}$ and thus a is non-quasi-decreasing.

We have now transposed conditional rewriting to an essentially equivalent relation on labeled terms, which enables us to keep track of failed computations.

4.3. Derivation Height and Complexity.

Now we show how labeled—or rather, complexity-conscious—reduction gives rise to conditional complexity. With failures now explicitly included in the reduction relation, the only hurdle to defining derivation height is the question of how exactly to handle the evaluation of conditions. To this end, we assign an evaluation cost to individual steps.

Definition 4.18. The *cost* $\mathsf{cost}(s \rightharpoonup^* t)$ of a complexity-conscious reduction $s \rightharpoonup^* t$ is the sum of the costs of its steps. The cost of a step $s \rightharpoonup t$ is 0 if $s \xrightarrow{\perp} t$,

$$1 + \sum_{i=1}^{k} \mathsf{cost}(\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma)$$

in case of a successful step $s \rightharpoonup t$, and

$$\sum_{i=1}^{j} \mathsf{cost}(\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma) + \mathsf{cost}(\mathsf{label}(a_{j+1})\sigma \rightharpoonup^* u\tau)$$

in case of a failed step $s \rightharpoonup t$.

Intuitively, the cost of a reduction measures the number of successful rewrite steps, both direct and in condition evaluations, but does not count the mere removal of a rule from a label. This is why the cost of a failed step is the cost to evaluate its conditions and conclude failure, while for successful steps we add one for the step itself.

Example 4.19. The cost of the reduction in Example 4.8 is $0 + 1 + 4 = 5$, where the $4 = 1 + 3$ includes the three steps in the conditions. The cost of the reduction in Example 4.9 is $1 + 0 + 2 = 3$. Note that in both cases, the cost is simply obtained by counting the number of successful rewrite steps, including those occurring in a condition evaluation.

Definition 4.20. The *derivation height* $\mathsf{dh}(s)$ of a labeled term s in a semi-finite CCTRS is defined as

$$\max\left(\{\mathsf{cost}(s \rightharpoonup^* t) \mid t \in \mathcal{T}(\mathcal{G}, \mathcal{V})\} \cup \{\infty \mid s \xrightarrow{\infty}\}\right)$$

where $\infty > n$ for all $n \in \mathbb{N}$.

That is, a labeled term s has infinite derivation height if $s \xrightarrow{\infty}$, and the maximum cost of any reduction starting in s otherwise. Since $\mathcal{R}$ is semi-finite, the set of possible values $\mathsf{cost}(s \rightharpoonup^* t)$ can only be unbounded if $s \xrightarrow{\infty}$, in which case $\mathsf{dh}(s) = \max(\langle\text{some infinite set}\rangle \cup \{\infty\}) = \infty$. In other cases, the set of costs is necessarily finite, and hence the derivation height is well-defined, and in $\mathbb{N}$. Note that for $t \in \mathcal{T}(\mathcal{F})$, the derivation height of $\mathsf{label}(t)$ is infinite if and only if t is quasi-decreasing, by Lemma 4.16.

We have limited interest to semi-finite CCTRSs primarily to follow the standard in complexity for unconditional term rewriting, where TRSs are assumed to be finite. It is certainly possible to extend the definition towards non-semi-finite CCTRSs, simply by taking the *infimum* instead of the *maximum* of the set in Definition 4.20, in which case we might obtain an infinite derivation height even for the labeled version of a quasi-decreasing term. This would happen both if there are reductions of arbitrarily high cost starting in $\mathsf{label}(s)$, or if we obtain an infinite reduction of rule-removal steps, e.g. $\mathsf{label}(s) \xrightarrow{\perp} s_1 \xrightarrow{\perp} s_2 \xrightarrow{\perp} \cdots$. One might argue that this is justified, as finding an appropriate rule to apply may take arbitrarily long. However, in the unconditional setting, it seems unnatural to assign an infinite derivation height to, for instance, a normal form. Given that non-semi-finite TRSs are of very little practical interest, we prefer to leave this discussion to another work.

Definition 4.21. The *conditional derivational complexity* of a semi-finite CCTRS $\mathcal{R}$ is defined as $\mathsf{cdc}_{\mathcal{R}}(n) = \max\{\mathsf{dh}(\mathsf{label}(t)) \mid |t| \leqslant n\}$. If we restrict t to basic terms we arrive at the *conditional runtime complexity* $\mathsf{crc}_{\mathcal{R}}(n)$.

Arguably, the case where the CCTRS $\mathcal{R}$ is not quasi-decreasing is not very interesting for complexity (unless perhaps all terms of interest, e.g. all basic terms, are quasi-decreasing). The main reason why we consider systems without this restriction is to show that the transformation methods we use preserve the fundamental properties of a CCTRS. Thus, we can for instance guarantee that the TRS obtained in the next section is terminating if and only if the original CCTRS is quasi-decreasing. This allows us to obtain completeness results, and to use complexity methods to prove quasi-decreasingness as well.

Continuing the discussion in Section 3, we claim that for a ground term $s \in \mathcal{T}(\mathcal{F})$, the derivation height $\mathsf{dh}(\mathsf{label}(s))$ gives a realistic and (in the absence of a reduction strategy) narrow bound on the time needed to normalize s. That is, we can always find a normal form of s in $\mathcal{O}(\mathsf{dh}(\mathsf{label}(s)))$ steps (by rewriting $\mathsf{label}(s)$ using $\rightharpoonup$). A worst-case derivation following the intuition laid out at the start of this section requires $\Omega(\mathsf{dh}(\mathsf{label}(s)))$ steps.

5. Complexity Transformation

The notion of complexity introduced in the preceding section has the downside that we cannot easily reuse existing complexity results and tools. Therefore, we will consider a transformation to unconditional rewriting where, rather than tracking rules in the labels of the defined function symbols, we will keep track of them in separate arguments, but restrict reduction by adopting a suitable *context-sensitive replacement map*. This transformation is based directly on the CCTRS $(\mathcal{F}, \mathcal{R})$, but in Section 5.2 we will see how it relates to the labeled system and the labeled rewrite relation $\rightharpoonup$. In particular, we will see that the unconditional rewrite relation defined in Section 5.1 both preserves and reflects complexity. To this end, however, we will have to limit interest to *strong* CCTRSs, as defined in Section 2, since we rely on (e.g.) left-linearity to be able to test when rules do *not* apply.

Our transformation builds on the ideas of the structure-preserving transformations in [1, 8], but differs in particular by its use of context-sensitivity, by forcing that the conditions for different rules are evaluated separately, and by using additional symbols f_i^j to mark when the evaluation of a condition is in progress—a change which significantly simplifies for instance the method of polynomial interpretations we shall employ in Section 6. Structure-preserving transformations are discussed in Section 9.2.

Context-sensitive rewriting restricts the positions in a term where rewriting is allowed. A (C)TRS is combined with a *replacement map* μ, which assigns to every n-ary symbol $f \in \mathcal{F}$ a subset $\mu(f) \subseteq \{1, \ldots, n\}$. A position p is *active* in a term t if either $p = \epsilon$, or $p = i\,q$, $t = f(t_1, \ldots, t_n)$, $i \in \mu(f)$, and q is active in t_i. The set of active positions in a term t is denoted by $\mathcal{P}os_\mu(t)$, and t may only be reduced at active positions.

5.1. The Unconditional TRS $\Xi(\mathcal{R})$.

Definition 5.1. Let $\mathcal{R}$ be a strong CCTRS over a signature $\mathcal{F}$. For $f \in \mathcal{F}$, let m_f be the number of rules in $\mathcal{R}{\restriction}f$ (so $m_f = 0$ for constructor symbols f) and fix an order $\mathcal{R}{\restriction}f = \{\rho_1^f, \ldots, \rho_{m_f}^f\}$. The context-sensitive signature $(\mathcal{H}, \mu)$ is defined as follows:
- $\mathcal{H}$ contains two constants $\bot$ and $\top$,
- for every symbol $f \in \mathcal{F}$ of arity n, $\mathcal{H}$ contains a symbol f of arity $n + m_f$ with $\mu(f) = \{1, \ldots, n\}$,
- for every defined symbol $f \in \mathcal{F}_\mathcal{D}$ of arity n, rule $\rho_i^f \colon \ell \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ in $\mathcal{R}{\restriction}f$, and $1 \leqslant j \leqslant k$, $\mathcal{H}$ contains a symbol f_i^j of arity $n + m_f + j - 1$ with $\mu(f_i^j) = \{n + i + j - 1\}$.

Terms in $\mathcal{T}(\mathcal{H}, \mathcal{V})$ which are involved in reducing $f(s_1, \ldots, s_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ will have one of two forms: $f(s_1, \ldots, s_n, t_1, \ldots, t_{m_f})$ with each $t_i \in \{\top, \bot\}$, indicating that rule ρ_i^f has been attempted (and failed) if and only if $t_i = \bot$, and

$$f_i^j(s_1, \ldots, s_n, t_1, \ldots, t_{i-1}, b_1\sigma, \ldots, b_{j-1}\sigma, u, t_{i+1}, \ldots, t_{m_f})$$

indicating that rule ρ_i^f is currently being evaluated and the first $j - 1$ conditions of ρ_i^f have succeeded; u records the current progress on the condition $a_j \approx b_j$.

In the following we drop the superscript f from ρ_i^f if no confusion arises.

Definition 5.2. The maps $\xi_\star\colon \mathcal{T}(\mathcal{F}, \mathcal{V}) \to \mathcal{T}(\mathcal{H}, \mathcal{V})$ with $\star \in \{\bot, \top\}$ are inductively defined as follows:

$$\xi_\star(t) = \begin{cases} t & \text{if } t \text{ is a variable,} \\ f(\xi_\star(t_1), \ldots, \xi_\star(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } f \text{ is a constructor symbol,} \\ f(\xi_\star(t_1), \ldots, \xi_\star(t_n), \star, \ldots, \star) & \text{if } t = f(t_1, \ldots, t_n) \text{ and } f \text{ is a defined symbol.} \end{cases}$$

Linear terms in the set $\{\xi_\bot(t) \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{V})\}$ are called $\bot$-*patterns*.

In the transformed system that we will define, a ground term is in normal form if and only if it is a $\bot$-pattern. This allows for syntactic "normal form" tests. Most importantly, it allows for purely syntactic anti-matching tests: If s does not reduce to an instance of some linear constructor term t, then $s \to^* u\sigma$ for some substitution σ and $\bot$-pattern u that does not unify with t. What is more, we only need to consider a finite number of $\bot$-patterns u.

Definition 5.3. Let t be a linear constructor term. The set of *anti-patterns* $\mathsf{AP}(t)$ is inductively defined as follows. If t is a variable then $\mathsf{AP}(t) = \varnothing$. If $t = f(t_1, \ldots, t_n)$ then $\mathsf{AP}(t)$ consists of the following $\bot$-patterns:

- $g(x_1, \ldots, x_m)$ for every m-ary constructor symbol g different from f,
- $g(x_1, \ldots, x_m, \bot, \ldots, \bot)$ for every defined symbol g of arity m in $\mathcal{F}$, and
- $f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n)$ for all $1 \leqslant i \leqslant n$ and $u \in \mathsf{AP}(t_i)$.

Here the x_j are fresh and pairwise distinct variables.

Example 5.4. Consider the CCTRS of Example 2.1. The set $\mathsf{AP}(\langle z, w \rangle)$ consists of the $\bot$-patterns 0, $\mathsf{s}(x)$, $\mathsf{fib}(x, \bot, \bot)$, and $+(x, y, \bot, \bot)$.

Lemma 5.5. *Let s be a $\bot$-pattern and t a linear constructor term with $\mathcal{V}\mathrm{ar}(s) \cap \mathcal{V}\mathrm{ar}(t) = \varnothing$. If s and t are not unifiable then s is an instance of an anti-pattern in $\mathsf{AP}(t)$.*

Proof. We use induction on the size of t. If s and t are not unifiable, neither can be a variable. So let $t = f(t_1, \ldots, t_n)$. If $s = g(s_1, \ldots, s_n)$ or $s = g(s_1, \ldots, s_n, \bot, \ldots, \bot)$ for some $g \neq f$ then s instantiates $g(x_1, \ldots, x_n)$ or $g(x_1, \ldots, x_n, \bot, \ldots, \bot)$ in $\mathsf{AP}(t)$. Otherwise, $s = f(s_1, \ldots, s_n)$. If s_i and t_i are not unifiable for some i, then by the induction hypothesis s_i is an instance of some $u \in \mathsf{AP}(t_i)$, so s instantiates $f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n) \in \mathsf{AP}(t)$. If no such i exists, there are substitutions $\sigma_1, \ldots, \sigma_n$ such that $s_i\sigma_i = t_i\sigma_i$ for all $1 \leqslant i \leqslant n$. Since s and t are linear terms without common variables, this implies that s and t are unifiable by the substitution $\sigma = \sigma_1 \cup \cdots \cup \sigma_n$, contradicting the assumption. $\square$

We are now ready to define the transformation from a CCTRS $(\mathcal{F}, \mathcal{R})$ to a context-sensitive TRS $(\mathcal{H}, \mu, \Xi(\mathcal{R}))$. Here, we will use the notation $\langle t_1, \ldots, t_n \rangle [u_1, \ldots, u_j]_i$ to denote the sequence $t_1, \ldots, t_{i-1}, u_1, \ldots, u_j, t_{i+1}, \ldots, t_n$ and we occasionally write $\vec{t}$ for a sequence $t_1, \ldots, t_n$.

Definition 5.6. Let $\mathcal{R}$ be a strong CCTRS over a signature $\mathcal{F}$. The TRS $\Xi(\mathcal{R})$ is defined over the context-sensitive signature $(\mathcal{H}, \mu)$ from Definition 5.1 as follows. Let $\rho_i\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ be the i-th rule in $\mathcal{R}{\upharpoonright}f$ (where $1 \leqslant i \leqslant m_f$).

- If $k = 0$ then $\Xi(\mathcal{R})$ contains the rule

$$f(\ell_1, \ldots, \ell_n, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i) \to \xi_\top(r) \tag{1_ρ}$$

- If $k > 0$ then $\Xi(\mathcal{R})$ contains the rules

$$f(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i) \to f_i^1(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\xi_\top(a_1)]_i) \tag{2_ρ}$$

$$f_i^k(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_k]_i) \to \xi_\top(r), \tag{3_ρ}$$

the rules

$$f_i^j(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_j]_i) \to$$
$$f_i^{j+1}(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_j, \xi_\top(a_{j+1})]_i) \tag{4_ρ}$$

for all $1 \leqslant j < k$, and the rules

$$f_i^j(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_{j-1}, v]_i) \to f(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\bot]_i) \tag{5_ρ}$$

for all $1 \leqslant j \leqslant k$ and $v \in \mathsf{AP}(b_j)$ (where $\mathcal{V}\mathrm{ar}(v) \cap \mathcal{V}\mathrm{ar}(f(\vec{\ell}, \vec{b}, \vec{x})) = \varnothing$).

- Regardless of k, $\Xi(\mathcal{R})$ contains the rules

$$f(\langle y_1, \ldots, y_n \rangle [v]_j, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i) \to f(\langle y_1, \ldots, y_n \rangle [v]_j, \langle x_1, \ldots, x_{m_f} \rangle [\bot]_i) \tag{6_ρ}$$

for all $1 \leqslant j \leqslant n$ and $v \in \mathsf{AP}(\ell_j)$ (where $\mathcal{V}\mathrm{ar}(v) \cap \mathcal{V}\mathrm{ar}(f(\vec{y}, \vec{x})) = \varnothing$).

Here $x_1, \ldots, x_{m_f}, y_1, \ldots, y_n$ are fresh and pairwise distinct variables. A step using rule (1_ρ) or rule (3_ρ) has cost 1; other rules—also called *administrative rules*—have cost 0.

Rule (1_ρ) simply adds the $\top$ labels to the right-hand sides of unconditional rules. To apply a conditional rule ρ_i, we mark the current function symbol as "in progress for ρ_i" with rule (2_ρ) and start evaluating the first condition of ρ_i by steps inside the argument for this condition. With rules (4_ρ) we move to the next condition and, after all conditions have succeeded, an application of rule (3_ρ) results in the right-hand side with $\top$ labels. If a condition fails (5_ρ) or the left-hand side of the rule does not match and will never match (6_ρ), then we replace the label for ρ_i by $\bot$, indicating that we do not need to try it again.

Note that the rules that do not produce the right-hand side of the originating conditional rewrite rule are considered *administrative* and hence do not contribute to the cost of a reduction. The anti-pattern sets result in many rules (5_ρ) and (6_ρ), but all of these are simple. We could generalize the system by replacing each $v \in \mathsf{AP}(\ell_j)$ by a fresh variable; the complexity of the resulting (smaller) TRS gives an upper bound for the original complexity. Indeed, all methods proposed in Sections 6–8, also apply to the transformation using variables instead. The primary purpose of anti-patterns is to ensure *completeness* (Theorem 5.12); by using anti-patterns instead of variables, we guarantee that a rule is only marked as unsuccessful (by replacing its parameter by $\bot$) if it truly cannot succeed anymore.

Note also that the resulting system $\Xi(\mathcal{R})$ is left-linear, which is advantageous for the potential applicability of various termination and complexity techniques.

Example 5.7. The (context-sensitive) TRS $\Xi(\mathcal{R}_{\mathsf{even}})$ consists of the rules below, with the numbers in square brackets indicating the cost of the rule: 0 for administrative rules and 1 for the others.

$$[1] \qquad \mathsf{even}(0, \top, y, z) \to \mathsf{true} \tag{1_1}$$

$$[0] \qquad \mathsf{even}(\star_1, \top, y, z) \to \mathsf{even}(\star_1, \bot, y, z) \tag{6_1}$$

$$[0] \qquad \mathsf{even}(\mathsf{s}(x), y, \top, z) \to \mathsf{even}_2^1(\mathsf{s}(x), y, \mathsf{odd}(x, \top, \top, \top), z) \tag{2_2}$$

$$[1] \qquad \mathsf{even}_2^1(\mathsf{s}(x), y, \mathsf{true}, z) \to \mathsf{true} \tag{3_2}$$

$$[0] \qquad \mathsf{even}_2^1(\mathsf{s}(x), y, \star_2, z) \to \mathsf{even}(\mathsf{s}(x), y, \bot, z) \tag{5_2}$$

$$[0] \qquad \mathsf{even}(\star_3, y, \top, z) \to \mathsf{even}(\star_3, y, \bot, z) \tag{6_2}$$

$$[0] \qquad \mathsf{even}(\mathsf{s}(x), y, z, \top) \to \mathsf{even}_3^1(\mathsf{s}(x), y, z, \mathsf{even}(x, \top, \top, \top)) \tag{2_3}$$

$$[1] \qquad \mathsf{even}_3^1(\mathsf{s}(x), y, z, \mathsf{true}) \to \mathsf{false} \tag{3_3}$$

$$[0] \qquad \mathsf{even}_3^1(\mathsf{s}(x), y, z, \star_2) \to \mathsf{even}(\mathsf{s}(x), y, z, \bot) \tag{5_3}$$

$$[0] \qquad \mathsf{even}(\star_3, y, z, \top) \to \mathsf{even}(\star_3, y, z, \bot) \tag{6_3}$$

$$[1] \qquad \mathsf{odd}(0, \top, y, z) \to \mathsf{false} \tag{1_4}$$

$$[0] \qquad \mathsf{odd}(\star_1, \top, y, z) \to \mathsf{odd}(\star_1, \bot, y, z) \tag{6_4}$$

$$[0] \qquad \mathsf{odd}(\mathsf{s}(x), y, \top, z) \to \mathsf{odd}_2^1(\mathsf{s}(x), y, \mathsf{odd}(x, \top, \top, \top), z) \tag{2_5}$$

$$[1] \qquad \mathsf{odd}_2^1(\mathsf{s}(x), y, \mathsf{true}, z) \to \mathsf{false} \tag{3_5}$$

$$[0] \qquad \mathsf{odd}_2^1(\mathsf{s}(x), y, \star_2, z) \to \mathsf{odd}(\mathsf{s}(x), y, \bot, z) \tag{5_5}$$

$$[0] \qquad \mathsf{odd}(\star_3, y, \top, z) \to \mathsf{odd}(\star_3, y, \bot, z) \tag{6_5}$$

$$[0] \qquad \mathsf{odd}(\mathsf{s}(x), y, z, \top) \to \mathsf{odd}_3^1(\mathsf{s}(x), y, z, \mathsf{even}(x, \top, \top, \top)) \tag{2_6}$$

$$[1] \qquad \mathsf{odd}_3^1(\mathsf{s}(x), y, z, \mathsf{true}) \to \mathsf{true} \tag{3_6}$$

$$[0] \qquad \mathsf{odd}_3^1(\mathsf{s}(x), y, z, \star_2) \to \mathsf{odd}(\mathsf{s}(x), y, z, \bot) \tag{5_6}$$

$$[0] \qquad \mathsf{odd}(\star_3, y, z, \top) \to \mathsf{odd}(\star_3, y, z, \bot) \tag{6_6}$$

for all

$$\star_1 \in \mathsf{AP}(0) = \{\mathsf{true}, \mathsf{false}, \mathsf{s}(x), \mathsf{even}(x, \bot, \bot, \bot), \mathsf{odd}(x, \bot, \bot, \bot)\}$$

$$\star_2 \in \mathsf{AP}(\mathsf{true}) = \{\mathsf{false}, 0, \mathsf{s}(x), \mathsf{even}(x, \bot, \bot, \bot), \mathsf{odd}(x, \bot, \bot, \bot)\}$$

$$\star_3 \in \mathsf{AP}(\mathsf{s}(x)) = \{\mathsf{true}, \mathsf{false}, 0, \mathsf{even}(x, \bot, \bot, \bot), \mathsf{odd}(x, \bot, \bot, \bot)\}$$

Following Definition 5.1, this TRS is equipped with the following replacement map μ:

$$\mu(\mathsf{even}) = \mu(\mathsf{odd}) = \{1\} \qquad \mu(\mathsf{even}_2^1) = \mu(\mathsf{odd}_2^1) = \{3\} \qquad \mu(\mathsf{s}) = \{1\}$$

$$\mu(\mathsf{even}_3^1) = \mu(\mathsf{odd}_3^1) = \{4\} \qquad \mu(0) = \mu(\mathsf{false}) = \mu(\mathsf{true}) = \varnothing$$

Instead of the current rules, which pass along the various ℓ_i and b_j unmodified throughout condition evaluation, we could have opted for a more fine-grained approach where we pass on their *variables*, and then only those which are needed later on, similar to what is done in the *optimized* unraveling [30]. Doing so, the example above would for instance have rules

$$\mathsf{even}(\mathsf{s}(x), y, \top, z) \to \mathsf{even}_2^1(x, y, \mathsf{odd}(x, \top, \top, \top), z) \tag{2_2}$$

and

$$\mathsf{even}_2^1(x, y, \star_2, z) \to \mathsf{even}(\mathsf{s}(x), y, \bot, z) \tag{5_2}$$

However, this would complicate the presentation for no easily discernible gain.

In an early version of this work [17], we employed a slightly different transformation in which the symbols f_ρ^i were *constructor symbols*, used in subterms corresponding to the rule whose conditions they evaluated. For instance, the above rules were rendered as

$$\mathsf{even}(\mathsf{s}(x), y, \top, z) \to \mathsf{even}_{\mathsf{active}}(x, y, \mathsf{even}_2^1(\mathsf{s}(x), \mathsf{odd}(x, \top, \top, \top)), z) \tag{2_2}$$

and

$$\mathsf{even}_{\mathsf{active}}(\mathsf{s}(x), y, \mathsf{even}_2^1(u, \star_2), z) \to \mathsf{even}(\mathsf{s}(x), y, \bot, z) \tag{5_2}$$

We simplifed this to our current definition because it is easier to work with when looking for interpretations to establish termination as in Section 6.

Definition 5.8. We define the derivation height of a terminating term s in the context-sensitive TRS $(\mathcal{H}, \Xi(\mathcal{R}))$ as the greatest number of non-administrative steps in any reduction starting in s, taking the replacement map into account:

$$\mathsf{dh}(s) = \max\left(\{\mathsf{cost}(s \to_{\mathcal{R},\mu}^* t) \mid t \in \mathcal{T}(\mathcal{H}, \mathcal{V})\}\right)$$

Letting $\mathsf{dh}(s) = \infty$ if s is non-terminating, the derivation and runtime complexities are defined accordingly:

$$\mathsf{dc}_{\Xi(\mathcal{R})}(n) = \max\{\mathsf{dh}(s) \mid s \in \mathcal{T}(\mathcal{H}) \text{ and } |s| \leqslant n\}$$

$$\mathsf{rc}_{\Xi(\mathcal{R})}(n) = \max\{\mathsf{dh}(s) \mid s \in \mathcal{T}(\mathcal{H}), |s| \leqslant n, \text{ and } s \text{ is basic}\}$$

5.2. Labeled reduction versus $\Xi(\mathcal{R})$.

In order to use the translated TRS $\Xi(\mathcal{R})$, we must understand how the conditional complexity of the original CCTRS relates to the unconditional complexity of $\Xi(\mathcal{R})$. To this end, we will define a translation ζ from *labeled* terms to terms over $\mathcal{H}$, which has the following properties:

(1) if s has (conditional) derivation height N then $\zeta(s)$ has (unconditional) derivation height at least N (Theorem 5.11),
(2) if $\zeta(s)$ has (unconditional) derivation height N then s has (conditional) derivation height at least N (Theorem 5.12).

Thus, we will be able to use the transformed system $\Xi(\mathcal{R})$ to obtain both upper and lower bounds for conditional complexity.

Even though $\to_{\Xi(\mathcal{R}),\mu}$ and $\rightharpoonup$ were designed to be intuitively equivalent, the proofs are rather technical. Before presenting the proof of the first result, we define the mapping ζ from terms in $\mathcal{T}(\mathcal{G}, \mathcal{V})$ to terms in $\mathcal{T}(\mathcal{H}, \mathcal{V})$. It resembles the earlier definition of $\xi_\star$, but also handles the labels.

Definition 5.9. For $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ we define

$$\zeta(t) = \begin{cases} t & \text{if } t \in \mathcal{V}, \\ f(\zeta(t_1), \ldots, \zeta(t_n)) & \text{if } t = f(t_1, \ldots, t_n) \text{ with } f \text{ a constructor symbol}, \\ f(\zeta(t_1), \ldots, \zeta(t_n), c_1, \ldots, c_{m_f}) & \text{if } t = f_R(t_1, \ldots, t_n) \text{ with } R \subseteq \mathcal{R}{\restriction}f \end{cases}$$

where $c_i = \top$ if ρ_i belongs to R and $c_i = \bot$ otherwise, for $1 \leqslant i \leqslant m_f$. For a substitution $\sigma \in \Sigma(\mathcal{G}, \mathcal{V})$ we denote the substitution $\zeta \circ \sigma$ by σ_ζ.

It is easy to see that $p \in \mathcal{P}\mathsf{os}_\mu(\zeta(t))$ if and only if $p \in \mathcal{P}\mathsf{os}(t)$, if and only if $p \in \mathcal{P}\mathsf{os}(\zeta(t))$ and $\zeta(t)|_p \notin \{\bot, \top\}$, for any $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$.

Lemma 5.10. *If $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ then $\zeta(\mathsf{label}(t)) = \xi_\top(t)$. If $t \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ and $\sigma \in \Sigma(\mathcal{G}, \mathcal{V})$ then $\zeta(t\sigma) = \zeta(t)\sigma_\zeta$. Moreover, if t is a linear labeled normal form then $\zeta(t) = \xi_\bot(\mathsf{erase}(t))$ is a $\bot$-pattern, and if $\zeta(t)$ is a $\bot$-pattern then t is a linear labeled normal form.*

Proof. All four properties are easily proved by induction on the size of t. $\qquad\square$

We are now ready for the first main result, which states that Ξ reflects complexity.

Theorem 5.11. *Let $\mathcal{R}$ be a strong CCTRS.*

    (1) *If $s \rightharpoonup^* t$ is a complexity-conscious reduction with cost N then there exists a context-sensitive reduction $\zeta(s) \rightarrow^*_{\Xi(\mathcal{R}),\mu} \zeta(t)$ with cost N.*

    (2) *If $s \xrightarrow{\infty}$ then there is an infinite $(\Xi(\mathcal{R}), \mu)$ reduction starting from $\zeta(s)$.*

Proof. We prove the first statement by induction on the number of steps in $s \rightharpoonup^* t$. The result is obvious when this number is zero, so suppose $s \rightharpoonup u \rightharpoonup^* t$ and let M be the cost of the step $s \rightharpoonup u$ and $N - M$ the cost of $u \rightharpoonup^* t$. The induction hypothesis yields a context-sensitive reduction $\zeta(u) \rightarrow^*_{\Xi(\mathcal{R}),\mu} \zeta(t)$ of cost $N - M$ and so it remains to show that there exists a context-sensitive reduction $\zeta(s) \rightarrow^*_{\Xi(\mathcal{R}),\mu} \zeta(u)$ of cost M. Let $\rho\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$ be the rule in $\mathcal{R}$ that gives rise to the step $s \rightharpoonup u$ and let i be its index in $\mathcal{R}{\restriction}f$. There exist a position $p \in \mathcal{P}\mathsf{os}(s)$, terms $s_1, \ldots, s_n$, and a subset $R \subseteq \mathcal{R}{\restriction}f$ such that $s|_p = f_R(s_1, \ldots, s_n)$ and $\rho \in R$. We have $\zeta(s)|_p = \zeta(s|_p) = f_R(\zeta(s_1), \ldots, \zeta(s_n), c_1, \ldots, c_{m_f})$ where $c_j = \top$ if the j-th rule of $\mathcal{R}{\restriction}f$ belongs to R and $c_j = \bot$ otherwise, for $1 \leqslant j \leqslant m_f$. In particular, $c_i = \top$. Note that p is an active position in $\zeta(s)$. We distinguish three cases.

- First suppose that $s \xrightarrow{\bot} u$. So $M = 0$, $u = s[f_{R \setminus \{\rho\}}(s_1, \ldots, s_n)]_p$, and—by linearity of $f(\ell_1, \ldots, \ell_n)$—there exist a linear labeled normal form v, a substitution σ, and an index $1 \leqslant j \leqslant n$ such that $s_j = v\sigma$ and $\mathsf{erase}(v)$ does not unify with ℓ_j. By Lemma 5.10, $\zeta(s_j) = \zeta(v\sigma) = \zeta(v)\sigma_\zeta = \xi_\bot(\mathsf{erase}(v))\sigma_\zeta$. By definition, $\xi_\bot(\mathsf{erase}(v))$ is a $\bot$-pattern, which cannot unify with ℓ_j because $\mathsf{erase}(v)$ does not. From Lemma 5.5 we obtain an anti-pattern $v' \in \mathsf{AP}(\ell_j)$ such that $\xi_\bot(\mathsf{erase}(v))$ is an instance of v'. Hence $\zeta(s) = \zeta(s)[f(\zeta(s_1), \ldots, \zeta(s_n), c_1, \ldots, c_{m_f})]_p$ with $\zeta(s_j)$ an instance of $v' \in \mathsf{AP}(\ell_j)$ and $c_i = \top$. Consequently, $\zeta(s)$ reduces to $\zeta(s)[f(\zeta(s_1), \ldots, \zeta(s_n), \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)]_p$ by an application of rule (6_ρ), which has cost zero. The latter term equals

$$\zeta(s[f_{R \setminus \{\rho\}}(s_1, \ldots, s_n)]_p) = \zeta(u)$$

and hence we are done.

- Next suppose that $s \rightharpoonup u$ is a successful step. So there exists a substitution σ such that $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$ with cost M_i for all $1 \leqslant i \leqslant k$, and $M = 1 + M_1 + \cdots + M_k$. The induction hypothesis yields reductions $\zeta(\mathsf{label}(a_i)\sigma) \rightarrow^*_{\Xi(\mathcal{R}),\mu} \zeta(b_i\sigma)$ with cost M_i. By Lemma 5.10, $\zeta(\mathsf{label}(a_i)\sigma) = \zeta(\mathsf{label}(a_i))\sigma_\zeta = \xi_\top(a_i)\sigma_\zeta$ and $\zeta(b_i\sigma) = b_i\sigma_\zeta$. Moreover, $\zeta(s)|_p = \zeta(s|_p) = f(\vec{\ell}, \langle c_1, \ldots, c_{m_f}\rangle[\top]_i)\sigma_\zeta$ and $\zeta(u) = \zeta(s)[\zeta(\mathsf{label}(r)\sigma)]_p$ with $\zeta(\mathsf{label}(r))\sigma_\zeta = \xi_\top(r)\sigma_\zeta$ by Lemma 5.10. So it suffices if $f(\vec{\ell}, \langle c_1, \ldots, c_{m_f}\rangle[\top]_i)\sigma_\zeta \rightarrow^*_{\Xi(\mathcal{R}),\mu} \xi_\top(r)\sigma_\zeta$ with cost M. If $k = 0$, we can use rule (1_ρ). Otherwise, we use the reductions $\xi_\top(a_i)\sigma_\zeta \rightarrow^*_{\Xi(\mathcal{R}),\mu} b_i\sigma_\zeta$, rules ($2_\rho$) and ($3_\rho$),

and $k - 1$ times a rule of type (4_ρ) to obtain

$$f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\top]_i) \sigma_\zeta \to_{\Xi(\mathcal{R}),\mu} f_i^1(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\xi_\top(a_1)]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu}^* f_i^1(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu} f_i^2(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \xi_\top(a_2)]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu}^* \cdots$$
$$\to_{\Xi(\mathcal{R}),\mu} f_i^k(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_k]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu} \xi_\top(r) \sigma_\zeta$$

Note that all steps take place at active positions, and that the steps with rules 2_ρ and 4_ρ are administrative. Therefore, the cost of this reduction equals M.

- The remaining case is a failed step $s \rightharpoonup u$. So there exist substitutions σ and τ, an index $1 \leqslant j < k$, and a linear labeled normal form v which does not unify with b_{j+1} such that $\mathsf{label}(a_i)\sigma \rightharpoonup^* b_i\sigma$ with cost M_i for all $1 \leqslant i \leqslant j$ and $\mathsf{label}(a_{j+1})\sigma \rightharpoonup^* v\tau$ with cost M_{j+1}. We obtain $\zeta(\mathsf{label}(a_i)\sigma) = \xi_\top(a_i)\sigma_\zeta$, $\zeta(b_i\sigma) = b_i\sigma_\zeta$, and $\zeta(s)|_p = f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\top]_i) \sigma_\zeta$ like in the preceding case. Moreover, like in the first case, we obtain an anti-pattern $v' \in \mathsf{AP}(b_{j+1})$ such that $\xi_\perp(\mathsf{erase}(v))$ is an instance of v'. We have $\zeta(v\tau) = \zeta(v)\tau_\zeta = \xi_\perp(\mathsf{erase}(v))\tau_\zeta$ by Lemma 5.10. Hence $\zeta(v\tau)$ is an instance of v'. Consequently,

$$f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\top]_i) \sigma_\zeta \to_{\Xi(\mathcal{R}),\mu}^* f_i^{j+1}(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_j, \xi_\top(a_{j+1})]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu}^* f_i^{j+1}(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_j, \zeta(v\tau)]_i) \sigma_\zeta$$
$$\to_{\Xi(\mathcal{R}),\mu} f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\perp]_i) \sigma_\zeta$$

where the last step uses an administrative rule of type (5_ρ). Again, all steps take place at active positions. Note that $f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\perp]_i) \sigma_\zeta = \zeta(f_{R \setminus \{\rho\}}(s_1, \ldots, s_n)) = \zeta(u|_p)$. Hence $\zeta(s) \to_{\Xi(\mathcal{R}),\mu}^* \zeta(u)$ as desired. The cost of this reduction is $M_1 + \cdots + M_{j+1}$, which coincides with the cost M of the step $s \rightharpoonup u$.

This concludes the proof of the first statement. As for the second statement, suppose $s \xrightarrow{\infty}$, so there exists an infinite sequence $(s)_{i \geqslant 0}$ of terms such that $s = s_0$ and $s_i \rightharpoonup s_{i+1}$ or $s_i \xrightarrow{\rhd} s_{i+1}$ for all $i \geqslant 0$. Fix $i \geqslant 0$. If $s_i \rightharpoonup s_{i+1}$ then $\zeta(s_i) \to_{\Xi(\mathcal{R}),\mu}^+ \zeta(s_{i+1})$ follows from the first statement. Suppose $s_i \xrightarrow{\rhd} s_{i+1}$. We show that $\zeta(s_i) \to_{\Xi(\mathcal{R}),\mu}^+ C[\zeta(s_{i+1})]$ for some context C whose hole is at an active position. There exist an active position $p \in \mathcal{P}\mathsf{os}(s_i)$, a rule $\rho \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$ in R, a substitution σ, and an index j such that $s_i|_p = f_R(\ell_1\sigma, \ldots, \ell_n\sigma)$, $\mathsf{label}(a_1)\sigma \rightharpoonup^* b_1\sigma$, ..., $\mathsf{label}(a_j)\sigma \rightharpoonup^* b_j\sigma$, and $s_{i+1} = \mathsf{label}(a_{j+1})\sigma$, so $\zeta(s_{i+1}) = \xi_\top(a_{j+1})\sigma_\zeta$. Let l be the index of ρ in $\mathcal{R}{\restriction}f$. We obtain $\zeta(s_i)|_p = \zeta(s_i|p) = f(\ell_1\sigma_\zeta, \ldots, \ell_n\sigma_\zeta, c_1, \ldots, c_{m_f})$ where $c_l = \top$, and $\xi_\top(a_d)\sigma_\zeta = \zeta(\mathsf{label}(a_d)\sigma) \to_{\Xi(\mathcal{R}),\mu}^* \zeta(b_d\sigma) = b_d\sigma_\zeta$ for $1 \leqslant d \leqslant j$, by the first statement. Hence

$$s_i = s_i[f(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [\top]_j) \sigma_\zeta]_p \to_{\Xi(\mathcal{R}),\mu}^* s_i[f_l^{j+1}(\vec{\ell}, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_j, \xi_\top(a_{j+1})]_l) \sigma_\zeta]_p$$

and thus we can take the context

$$C = s_i[f_l^{j+1}(\ell_1\sigma, \ldots, \ell_n\sigma, c_1, \ldots, c_{l-1}, b_1, \ldots, b_j, \square, c_{l+1}, \ldots, c_{m_f})]_p$$

The hole is at an active position, since p is active in s_i and $n + l + j$ in $\mu(f_l^{j+1})$. $\qquad \square$

Theorem 5.11 provides a way to establish conditional complexity: If $\Xi(\mathcal{R})$ has complexity $\mathcal{O}(\varphi(n))$ then the conditional complexity of $\mathcal{R}$ is at $\mathcal{O}(\varphi(n))$. This is the important direction as it allows us to obtain an upper bound for complexity by transforming the conditional system into an unconditional one. However, we have more. The following result shows that complexity bounds thus obtained can be *sharp*.

Theorem 5.12. *Let $\mathcal{R}$ be a strong CCTRS and $s \in \mathcal{T}(\mathcal{G})$.*

  (1) *If $\zeta(s)$ is terminating and there exists a context-sensitive reduction $\zeta(s) \to^*_{\Xi(\mathcal{R}),\mu} t$ for some t with cost N, then there exists a complexity-conscious reduction $s \rightharpoonup^* t'$ for some t' with cost at least N.*
  (2) *If there exists an infinite $(\Xi(\mathcal{R}), \mu)$ reduction starting from $\zeta(s)$ then $s \xrightarrow{\infty}$.*

Proof Idea. First of all, we may safely assume that t is in normal form; if it is not, we simply extend the reduction (which can only increase the cost). Due to the context-sensitivity restrictions and the form of the rules $\Xi(\mathcal{R})$, any such normal form t must be a $\perp$-pattern.

Next we transform the reduction $\zeta(s) \to^*_{\Xi(\mathcal{R}),\mu} t$ (resp. $\zeta(s) \to_{\Xi(\mathcal{R})} \dots$) to a reduction with at least the same cost (resp. an infinite reduction) which is well-behaved in the sense that for any rule application $u[\ell\sigma]_p \to u[r\sigma]_p$, the substitution σ can be written as $\zeta \circ \tau$. This is done by a reordering argument, either postponing steps in subterms (if the result of the step is used later), or eagerly evaluating the corresponding subterm to normal form.

Having a well-behaved reduction, steps using rules (1_ρ) can be translated directly to unconditional $\rightharpoonup$ steps, and (6_ρ) translates to $\xrightarrow{\perp}$. Combined steps (2_ρ) followed by some (4_ρ) applications and ending with (3_ρ) or (5_ρ) correspond to successful or failed applications; the restrictions of context-sensitivity guarantee that any reduction steps in between these rule applications are either at independent positions—in which case they can be postponed—or inside the argument for the condition in progress. Since t is assumed to be in normal form, all such combinations are either completed—in which case they can be transformed—or give rise to an infinite reduction inside the accessible argument of a f_i^j symbol—in which case we can reduce with a $\xrightarrow{\triangleright}$ step to a non-terminating term $\zeta(a_j)$. Either way we are done. $\square$

We refer to Appendix A for the full and rather intricate proof. Note that Theorems 5.11 and 5.12 together with Lemma 5.10 tell us that for terms s in $\mathcal{T}(\mathcal{F}, \mathcal{V})$, the "conditional complexity cost" of $\mathsf{label}(s)$ is the same as the derivation height of $\xi_\top(s)$. Consequently, complexity notions between the original CTRS and the resulting context-sensitive TRS are interchangeable, but only so long as we limit interest to starting terms where the additional m_f arguments of every defined symbol f are set to $\top$:

$$\mathsf{cdc}(n) = \max \{\mathsf{dh}(\xi_\top(t)) \mid |t| \leqslant n\}$$
$$\mathsf{crc}(n) = \max \{\mathsf{dh}(\xi_\top(t)) \mid |t| \leqslant n \text{ and } t \text{ is basic}\}$$

What is more, we have gained an additional result: The transformation does not merely relate *complexity* notions, but conservatively translates *quasi-decreasingness* to termination.

Corollary 5.13. *A strong CCTRS $\mathcal{R}$ is quasi-decreasing if and only if the corresponding context-sensitive TRS $\Xi(\mathcal{R})$ is terminating on all terms in the set $\{\zeta(s) \mid s \in \mathcal{T}(\mathcal{G}, \mathcal{V})\}$.*

Thus, we can use the same transformation to prove quasi-decreasingness of CCTRSs. Although there are no complexity tools yet which take context-sensitivity into account, we can obtain an upper bound by simply ignoring the replacement map. Similarly, although existing tools do not accommodate administrative rules we can count all rule applications

equally. Since for every non-administrative step reducing a term $f_R(\cdots)$ at a position p, at most (number of rules) $\times$ (greatest number of conditions $+\ 1$) administrative steps at position p can be done, the difference is only a constant factor. Moreover, these rules are an instance of *relative rewriting*, for which advanced complexity methods do exist. Thus, it is likely that there will be direct tool support in the future.

6. INTERPRETATIONS IN $\mathbb{N}$

A common method to derive complexity bounds for a TRS is the use of interpretations in $\mathbb{N}$. Such an interpretation $\mathcal{I}$ maps function symbols of arity n to functions from $\mathbb{N}^n$ to $\mathbb{N}$, giving a value $[t]_\mathcal{I}$ for every ground term t, which is shown to decrease in each reduction step. The method is easily adapted to support context-sensitive rewriting and administrative rules.

As we will consider interpretations on different domains later on, we define interpretations in a general way. Let $\mathcal{A}$ be a set (such as $\mathbb{N}$) and let $>$ be a well-founded order on this set, and $\geqslant$ a quasi-order compatible with $>$ (i.e., $\geqslant \cdot > \ \subseteq\ >$ and $> \cdot \geqslant\ \subseteq\ >$). A function f from $\mathcal{A}^n$ to $\mathcal{A}$ is *strictly monotone* in its i-th argument if $f(s_1, \ldots, s_i, \ldots, s_n) > f(s_1, \ldots, s_i', \ldots, s_n)$ whenever $s_i > s_i'$ and *weakly monotone* in its i-th argument if $f(s_1, \ldots, s_i, \ldots, s_n) \geqslant f(s_1, \ldots, s_i', \ldots, s_n)$ whenever $s_i \geqslant s_i'$.

Definition 6.1. A context-sensitive interpretation over $\mathcal{A}$ is a function $\mathcal{I}$ mapping each symbol $f \in \mathcal{F}$ of arity n to a function $\mathcal{I}_f$ from $\mathcal{A}^n$ to $\mathcal{A}$, such that $\mathcal{I}_f$ is strictly monotone in its i-th argument for all $i \in \mu(f)$. Given a valuation α mapping each variable to an element of $\mathcal{A}$, the value $[t]_\mathcal{I}^\alpha \in \mathcal{A}$ of a term t is defined as usual:

- $[x]_\mathcal{I}^\alpha = \alpha(x)$ for $x \in \mathcal{V}$,
- $[f(s_1, \ldots, s_n)]_\mathcal{I}^\alpha = \mathcal{I}_f([s_1]_\mathcal{I}^\alpha, \ldots, [s_n]_\mathcal{I}^\alpha)$ for $f \in \mathcal{F}$.

We say $\mathcal{I}$ is compatible with a set of unconditional rules $\mathcal{R}$ if for all rules $\ell \to r \in \mathcal{R}$ and valuations α, $[\ell]_\mathcal{I}^\alpha > [r]_\mathcal{I}^\alpha$ if $\ell \to r \in \mathcal{R}$ is non-administrative and $[\ell]_\mathcal{I}^\alpha \geqslant [r]_\mathcal{I}^\alpha$ otherwise.

We easily see that if $s \to_{\mathcal{R},\mu} t$ then $[s]_\mathcal{I}^\alpha \geqslant [t]_\mathcal{I}^\alpha$, and $[s]_\mathcal{I}^\alpha > [t]_\mathcal{I}^\alpha$ if the employed rule is non-administrative. Consequently, if $\mathcal{A} = \mathbb{N}$, then $\mathsf{dh}(s, \to_{\mathcal{R},\mu}) \leqslant [s]_\mathcal{I}^\alpha$ for any valuation α. Having a derivation height for all terms, we can obtain the derivational and runtime complexity of the original system. To take advantage of the fact that we only need to consider terms $\xi_\top(s)$, we can limit interest to "$\top$-terms": ground terms which have the property that $t_1 = \cdots = t_{m_f} = \top$ and $s_1, \ldots, s_n \notin \{\bot, \top\}$ for all subterms $f(s_1, \ldots, s_n, t_1, \ldots, t_{m_f})$. For runtime complexity, we only have to consider basic $\top$-terms. We let $|s|$ denote the number of function symbols in s *not counting* $\top$. Then $|s| = |\xi_\top(s)|$.

Example 6.2. Continuing Example 5.7, we define the following interpretation over $\mathbb{N}$:

$$\mathcal{I}_\top = 1 \qquad \mathcal{I}_\bot = \mathcal{I}_{\mathsf{true}} = \mathcal{I}_{\mathsf{false}} = \mathcal{I}_0 = 0 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1$$

$$\mathcal{I}_{\mathsf{even}}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}}(x, u, v, w) = 1 + x + v \cdot 3^x + w \cdot 3^x$$

$$\mathcal{I}_{\mathsf{even}_2^1}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}_2^1}(x, u, v, w) = 1 + x + v + w \cdot 3^x$$

$$\mathcal{I}_{\mathsf{even}_3^1}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}_3^1}(x, u, v, w) = 1 + x + v \cdot 3^x + w$$

One easily checks that $\mathcal{I}$ satisfies the required monotonicity constraints: $\mathcal{I}_{\mathsf{s}}$ is monotone in its only argument, $\mathcal{I}_{\mathsf{even}}$ and $\mathcal{I}_{\mathsf{odd}}$ are monotone in x, while $\mathcal{I}_{\mathsf{even}_2^1}$, $\mathcal{I}_{\mathsf{odd}_2^1}$ are monotone in v and $\mathcal{I}_{\mathsf{even}_3^1}$, $\mathcal{I}_{\mathsf{odd}_3^1}$ in w. Moreover, all rules in $\Xi(\mathcal{R}_{\mathsf{even}})$ are oriented as required. For example, the rules generated by the unconditional rule (3.1) give the following obligations:

(1_ρ) $[\mathsf{even}(0, \top, y, z)]_\mathcal{I} = 1 + 0 + y \cdot 3^0 + z \cdot 3^0 > 0 = [\mathsf{true}]_\mathcal{I}$,

(6_ρ) $[\mathsf{even}(\star_1, \top, y, z)]_\mathcal{I} = 1 + \varphi + y \cdot 3^\varphi + z \cdot 3^\varphi \geqslant 1 + \varphi + y \cdot 3^\varphi + z \cdot 3^\varphi = [\mathsf{even}(\star_1, \bot, y, z)]_\mathcal{I}$
with $\varphi = [\star_1]_\mathcal{I}$.

The rules corresponding to the unconditional odd rule (3.4) give the same inequalities. As for the other four rules, their translations and interpretations are all very similar, so we will show only the interpretations and proof obligations for rule (3.2):

(2_ρ) $[\mathsf{even}(\mathsf{s}(x), y, \top, z)]_\mathcal{I} = 2 + x + 3^{x+1} + z \cdot 3^{x+1} \geqslant 2 + x + (1 + x + 2 \cdot 3^x) + z \cdot 3^{x+1} = [\mathsf{even}_2^1(\mathsf{s}(x), y, \mathsf{odd}(x, \top, \top, \top), z)]_\mathcal{I}$, which follows from $3^{x+1} = 3^x + 2 \cdot 3^x \geqslant (1 + x) + 2 \cdot 3^x$,

(3_ρ) $[\mathsf{even}_2^1(\mathsf{s}(x), y, \mathsf{true}, z)]_\mathcal{I} = 2 + x + 0 + z \cdot 3^{x+1} > 0 = [\mathsf{true}]_\mathcal{I}$,

(5_ρ) $[\mathsf{even}_2^1(s(x), y, \star_2, z)]_\mathcal{I} = 2 + x + \varphi + z \cdot 3^{x+1} \geqslant 2 + x + 0 + z \cdot 3^{x+1} = [\mathsf{even}_2^1(s(x), y, \bot, z)]_\mathcal{I}$
with $\varphi = [\star_2]_\mathcal{I} \geqslant 0$,

(6_ρ) $[\mathsf{even}(\star_3, y, \top, z)]_\mathcal{I} = 1 + \varphi + 3^\varphi + z \cdot 3^\varphi \geqslant 1 + \varphi + z \cdot 3^\varphi = [\mathsf{even}(\star_3, y, \bot, z)]_\mathcal{I}$.

Now, towards runtime complexity, we observe that for all ground constructor terms s with $|s| \leqslant n$ we also have $[s]_\mathcal{I} \leqslant n$ as $\mathcal{I}_f(x_1, \ldots, x_m) \leqslant x_1 + \cdots + x_m + 1$ for all constructor symbols f. Therefore, the conditional runtime complexity $\mathsf{crc}_{\mathcal{R}_{\mathsf{even}}}(n)$ is bounded by $\mathcal{O}(3^n)$:

$$\max(\{[f(s_1, \ldots, s_m, \top, \ldots, \top)]_\mathcal{I} \mid f \in \mathcal{F_D} \text{ and } s_1, \ldots, s_m \text{ are ground constructor}$$
$$\text{terms with } |s_1| + \cdots + |s_m| < n\})$$
$$\leqslant \max(\{\mathcal{I}_f(x_1, \ldots, x_m, 1, \ldots, 1) \mid f \in \mathcal{F_D} \text{ and } x_1 + x_2 + x_3 + x_4 < n\})$$
$$= \max(\{1 + x + 2 \cdot 3^x \mid x < n\}) = n + 2 \cdot 3^{n-1} \leqslant 3^n \quad \text{for } n \geqslant 1$$

As to derivational complexity, we observe that $[t]_\mathcal{I} \leqslant {}^n 3$ (tetration,[2] or $3 \uparrow\uparrow n$ in Knuth's up-arrow notation) when t is an arbitrary ground $\top$-term of size n.

To obtain a more elementary bound we will need more sophisticated methods, for instance assigning a compatible sort system and using the fact that all terms of sort int are necessarily constructor terms. A method based on separating size and space complexity is discussed in Section 8.

The interpretations in Example 6.2 may appear somewhat arbitrary, but in fact there is a recipe that we can most likely apply to many TRSs obtained from CCTRSs using Definitions 5.1 and 5.6. The idea is to define the interpretation $\mathcal{I}$ as an extension of a "basic" interpretation $\mathcal{J}$ over $\mathbb{N}$ with a fixed way of handling the additional arguments.

Definition 6.3 (Recipe A). Given

- a strictly monotone interpretation function $\mathcal{J}_f^0 \colon \mathbb{N}^n \to \mathbb{N}$ for every symbol f of arity n in the original signature $\mathcal{F}$,
- weakly monotone interpretation functions $\mathcal{J}_f^1, \ldots, \mathcal{J}_f^{m_f} \colon \mathbb{N}^n \to \mathbb{N}$ for every $f \in \mathcal{F_D}$,
- interpretation functions $\mathcal{J}_{f,i}^1, \ldots, \mathcal{J}_{f,i}^k$ with $\mathcal{J}_{f,i}^j \colon \mathbb{N}^{n+j} \to \mathbb{N}$ that are strictly monotone in their last argument position $(n + j)$, for each rule $\rho_i \in \mathcal{R} \restriction f$ with $k > 0$ conditions,

we construct an interpretation $\mathcal{I}$ for $\mathcal{H}$ as follows: $\mathcal{I}_\top = 1$ and $\mathcal{I}_\bot = 0$, $\mathcal{I}_f(x_1, \ldots, x_n) = \mathcal{J}_f^0(x_1, \ldots, x_n)$ for every $f \in \mathcal{F_C}$ of arity n,

$$\mathcal{I}_f(x_1, \ldots, x_n, c_1, \ldots, c_{m_f}) = \mathcal{J}_f^0(x_1, \ldots, x_n) + \sum_{k=1}^{m_f} c_k \cdot \mathcal{J}_f^k(x_1, \ldots, x_n)$$

---

[2]Tetration is the next hyperoperation after exponentiation, defined as iterated exponentiation.

for every $f \in \mathcal{F}_\mathcal{D}$ of arity n, and finally

$$\mathcal{I}_{f_i^j}(x_1, \ldots, x_n, c_1, \ldots, c_{i-1}, y_1, \ldots, y_j, c_{i+1}, \ldots, c_{m_f}) =$$

$$\mathcal{J}_f^0(x_1, \ldots, x_n) + \mathcal{J}_{f,i}^j(x_1, \ldots, x_n, y_1, \ldots, y_j) + \sum_{k=1,\, k\neq i}^{m_f} c_k \cdot \mathcal{J}_f^k(x_1, \ldots, x_n)$$

for every symbol f_i^j.

Using the interpretation of Recipe A for the rules in Definition 5.6, the inequalities we obtain can be greatly simplified, and in many cases removed.

Definition 6.4. The *compatibility constraints* for $\mathcal{J}$ comprise the following inequalities, for every rule $\rho_i\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ in the *original* system $\mathcal{R}$:

(1_ρ) $\mathcal{J}_f^0(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) + \mathcal{J}_f^i(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) > [\xi_\top(r)]_\mathcal{I}^\alpha$ if $k = 0$,

(2_ρ) $\mathcal{J}_f^i(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) \geqslant \mathcal{J}_{f,i}^1(\overrightarrow{[\ell]_\mathcal{I}^\alpha}, [\xi_\top(a_1)]_\mathcal{I}^\alpha)$ if $k > 0$,

(3_ρ) $\mathcal{J}_f^0(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) + \mathcal{J}_{f,i}^k(\overrightarrow{[\ell]_\mathcal{I}^\alpha}, [b_1]_\mathcal{I}^\alpha, \ldots, [b_k]_\mathcal{I}^\alpha) > [\xi_\top(r)]_\mathcal{I}^\alpha$,

(4_ρ) $\mathcal{J}_{f,i}^j(\overrightarrow{[\ell]_\mathcal{I}^\alpha}, [b_1]_\mathcal{I}^\alpha, \ldots, [b_j]_\mathcal{I}^\alpha) \geqslant \mathcal{J}_{f,i}^{j+1}(\overrightarrow{[\ell]_\mathcal{I}^\alpha}, [b_1]_\mathcal{I}^\alpha, \ldots, [b_j]_\mathcal{I}^\alpha, [\xi_\top(a_{j+1})]_\mathcal{I}^\alpha)$ for $1 \leqslant j < k$.

Here $\overrightarrow{[\ell]_\mathcal{I}^\alpha}$ denotes the sequence $[\ell_1]_\mathcal{I}^\alpha, \ldots, [\ell_n]_\mathcal{I}^\alpha$.

Lemma 6.5. *The interpretation $\mathcal{I}$ from Recipe A is a context-sensitive interpretation for $(\mathcal{H}, \mu)$. If its interpretation functions satisfy the compatibility constraints then $\mathcal{I}$ is compatible with $\mathcal{H}$, so*

$$\mathsf{cdc}_\mathcal{R}(n) = \max\{[\xi_\top(t)]_\mathcal{I} \mid t \in \mathcal{T}(\mathcal{F}) \text{ and } |t| \leqslant n\}$$
$$\mathsf{crc}_\mathcal{R}(n) = \max\{[\xi_\top(t)]_\mathcal{I} \mid t \in \mathcal{T}(\mathcal{F}), \ |t| \leqslant n, \text{ and } t \text{ is basic}\}$$

Moreover,

$$[\xi_\top(f(t_1, \ldots, t_n))]_\mathcal{I}^\alpha = \sum_{i=0}^{m_f} \mathcal{J}_f^i([\xi_\top(t_1)]_\mathcal{I}^\alpha, \ldots, [\xi_\top(t_n)]_\mathcal{I}^\alpha)$$

Proof. It is not hard to see that $\mathcal{I}$ satisfies the monotonicity requirements of Definition 6.1. Hence it is a context-sensitive interpretation for $(\mathcal{H}, \mu)$. The statements on $\mathsf{cdc}_\mathcal{R}$ and $\mathsf{crc}_\mathcal{R}$ follow by compatibility and the observations at the end of Section 5, because of the inequality $\mathsf{dh}(s, \to_{\Xi(\mathcal{R}),\mu}) \leqslant [s]_\mathcal{I}$. The final equality claim is obtained by writing out definitions. For the compatibility claim, note that rules obtained from clause (6_ρ) are obviously oriented as $[\bot]_\mathcal{I} = 0$. Compatibility is also satisfied for rules obtained from clause (5_ρ), as

$$\mathcal{J}_{f,i}^j(s_1, \ldots, s_n, t_1, \ldots, t_j) \geqslant [\bot]_\mathcal{I}^\alpha \cdot \mathcal{J}_f^i(s_1, \ldots, s_n) = 0$$

always holds. The requirements for the other rules follow from the compatibility constraints, by expanding the inequality ($[\ell]_\mathcal{I}^\alpha \geqslant [r]_\mathcal{I}^\alpha$ or $[\ell]_\mathcal{I}^\alpha > [r]_\mathcal{I}^\alpha$) and removing unhelpful terms on the left. For instance, rules obtained from (1_ρ) impose the inequality

$$\mathcal{J}_f^0(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) + \mathcal{J}_f^i(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) + \sum_{k=1,\, k\neq i}^{m_f} x_k \cdot \mathcal{J}_f^k(\overrightarrow{[\ell]_\mathcal{I}^\alpha}) > [\xi_\top(r)]_\mathcal{I}^\alpha$$

which follows from clause (1_ρ) in Definition 6.4; we omitted the summation because the x_i do not appear on the right, and could well be 0. $\qquad\qquad\square$

By the final part of Lemma 6.5, which recursively defines $[\xi_\top(f(t_1, \ldots, t_n))]^\alpha_\mathcal{I}$ purely in terms of $\mathcal{J}$, we can obtain bounds on derivation heights without ever calculating $\xi_\top(t)$. Thus, we do not even need to consider the labeled or translated systems.

Example 6.6. To demonstrate the use of the recipe, recall the CCTRS from Example 3.2:

$$\mathsf{f}(x) \to x \qquad\qquad\qquad \mathsf{g}(x) \to \mathsf{a} \Leftarrow x \approx \mathsf{b}$$

The recipe gives the following proof obligations:

$$\mathcal{J}_\mathsf{f}^0(x) + \mathcal{J}_\mathsf{f}^1(x) > x \qquad\qquad\qquad \mathcal{J}_\mathsf{g}^1(x) \geqslant \mathcal{J}_{\mathsf{g},1}^1(x, x)$$
$$\mathcal{J}_\mathsf{g}^0(x) + \mathcal{J}_{\mathsf{g},1}^1(x, \mathcal{J}_\mathsf{b}) > \mathcal{J}_\mathsf{a}$$

Here, $\mathcal{J}_\mathsf{f}^0$ and $\mathcal{J}_\mathsf{g}^0$ must be strictly monotone in their first argument, $\mathcal{J}_\mathsf{f}^1$ and $\mathcal{J}_\mathsf{g}^1$ weakly monotone, and $\mathcal{J}_{\mathsf{g},1}^1$ must be strictly monotone in its *second* argument. These monotonicity requirements are satisfied by choosing

$$\mathcal{J}_\mathsf{a} = 0 \qquad \mathcal{J}_\mathsf{f}^0(x) = x \qquad \mathcal{J}_\mathsf{g}^0(x) = x \qquad \mathcal{J}_{\mathsf{g},1}^1(x, y) = y$$
$$\mathcal{J}_\mathsf{b} = 1 \qquad \mathcal{J}_\mathsf{f}^1(x) = 1 \qquad \mathcal{J}_\mathsf{g}^1(x) = x$$

With these interpretations, the proof obligations are simplified to

$$x + 1 > x \qquad\qquad\qquad x \geqslant x$$
$$x + 1 > 0$$

and obviously satisfied.

In order to bound the derivational complexity in Example 6.6, we make the following general observation.

Lemma 6.7. *If for every symbol h of arity n in some strong CCTRS we have*

$$\mathcal{J}_h^0(x_1, \ldots, x_n) + \cdots + \mathcal{J}_h^{m_h}(x_1, \ldots, x_n) \leqslant K \cdot (x_1 + \cdots + x_n) + M$$

then $[\xi_\top(s)]_\mathcal{I} \leqslant M \cdot (K^0 + \cdots + K^{|s|-1})$ for all ground terms s.

Recall that $m_h = 0$ for constructor symbols, so the above requirement is well-defined.

Proof. We use induction on $|s|$. If $|s| = 1$ then s is a constant and

$$[\xi_\top(s)]_\mathcal{I} = \mathcal{J}_s^0 + \cdots + \mathcal{J}_s^{m_s} \leqslant K \cdot 0 + M = M = M \cdot K^0$$

If $|s| = m + 1$ then $s = h(t_1, \ldots, t_n)$ with $|t_1| + \cdots + |t_n| = m$ and

$$
\begin{aligned}
[\xi_\top(s)]_\mathcal{I} &= \sum_{i=0}^{m_f} \mathcal{J}_h^i([\xi_\top(t_1)]_\mathcal{I}, \ldots, [\xi_\top(t_n)]_\mathcal{I}) \\
&\leqslant K \cdot ([\xi_\top(t_1)]_\mathcal{I} + \cdots + [\xi_\top(t_n)]_\mathcal{I}) + M \\
&\leqslant K \cdot M \cdot (K^0 + \cdots + K^m) + M \\
&= M \cdot (K^1 + \cdots + K^{m+1}) + M \\
&= M \cdot (K^0 + \cdots + K^{m+1})
\end{aligned}
$$

$\square$

Since, for $K \geqslant 2$, we have $K^0 + \cdots + K^m \leqslant K^{m+1}$, a linear interpretation satisfying the premise of Lemma 6.7 gives $\mathsf{cdc}_{\mathcal{R}}(n) = \mathcal{O}(K^n)$ by Lemma 6.5. With this understanding, we can complete the example.

Example 6.6 (continued). We thus obtain an exponential $\mathcal{O}(2^n)$ bound. This may not seem like an impressive result, but in fact, this bound is tight! Consider a term $\mathsf{g}^n(\mathsf{b})$. To evaluate this term to normal form, we obtain a cost of 2^{n-1} if we simply evaluate outside-in:

- $\mathsf{g}_{\{(3.8)\}}(\mathsf{b}) \rightharpoonup \mathsf{a}$ with cost $1 = 2^{1-1}$,
- $\mathsf{g}_{\{(3.8)\}}(\mathsf{g}^n_{\{(3.8)\}}(\mathsf{b})) \rightharpoonup \mathsf{g}_{\varnothing}(\mathsf{g}^n_{\{(3.8)\}}(\mathsf{b}))$ with cost 2^{n-1} (the cost to reduce the left-hand side of the condition, $\mathsf{g}^n_{\{(3.8)\}}(\mathsf{b})$, to normal form), and $\mathsf{g}_{\varnothing}(\mathsf{g}^n_{\{(3.8)\}}(\mathsf{b}))$ reduces to normal form with cost 2^{n-1} (the cost to evaluate the subterm), amounting to a total cost of $2^{n-1} + 2^{n-1} = 2^n$.

However, we do have

$$
\begin{aligned}
\mathsf{dh}(\xi_{\top}(\mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a}))))) &\leqslant [\xi_{\top}(\mathsf{f}^n(\mathsf{g}(\mathsf{f}^m(\mathsf{a}))))]_{\mathcal{I}} \\
&= [\xi_{\top}(\mathsf{g}(\mathsf{f}^m(\mathsf{a})))]_{\mathcal{I}} + n \\
&= 2 \cdot [\xi_{\top}(\mathsf{f}^m(\mathsf{a}))]_{\mathcal{I}} + n \\
&= 2 \cdot m + n
\end{aligned}
$$

which gives the expected linear bound for the collection of terms considered in Example 3.2.

7. Using Context-Sensitivity to Improve Runtime Complexity Bounds

As observed before, the actual runtime complexity for the system in Example 6.2 is $\mathcal{O}(2^n)$. In order to obtain this more realistic bound, we will need more sophisticated methods than simply polynomial interpretations. This is not a problem specific to our transformed systems $\Xi(\mathcal{R})$; rather, giving tight complexity bounds is a hard problem, which has been studied extensively in the literature. Consequently, many different complexity methods have been developed (e.g. matrix interpretations [27, 38, 25], arctic interpretations [18], polynomial path orders [2, 3], match bounds [11], dependency tuples [29]) and it seems likely that most of these methods can easily be adapted to context-sensitive and relative rewriting.

In order to demonstrate that the systems we obtain using our transformation are not inherently problematic, we will show two improvements which allow us to obtain better bounds. The first one, which is treated in this section, employs a technique from [15].

Definition 7.1. A replacement map υ is *usable* for a strong CCTRS $(\mathcal{F}, \mathcal{R})$ if for every rewrite rule $b_0 \rightarrow a_{k+1} \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ in $\mathcal{R}$ and all $1 \leqslant i \leqslant k+1$ and $p \in \mathcal{P}\mathsf{os}(a_i)$ we have $p \in \mathcal{P}\mathsf{os}_{\upsilon}(a_i)$ if either $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}_{\mathcal{D}}}(a_i)$, or p is a variable position in a_i and there exist $0 \leqslant j < i$ and $q \in \mathcal{P}\mathsf{os}_{\upsilon}(b_j)$ such that $(a_i)|_p = (b_j)|_q$.

Note that the requirement on $p \in \mathcal{P}\mathsf{os}_{\upsilon}(a_i)$ is a sufficient condition only; it is allowed for $\mathcal{P}\mathsf{os}_{\upsilon}(a_i)$ to contain also p which satisfy neither premise. Therefore, the full replacement map, with $\nu(f) = \{1, \ldots, n\}$ for f of arity n, is always usable.

Example 7.2. We derive a usable replacement map υ for the CCTRS $\mathcal{R}_{\mathsf{fib}}$ of Example 2.1:

$$
\begin{aligned}
0 + y &\rightarrow y & \mathsf{fib}(0) &\rightarrow \langle 0, \mathsf{s}(0) \rangle \\
\mathsf{s}(x) + y &\rightarrow \mathsf{s}(x + y) & \mathsf{fib}(\mathsf{s}(x)) &\rightarrow \langle z, w \rangle \Leftarrow \mathsf{fib}(x) \approx \langle y, z \rangle, \; y + z \approx w
\end{aligned}
$$

From the rule $\mathsf{s}(x) + y \to \mathsf{s}(x + y)$ we obtain $1 \in \upsilon(\mathsf{s})$. The other constraints are obtained from the conditional rule for fib. The variable w appears at an active (root) position in the right-hand side of a condition and also at position 2 in $\langle z, w \rangle$. Hence we obtain $2 \in \upsilon(\langle \cdot, \cdot \rangle)$, which causes the variable z to appear at an active position in $\langle y, z \rangle$ and thus $2 \in \upsilon(+)$ and $1 \in \upsilon(\langle \cdot, \cdot \rangle)$. The latter activates the variable y in $\langle y, z \rangle$ and thus we also need $1 \in \upsilon(+)$. There are no other demands and hence the replacement map υ defined by $\upsilon(\mathsf{s}) = \{1\}$, $\upsilon(+) = \upsilon(\langle \cdot, \cdot \rangle) = \{1, 2\}$, and $\upsilon(\mathsf{fib}) = \varnothing$ is usable.

Definition 7.3. Let υ be a usable replacement map for a strong CCTRS $(\mathcal{F}, \mathcal{R})$. Let μ be the replacement map defined in Definition 5.1 for the signature $\mathcal{H}$. We define a new replacement map $\mu\upsilon$ for $\mathcal{H}$ as follows: $\mu\upsilon(f) = \upsilon(f)$ for every $f \in \mathcal{H} \cap \mathcal{F}$ and $\mu\upsilon(f) = \mu(f)$ for every $f \in \mathcal{H} \setminus \mathcal{F}$.

Theorem 7.4. *If υ is a usable replacement map for a strong CCTRS $(\mathcal{F}, \mathcal{R})$ then $\mathsf{crc}_{\mathcal{R}}(n) \leqslant \mathsf{rc}_{\Xi(\mathcal{R}), \mu\upsilon}(n)$ for all $n \geqslant 0$.*

Proof. We define an intermediate replacement map υ' as follows: $\upsilon'(f) = \upsilon(f)$ for every $f \in \mathcal{H} \cap \mathcal{F}$ and $\upsilon'(f_i^j) = \upsilon(f) \cup \{n+i, \ldots, n+i+j-1\}$ for every $f_i^j \in \mathcal{H} \setminus \mathcal{F}$ such that the arity of f in $\mathcal{F}$ is n. It is not difficult to prove that $\mu\upsilon(f) = \upsilon'(f) \cap \mu(f)$ for every $f \in \mathcal{H}$.

We prove that $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(t) \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(t)$ whenever $s \to^*_{\Xi(\mathcal{R}), \mu} t$ and s is basic, by induction on the length. Since $\mathcal{P}\mathsf{os}_{\upsilon'}(t) \cap \mathcal{P}\mathsf{os}_{\mu}(t) = \mathcal{P}\mathsf{os}_{\mu\upsilon}(t)$, this implies that any $(\Xi(\mathcal{R}), \mu)$ reduction sequence starting from a basic term is a reduction sequence in $(\Xi(\mathcal{R}), \mu\upsilon)$, and hence the statement of the theorem follows from Theorem 5.11. The base case is obvious since $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(t) = \{\epsilon\} \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(s)$ if t is basic. For the induction step we consider

$$s \to^*_{\Xi(\mathcal{R}), \mu} s' \to_{\Xi(\mathcal{R}), \mu} t$$

We obtain $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(s') \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(s')$ from the induction hypothesis. Suppose the step from s' to t employs the rule $u \to v$ from $\Xi(\mathcal{R})$ at position $p \in \mathcal{P}\mathsf{os}_{\mu}(s')$ with substitution σ. We have $p \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(s')$ and thus also $p \in \mathcal{P}\mathsf{os}_{\upsilon'}(s')$. Since $s'|_p = u\sigma$ we also have

$$\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(u\sigma) \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(u\sigma) \tag{7.1}$$

Furthermore, because $t = s'[v\sigma]_p$, $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(t) \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(t)$ follows from $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(v\sigma) \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(v\sigma)$. The latter inclusion we prove by a case analysis on $u \to v$. Let $q \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(v\sigma)$.

(1_{ρ}) We have $u = f(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i)$ and $v = \xi_{\top}(r)$ with $\ell = f(\vec{\ell}) \to r$ a rule of $\mathcal{R}$. If $q \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(\xi_{\top}(r))$ then $q \in \mathcal{P}\mathsf{os}_{\mathcal{F}_{\mathcal{D}}}(r)$ and thus $q \in \mathcal{P}\mathsf{os}_{\upsilon}(r) = \mathcal{P}\mathsf{os}_{\upsilon'}(\xi_{\top}(r)) \subseteq \mathcal{P}\mathsf{os}_{\upsilon'}(\xi_{\top}(r)\sigma)$ since υ is a usable replacement map. Otherwise $q = q_1 q_2$ with $q_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\xi_{\top}(r)) = \mathcal{P}\mathsf{os}_{\mathcal{V}}(r)$ and $q_2 \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(z\sigma)$ where $z = \xi_{\top}(r)|_{q_1}$. Since $z \in \mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$, there exists a position $q_3 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell) \subseteq \mathcal{P}\mathsf{os}_{\mathcal{V}}(u)$ such that $\ell|_{q_3} = u|_{q_3} = z$. We have $q_3 q_2 \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(u\sigma)$ and thus $q_3 q_2 \in \mathcal{P}\mathsf{os}_{\upsilon'}(u\sigma)$ by (7.1). Hence both $q_3 \in \mathcal{P}\mathsf{os}_{\upsilon'}(u)$ and $q_2 \in \mathcal{P}\mathsf{os}_{\upsilon'}(z\sigma)$. Since $q_3 \in \mathcal{P}\mathsf{os}_{\upsilon'}(u) = \mathcal{P}\mathsf{os}_{\upsilon}(\ell)$, we obtain $q_1 \in \mathcal{P}\mathsf{os}_{\upsilon}(r) = \mathcal{P}\mathsf{os}_{\upsilon'}(r) = \mathcal{P}\mathsf{os}_{\upsilon'}(v)$ from the usability of υ. Hence $q \in \mathcal{P}\mathsf{os}_{\upsilon'}(v\sigma)$ as desired.

(2_{ρ}) We have $u = f(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i)$ and $v = f_i^1(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\xi_{\top}(a_1)]_i)$. Comparing $u\sigma$ and $v\sigma$ and observing that $\upsilon'(f)$ and $\upsilon'(f_i^1)$ agree on $\{1, \ldots, n+m_f\} \setminus \{n+i\}$, the only interesting case is $q = (n+i)\,q'$ with $q' \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(\xi_{\top}(a_1)\sigma)$. We distinguish two subcases. If $q' \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(\xi_{\top}(a_1)) = \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(a_1)$ then $q' \in \mathcal{P}\mathsf{os}_{\upsilon}(a_1) = \mathcal{P}\mathsf{os}_{\upsilon'}(\xi_{\top}(a_1))$ and, since $n+i \in \upsilon'(f_i^1)$, $q \in \mathcal{P}\mathsf{os}_{\upsilon'}(v\sigma)$. Otherwise $q' = q_1 q_2$ with

$q_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\xi_\top(a_1)) = \mathcal{P}\mathsf{os}_{\mathcal{V}}(a_1)$ and $q_2 \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(z\sigma)$ where $z = (a_1)|_{q_1}$. Sinze z must occur in $\mathcal{V}\mathsf{ar}(\vec{\ell}) = \mathcal{V}\mathsf{ar}(b_0)$, we conclude as in case (1_ρ).

(3_ρ) We have $u = f_i^k(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_k]_i)$ and $v = \xi_\top(r)$. Like in case (1_ρ) we distinguish two cases. The case $q \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(\xi_\top(r))$ is dealt with as before. Suppose $q = q_1 q_2$ with $q_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\xi_\top(r)) = \mathcal{P}\mathsf{os}_{\mathcal{V}}(r)$ and let $z = r|_{q_1}$. The variable z occurs in $\ell_1, \ldots, \ell_n$ or $b_1, \ldots, b_k$. The former is treated as before. Suppose $z = (b_l)|_{q_3}$ with $1 \leqslant l \leqslant k$. We have $(n + i + l - 1)\, q_3 q_2 \in \mathcal{P}\mathsf{os}_{v'}(u\sigma)$ according to (7.1). Hence $q_3 \in \mathcal{P}\mathsf{os}_{v'}(b_l) = \mathcal{P}\mathsf{os}_{v}(b_l)$ and thus $q_1 \in \mathcal{P}\mathsf{os}_{v}(r)$ by the usability of v. We obtain $q \in \mathcal{P}\mathsf{os}_{v'}(v\sigma)$ as before.

(4_ρ) We have $u = f_i^j(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_j]_i)$ and $v = f_i^{j+1}(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_j, \xi_\top(a_{j+1})]_i)$. Comparing $u\sigma$ and $v\sigma$ as well as $v'(f_i^j)$ and $v'(f_i^{j+1})$ allows us to focus on the interesting case: $q = (n + i + j)\, q'$ with $q' \in \mathcal{P}\mathsf{os}_{\mathcal{H}_D}(\xi_\top(a_{j+1})\sigma)$. We obtain $q' \in \mathcal{P}\mathsf{os}_{v'}(\xi_\top(a_{j+1})\sigma)$ and thus $q \in \mathcal{P}\mathsf{os}_{v'}(v\sigma)$ by repeating the reasoning performed in the preceding cases.

(5_ρ) We have $u = f_i^j(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [b_1, \ldots, b_{j-1}, v']_i)$ and $v = f(\vec{\ell}, \langle x_1, \ldots, x_{m_f} \rangle [\perp]_i)$. We distinguish three cases. If $q = \epsilon$ then obviously $q \in \mathcal{P}\mathsf{os}_{v'}(v\sigma)$. Let $q = i'q'$. If $i' \in \{1, \ldots, n\}$ then $q \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(u\sigma)$ and thus $q \in \mathcal{P}\mathsf{os}_{v'}(u\sigma)$ by (7.1). Hence also $q \in \mathcal{P}\mathsf{os}_{v'}(v\sigma)$ since $v'(f) = v(f) = v'(f_i^j) \cap \{1, \ldots, n\}$. In the remaining case we have $i' \in \{n+1, \ldots, n+m_f\} \setminus \{n+i\}$ and thus $v\sigma|_q = \sigma(x_{i'-n})|_{q'}$. However, this subterm appears in $u\sigma$ at a position not in $\mathcal{P}\mathsf{os}_{v'}(u\sigma)$ and thus cannot contain defined symbols according to (7.1), contradicting the assumption $q \in \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(v\sigma)$.

(6_ρ) We have $u = f(\langle y_1, \ldots, y_n \rangle [v']_j, \langle x_1, \ldots, x_{m_f} \rangle [\top]_i)$ for some $v' \in \mathsf{AP}(\ell_j)$ and $v = f(\langle y_1, \ldots, y_n \rangle [v']_j, \langle x_1, \ldots, x_{m_f} \rangle [\perp]_i)$. In this case we obviously have $\mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(u\sigma) = \mathcal{P}\mathsf{os}_{\mathcal{H}_{\mathcal{D}}}(v\sigma)$ and $\mathcal{P}\mathsf{os}_{v'}(u\sigma) = \mathcal{P}\mathsf{os}_{v'}(v\sigma)$. Hence $q \in \mathcal{P}\mathsf{os}_{v'}(v\sigma)$ is a consequence of (7.1). $\qquad\square$

Using Theorem 5.12, the inequality in Theorem 7.4 becomes an equality if we restrict the terms to consider for $\mathsf{rc}_{\Xi(\mathcal{R}),\mu v}(n)$ to those that correspond to labeled basic terms.

Theorem 7.4 is highly relevant when using interpretations since the (strong or weak) monotonicity requirements are only imposed on the active arguments of the interpretation functions.

Example 7.5. For the CCTRS $\mathcal{R}_{\mathsf{even}}$ we can take the (empty) usable replacement map $v(f) = \varnothing$ for all function symbols because even and odd do not appear below the root in the right-hand side or left-hand side of a condition of any rule. This implies that $\mathcal{I}_{\mathsf{even}}$ and $\mathcal{I}_{\mathsf{odd}}$ do not need to be monotone in their first arguments. Hence we can simplify the interpretation of Example 6.2 to

$$\mathcal{I}_\top = 1 \qquad \mathcal{I}_\perp = \mathcal{I}_{\mathsf{true}} = \mathcal{I}_{\mathsf{false}} = \mathcal{I}_0 = 0 \qquad \mathcal{I}_{\mathsf{s}}(x) = x + 1$$

$$\mathcal{I}_{\mathsf{even}}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}}(x, u, v, w) = 1 + v \cdot (2^x - 1) + w \cdot (2^x - 1)$$

$$\mathcal{I}_{\mathsf{even}_2^1}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}_2^1}(x, u, v, w) = 1 + v + w \cdot (2^x - 1)$$

$$\mathcal{I}_{\mathsf{even}_3^1}(x, u, v, w) = \mathcal{I}_{\mathsf{odd}_3^1}(x, u, v, w) = 1 + v \cdot (2^x - 1) + w$$

The rules are still oriented; for example, rule (2_2) gives rise to the inequality

$$1 + 2^{x+1} - 1 + z \cdot (2^{x+1} - 1) \geqslant 1 + (1 + 2 \cdot (2^x - 1)) + z \cdot (2^{x+1} - 1)$$

which holds because $2^{x+1} - 1 = 1 + 2 \cdot (2^x - 1)$. The above interpretation induces a runtime complexity of $\mathcal{O}(2^n)$. This is a *tight* bound, as we observed earlier.

Definition 7.6 (Recipe B: Extension for Runtime Complexity). Recipe A is altered as follows, assuming we are given a usable replacement map υ for $(\mathcal{F}, \mathcal{R})$. Rather than demanding (strict or weak) monotonicity of the functions $\mathcal{J}_f^i$ in *all* arguments, we merely demand that

- $\mathcal{J}_f^0$ is strictly monotone in the arguments in $\upsilon(f)$, for all $f \in \mathcal{F}$,
- $\mathcal{J}_f^i$ is weakly monotone in the arguments in $\upsilon(f)$, for all $f \in \mathcal{F}_{\mathcal{D}}$ and $1 \leqslant i \leqslant m_f$,
- as before, $\mathcal{J}_{f,i}^j$ is strictly monotone in argument $n + i + j - 1$, where n is the arity of $f \in \mathcal{F}_{\mathcal{D}}$.

Given $\mathcal{J}$, the definition of $\mathcal{I}$ remains the same.

Recipe B can be used like Recipe A, but only for runtime complexity.

Lemma 7.7. *The interpretation $\mathcal{I}$ from Recipe B is a context-sensitive interpretation for $(\mathcal{H}, \mu\upsilon)$. If its interpretation functions satisfy the compatibility constraints from Definition 6.4, then $\mathcal{I}$ is compatible with $\mathcal{H}$ and $\mathsf{crc}_{\mathcal{R}}(n) = \max\{[\xi_{\top}(t)]_{\mathcal{I}} \mid t \in \mathcal{T}(\mathcal{F}), |t| \leqslant n, \text{ and } t \text{ is basic}\}$. Moreover,*

$$[\xi_{\top}(f(t_1, \ldots, t_n))]_{\mathcal{I}}^{\alpha} = \sum_{i=0}^{m_f} \mathcal{J}_f^i([\xi_{\top}(t_1)]_{\mathcal{I}}^{\alpha}, \ldots, [\xi_{\top}(t_n)]_{\mathcal{I}}^{\alpha})$$

Proof. It is not hard to see that if the restrictions in the recipe are satisfied, then indeed all interpretation functions $\mathcal{I}_f$ are strictly monotone in all arguments $i \in \mu\upsilon(f)$. The result for $\mathsf{crc}_{\mathcal{R}}$ follows by Theorem 7.4. Compatibility and equivalence are obtained from Lemma 6.5, as changing the monotonicity requirements does not affect either property. $\qquad\square$

Example 7.8. We use Recipe B to derive an upper bound for the runtime complexity of $\mathcal{R}_{\mathsf{fib}}$. From Example 7.2 we know that the replacement map υ_{rc} defined by $\upsilon_{\mathsf{rc}}(\mathsf{s}) = \{1\}$, $\upsilon_{\mathsf{rc}}(+) = \upsilon_{\mathsf{rc}}(\langle \cdot, \cdot \rangle) = \{1, 2\}$, and $\upsilon_{\mathsf{rc}}(\mathsf{fib}) = \varnothing$ is usable. For the interpretations, we assign:

$$\mathcal{J}_0^0 = 0 \qquad \mathcal{J}_{\mathsf{s}}^0(x) = x + 1 \qquad \mathcal{J}_{\langle \cdot, \cdot \rangle}^0(x, y) = x + y + 1 \qquad \mathcal{J}_+^0(x, y) = 2x + y + 1$$

$$\mathcal{J}_+^1(x, y) = \mathcal{J}_+^2(x, y) = 0 \qquad \mathcal{J}_{\mathsf{fib}}^0(x) = 3 \qquad \mathcal{J}_{\mathsf{fib}}^1(x) = 0 \qquad \mathcal{J}_{\mathsf{fib}}^2(x) = 5 \cdot (3^x - 1)$$

$$\mathcal{J}_{\mathsf{fib},2}^1(x, a) = 3a \qquad \mathcal{J}_{\mathsf{fib},2}^2(x, a, b) = a + b$$

One easily verifies that these interpretations are strictly monotone in the required argument positions, and weakly monotone in all argument positions. Omitting the (automatically satisfied) proof obligations for rules (5_ρ) and (6_ρ), this leaves

$$
\begin{aligned}
[+(0, y, \top, z)]_{\mathcal{I}} &= & y + 1 &> y & &= [y]_{\mathcal{I}} \\
[+(\mathsf{s}(x), y, z, \top)]_{\mathcal{I}} &= & 2x + y + 3 &> 2x + y + 2 & &= [\mathsf{s}(+(x, y, \top, \top))]_{\mathcal{I}} \\
[\mathsf{fib}(0, \top, u)]_{\mathcal{I}} &= & 3 + u \cdot 5 \cdot 0 &> 2 & &= [\langle 0, \mathsf{s}(0) \rangle]_{\mathcal{I}} \\
[\mathsf{fib}(\mathsf{s}(x), u, \top)]_{\mathcal{I}} &= & 5 \cdot 3^{x+1} - 2 &\geqslant 5 \cdot 3^{x+1} - 3 & &= [\mathsf{fib}_2^1(\mathsf{s}(x), u, \mathsf{fib}(x, \top, \top))]_{\mathcal{I}} \\
[\mathsf{fib}_2^1(\mathsf{s}(x), u, \langle y, z \rangle)]_{\mathcal{I}} &= & 3y + 3z + 6 &\geqslant 3y + 2z + 5 & &= [\mathsf{fib}_2^2(\mathsf{s}(x), u, \langle y, z \rangle, +(y, z, \top, \top))]_{\mathcal{I}} \\
[\mathsf{fib}_2^2(\mathsf{s}(x), u, \langle y, z \rangle, w)]_{\mathcal{I}} &= y + z + w + 4 &> z + w + 1 & &= [\langle z, w \rangle]_{\mathcal{I}}
\end{aligned}
$$

which holds for all values of x, y, z, u, and w. From this we conclude $\mathcal{O}(3^n)$ runtime complexity by Lemma 7.7.

Note that Recipe B may not be used for derivational complexity.

Example 7.9. The system $\mathcal{R}_{\mathsf{odd}}$ is a variation of $\mathcal{R}_{\mathsf{even}}$ defined by the following rules:

$$\mathsf{odd}(0) \to \mathsf{false} \qquad\qquad \mathsf{not}(\mathsf{true}) \to \mathsf{false}$$
$$\mathsf{odd}(\mathsf{s}(x)) \to \mathsf{not}(y) \;\Leftarrow\; \mathsf{odd}(x) \approx y \qquad\qquad \mathsf{not}(\mathsf{false}) \to \mathsf{true}$$

We will use Recipe B to derive an upper bound for the runtime complexity of this CCTRS, giving a bit more detail as to how the interpretations are chosen. The replacement map υ with $\upsilon(\mathsf{odd}) = \upsilon(\mathsf{s}) = \varnothing$ and $\upsilon(\mathsf{not}) = \{1\}$ is usable. Since the unconditional rules will be taken care of by the choice of $\mathcal{J}^0_{\mathsf{odd}}$ and $\mathcal{J}^0_{\mathsf{not}}$, we let $\mathcal{J}^1_{\mathsf{odd}}(x) = \mathcal{J}^1_{\mathsf{not}}(x) = \mathcal{J}^2_{\mathsf{not}}(x) = 0$. For clarity, we assign different names to the remaining interpretation functions:

$$\mathcal{J}^0_{\mathsf{true}} = T \qquad \mathcal{J}^0_0 = Z \qquad \mathcal{J}^0_{\mathsf{odd}} = O \qquad \mathcal{J}^1_{\mathsf{odd},2} = C$$
$$\mathcal{J}^0_{\mathsf{false}} = F \qquad \mathcal{J}^0_{\mathsf{s}} = S \qquad \mathcal{J}^2_{\mathsf{odd}} = D \qquad \mathcal{J}^0_{\mathsf{not}} = N$$

Here T, F, and Z are (unknown) constants and C, D, N, O, and S are (unknown) unary functions, and N must be strictly monotone. The recipe gives rise to the following constraints:

$$O(Z) > F \qquad\qquad N(T) > F \qquad\qquad N(F) > T$$

for the unconditional rules and

$$D(S(x)) \geqslant C(S(x), O(x) + D(x))$$
$$O(S(x)) + C(S(x), y) > N(y)$$

for the conditional rule. The constraints $N(T) > F$ and $N(F) > T$ are satisfied by simply taking $F = T = 0$ and $N(x) = x + 1$ (recall that N must be strictly monotone). As O is not required to be (strictly) monotone, and the constraints give little reason for O to regard its argument, we let $O(x) = A$ for some constant A. Hence the remaining constraints reduce to

$$A > 0$$
$$D(S(x)) \geqslant C(S(x), A + D(x))$$
$$A + C(S(x), y) > y + 1$$

By taking $A = 2$ and $C(x,y) = y$ we are left with

$$D(S(x)) \geqslant 2 + D(x)$$

which is easily satisfied by choosing $D(x) = x$ and $S(x) = x + 2$. With these choices, we have $[s]_{\mathcal{I}} \leqslant 2 \cdot |s|$ for all terms s, so we obtain linear runtime complexity by Lemma 7.7.

Note that the use of the replacement map υ was essential to obtain linear runtime complexity; if $\mathcal{J}^0_{\mathsf{odd}} = O$ was required to be monotone in its first argument, we would have had to choose $O(x) = x + 1$ or worse. While this would allow us to choose the tighter interpretation $S(x) = x + 1$, it would have produced the constraint $D(x+1) \geqslant D(x) + x + 1$, which can be satisfied with a *quadratic* interpretation $D(y) = y^2$, but not with a linear one.

8. Splitting Time and Space Complexity

Another method to improve interpretations is to separate *time* and *space* complexity. To understand the motivation, consider Example 7.8. Since the rules for addition had to be oriented strictly, the interpretation $\mathcal{J}^0_+(x,y) = 2x + y + 1$ was chosen rather than the simpler $\mathcal{J}^0_+(x,y) = x + y$. However, this does not accurately reflect the number of steps it takes to evaluate an addition. Rather, it reflects the sum of the number of steps *plus* the size of the result. This high value for the interpretation also affects the interpretations

for other symbols. And while the difference is only a constant factor, which is not an issue in *polynomial* interpretations, it is a cause for concern when considering *exponential* complexities; compare $\mathcal{O}(2^n)$ and $\mathcal{O}(2^{(an)}) = \mathcal{O}((2^a)^n)$.

Thus, as an alternative, let us consider interpretations not in $\mathbb{N}$, but rather in $\mathbb{N}^2$: pairs (n, m), where n records the number of steps to evaluate a term to constructor normal form, and m the size of the result. These pairs are equipped with the following orders: $(n_1, m_1) > (n_2, m_2)$ if $n_1 > n_2$ and $m_1 \geqslant m_2$, and $(n_1, m_1) \geqslant (n_2, m_2)$ if $n_1 \geqslant n_2$ and $m_1 \geqslant m_2$. We suggestively write $cost((n, m)) = n$ and $size((n, m)) = m$, and note that $cost(x) > cost(y)$ if $x > y$. Consequently, $\mathsf{dh}(s, \rightarrow_{\Xi(\mathcal{R}),\mu}) \leqslant cost([s]_{\mathcal{I}}^{\alpha})$ for any valuation α over $\mathbb{N}^2$.

Example 8.1. We revisit Example 5.7 and define

$$\mathcal{I}_{\top} = (0,1) \qquad \mathcal{I}_{\bot} = \mathcal{I}_{\mathsf{true}} = \mathcal{I}_{\mathsf{false}} = \mathcal{I}_0 = (0,0) \qquad \mathcal{I}_{\mathsf{s}}((c,s)) = (c, s+1)$$

$$\mathcal{I}_{\mathsf{even}}(x,u,v,w) = \mathcal{I}_{\mathsf{odd}}(x,u,v,w) = (1 + cost(x) + (size(v) + size(w)) \cdot A(x), 0)$$

$$\mathcal{I}_{\mathsf{even}_2^1}(x,u,v,w) = \mathcal{I}_{\mathsf{odd}_2^1}(x,u,v,w) = (1 + cost(x) + cost(v) + size(w) \cdot A(x), 0)$$

$$\mathcal{I}_{\mathsf{even}_3^1}(x,u,v,w) = \mathcal{I}_{\mathsf{odd}_3^1}(x,u,v,w) = (1 + cost(x) + size(v) \cdot A(x) + cost(w), 0)$$

where

$$A(x) = (cost(x) + 1) \cdot (2^{size(x)} - 1)$$

All interpretations are weakly monotone in all arguments because $x \geqslant y$ implies both $cost(x) \geqslant cost(y)$ and $size(x) \geqslant size(y)$, and in all interpretation functions $cost(\cdot)$ and $size(\cdot)$ are only used positively. If $x > x'$ then $\mathcal{I}_{\mathsf{even}}(x,u,v,w) > I_{\mathsf{even}}(x',u,v,w)$ since $cost(\mathcal{I}_{\mathsf{even}}(x,u,v,w))$ has a $cost(x)$ summand. The same holds for $\mathcal{I}_{\mathsf{odd}}(x,u,v,w)$ and $\mathcal{I}_{\mathsf{s}}(x)$. Both $\mathcal{I}_{\mathsf{even}_2^1}(x,u,v,w)$ and $\mathcal{I}_{\mathsf{odd}_2^1}(x,u,v,w)$ have a $cost(v)$ summand and hence are strictly monotone in their third arguments, and $\mathcal{I}_{\mathsf{even}_3^1}(x,u,v,w)$ and $\mathcal{I}_{\mathsf{odd}_3^1}(x,u,v,w)$ have a $cost(w)$ summand. Hence $\mathcal{I}$ satisfies the monotonicity requirements.

Furthermore, all rules of $\Xi(\mathcal{R}_{\mathsf{even}})$ are oriented as required. For the size component this is clear as $size([\ell]_{\mathcal{I}}^{\alpha}) = 0 = size([r]_{\mathcal{I}}^{\alpha})$ for all rules $\ell \to r$. For the cost component, we see that rules of the form (5_ρ) are oriented because $cost(v) \geqslant 0 = size([\bot]_{\mathcal{I}}^{\alpha}) \cdot A(x)$, and rules of the form (6_ρ) are oriented by monotonicity since $[\top]_{\mathcal{I}}^{\alpha} = (0,1) \geqslant (0,0) = [\bot]_{\mathcal{I}}^{\alpha}$. Rules (1_1), (1_4), (3_2), (3_3), (3_5), and (3_6) are strictly oriented since their left-hand sides evaluate to 1 whereas the right-hand sides evaluate to 0. The only rules where the orientation is non-trivial are (2_2), (2_3), (2_5), and (2_6). We consider (2_2):

$$1 + cost(x) + (1 + size(w)) \cdot A((cost(x), size(x) + 1))$$

$$\geqslant 1 + cost(x) + (1 + cost(x) + 2 \cdot A(x)) + size(w) \cdot A((cost(x), size(x) + 1))$$

Removing equal parts from both sides of the equation and inserting the definition of A yields

$$(cost(x) + 1) \cdot (2^{size(x)+1} - 1) \geqslant 1 + cost(x) + 2 \cdot (cost(x) + 1) \cdot (2^{size(x)} - 1)$$

and one easily checks that both sides are equal.

Now, towards runtime complexity, an easy induction proof shows that $cost([s]_{\mathcal{I}}) = 0$ and $size([s]_{\mathcal{I}}) \leqslant n$ for all ground constructor terms s with $|s| \leqslant n$. Therefore, the conditional

runtime complexity $\mathsf{crc}_{\mathcal{R}_\mathsf{even}}(n)$ is bounded by

$$\max\{cost([f(s_1,\ldots,s_m,\top,\ldots,\top)]_{\mathcal{I}}) \mid f \in \mathcal{F}_\mathcal{D} \text{ and } s_1,\ldots,s_m \text{ are ground constructor}$$
$$\text{terms with } |s_1| + \cdots + |s_m| < n\}$$
$$= \max\{cost(\mathcal{I}_f((0,x_1),\ldots,(0,x_m),(0,1),\ldots,(0,1))) \mid f \in \mathcal{F}_\mathcal{D} \text{ and } x_1 + x_2 + x_3 + x_4 < n\}$$
$$= \max\{1 + 0 + 2 \cdot 1 \cdot (2^x - 1) \mid x < n\} = 2^n - 1 \leqslant 2^n$$

This is the same bound that we obtained in Example 7.5, but without employing context-sensitivity.

Interestingly, we can obtain the same tight bound for derivational complexity.

Example 8.2. We prove by induction that for all ground $\top$-terms s there exist $K, N \geqslant 0$ with $K + N \leqslant |s|$ such that $cost([s]_{\mathcal{I}}) \leqslant 2^N - 1$ and $size([s]_{\mathcal{I}}) \leqslant K$.

- If s is 0, true, or false then $cost([s]_{\mathcal{I}}) = 0 = size([s]_{\mathcal{I}})$, so we can take $K = N = 0$.
- If $s = \mathsf{s}(t)$ and t is bounded by (K, N), then $cost([s]_{\mathcal{I}}) = cost([t]_{\mathcal{I}}) \leqslant 2^N - 1$ and $size([s]_{\mathcal{I}})) = 1 + size([t]_{\mathcal{I}}) \leqslant K + 1$, so we can take $(K+1, N)$.
- If $s = \mathsf{even}(t, \top, \top, \top)$ or $s = \mathsf{odd}(t, \top, \top, \top)$ with t bounded by (K, N) then $size([s]_{\mathcal{I}}) = 0$ and

$$\begin{aligned} cost([s]_{\mathcal{I}}) &= 1 + cost([t]_{\mathcal{I}}) + 2 \cdot (cost([t]_{\mathcal{I}}) + 1) \cdot (2^{size([t]_{\mathcal{I}})} - 1) \\ &\leqslant 1 + (2^N - 1) + 2 \cdot 2^N \cdot (2^K - 1) \\ &= 2^N + 2^{N+K+1} - 2^{N+1} \\ &= 2^{N+K+1} - 2^N \\ &\leqslant 2^{N+K+1} - 1 \end{aligned}$$

and so we can take $(0, N + K + 1)$.

It follows that $\mathsf{cdc}_{\mathcal{R}_\mathsf{even}}(n) = \mathcal{O}(2^n)$.

Separating the "cost" and "size" component made it possible to obtain an exponential bound for the derivational complexity of $\mathcal{R}_\mathsf{even}$. However, a downside is that the derivation of this bound is ad-hoc, and it would require a more systematic analysis of various systems with the separated $cost/size$ approach to obtain a strategy to find such bounds. For runtime complexity, the approach is more straightforward. If for all $f \in \mathcal{F}_\mathcal{C}$ the result $\mathcal{I}_f(x_1, \ldots, x_n)$ has the form

$$(c(cost(x_1), \ldots, cost(x_n)), s(size(x_1), \ldots, size(x_n)))$$

where c is a linear polynomial with coefficients in $\{0, 1\}$ and constant part 0, and s is a linear polynomial with coefficients in $\{0, 1\}$ and a constant part at most K, then all ground constructor terms s have cost 0 and size at most $K \cdot |s|$, so $\mathsf{crc}_\mathcal{R}(n)$ is bounded by the maximum value of $\mathcal{I}_f((0, s_1), \ldots, (0, s_m), (0, 1), \ldots, (0, 1))$ where $f \in \mathcal{F}_\mathcal{D}$ and $s_1 + \cdots + s_m < K \cdot n$. This mirrors the corresponding notion of "strongly linear polynomials" in the setting with interpretations over $\mathbb{N}$, and is what we used in Example 8.1 (with $K = 1$).

As before, we will use a standard recipe to find such interpretations. To this end, we adapt the ideas from Recipes A and B.

Definition 8.3 (Recipe C: Cost/Size Version)**.** Given a usable replacement map υ, we consider the replacement map $\mu\upsilon$ where, for f of arity n in the original signature $\mathcal{F}$, $\mu\upsilon(f) = \upsilon(f)$ when considering runtime complexity and $\mu\upsilon(f) = \{1, \ldots, n\}$ otherwise. Given interpretation functions

- $\mathcal{S}_f \colon \mathbb{N}^n \to \mathbb{N}$ and $\mathcal{C}_f^0, \ldots, \mathcal{C}_f^{m_f} \colon \mathbb{N}^{2n} \to \mathbb{N}$ for every symbol f of arity n in $\mathcal{F}$ such that $\mathcal{R} {\restriction} f$ consists of m_f rules,
- $\mathcal{S}_{f,i}^1, \ldots, \mathcal{S}_{f,i}^k$ with $\mathcal{S}_{f,i}^j \colon \mathbb{N}^{n+j} \to \mathbb{N}$ and $\mathcal{C}_{f,i}^1, \ldots, \mathcal{C}_{f,i}^k$ with $\mathcal{C}_{f,i}^j \colon \mathbb{N}^{2(n+j)} \to \mathbb{N}$ for every rule $\rho_i \in \mathcal{R} {\restriction} f$ with $k > 0$ conditions

such that the following monotonicity constraints are satisfied:

- $\mathcal{S}_f$ is weakly monotone in all arguments in $\mu\upsilon(f)$,
- $\mathcal{C}_f^0$ is strictly monotone in all arguments in $\mu\upsilon(f)$ and weakly monotone in all arguments in $\{n + j \mid j \in \mu\upsilon(f)\}$,
- $\mathcal{C}_f^i$ is weakly monotone in all arguments in $\{j, n + j \mid j \in \mu\upsilon(f)\}$,
- $\mathcal{S}_{f,i}^j$ is weakly monotone in its last argument $n + j$,
- $\mathcal{C}_{f,i}^j$ is strictly monotone in argument $n+j$ and weakly monotone in argument $2(n+j)$,

we construct an interpretation $\mathcal{I}$ for $\mathcal{H}$ as follows: $\mathcal{I}_\top = (0, 1)$ and $\mathcal{I}_\bot = (0, 0)$,

$$\mathcal{I}_f(x_1, \ldots, x_n, c_1, \ldots, c_{m_f}) = \big(\mathcal{C}_f^0(cost(\vec{x}), size(\vec{x})) + \sum_{k=1}^{m_f} size(c_k) \cdot \mathcal{C}_f^k(cost(\vec{x}), size(\vec{x})),$$
$$\mathcal{S}_f(size(\vec{x})) \big)$$

for every $f \in \mathcal{F}_\mathcal{C} \cup \mathcal{F}_\mathcal{D}$ of arity n, and finally

$$\mathcal{I}_{f_i^j}(x_1, \ldots, x_n, c_1, \ldots, c_{i-1}, y_1, \ldots, y_j, c_{i+1}, \ldots, c_{m_f}) =$$
$$\big(\mathcal{C}_f^0(cost(\vec{x}), size(\vec{x})) + \mathcal{C}_{f,i}^j(cost(\vec{x}), cost(\vec{y}), size(\vec{x}), size(\vec{y}))$$
$$+ \sum_{k=1,\, k\neq i}^{m_f} size(c_k) \cdot \mathcal{C}_f^k(cost(\vec{x}), size(\vec{x})), \, \max(\mathcal{S}_f(size(\vec{x})), \mathcal{S}_{f,i}^j(size(\vec{x}), size(\vec{y}))) \big)$$

Here $cost(\vec{x})$ and $size(\vec{x})$ stand for $cost(x_1), \ldots, cost(x_n)$ and $size(x_1), \ldots, size(x_n)$, and similar for $cost(\vec{y})$ and $size(\vec{y})$.

The following remarks are helpful to understand the intuition behind the interpretations defined in the above recipe.

- The "size" of a term s is intended to reflect—or at least bound—how large a normal form of s may be, where different constructor symbols count differently towards the size. In a term $f(s_1, \ldots, s_n, t_1, \ldots, t_{m_f})$, the size is only affected by the sizes of $s_1, \ldots, s_n$; the additional arguments merely indicate our progress in trying to reduce the term. In a term of the shape $f_i^j(s_1, \ldots, s_n, \langle t_1, \ldots, t_{m_f}\rangle[y_1, \ldots, y_j]_i)$ the size should similarly not be affected by the progress on testing the applicability of the rule $\rho_i \in \mathcal{R} {\restriction} f$. However, here a rule-specific size function is included in a max expression for technical reasons; in practice, we will always have $\mathcal{S}_f(\cdots) \geqslant \mathcal{S}_{f,i}^j(\cdots)$, but the latter will have more variables that can be used to orient rules of the form (3_ρ).
- The "cost" of $f(s_1, \ldots, s_n, t_1, \ldots, t_{m_f})$ reflects how many steps we may take to reach a normal form. This is affected by the cost of evaluating each of the rule conditions where $t_i = (0, 1)$ is the value of $\top$, as well as the cost of evaluating whatever we may reduce to; the sizes of the arguments may affect both those costs (since it will take longer to evaluate $\mathsf{even}(\mathsf{s}^{100}(0))$ than $\mathsf{even}(0)$, for instance).

As before, using this interpretation for the rules in Definition 5.6, the obtained inequalities can be greatly simplified.

Definition 8.4. The *compatibility constraints* for $\mathcal{C}$ and $\mathcal{S}$ comprise the following inequalities, for every rule $\rho_i \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow a_1 \approx b_1, \ldots, a_k \approx b_k$ in $\mathcal{R}$:

(1_ρ) $\mathcal{S}_f(\overrightarrow{[\ell]_\mathcal{S}}) \geqslant [\xi_\top(r)]_\mathcal{S}$,

(2_ρ) $\mathcal{S}_f(\overrightarrow{[\ell]_\mathcal{S}}) \geqslant \mathcal{S}_{f,i}^1(\overrightarrow{[\ell]_\mathcal{S}}, [\xi_\top(a_1)]_\mathcal{S})$,

(3_ρ) $\mathcal{S}_{f,i}^k(\overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_k]_\mathcal{S}) \geqslant [\xi_\top(r)]_\mathcal{S}$,

(4_ρ) $\mathcal{S}_{f,i}^j(\overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_j]_\mathcal{S}) \geqslant \mathcal{S}_{f,i}^{j+1}(\overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_j]_\mathcal{S}, [\xi_\top(a_{j+1})]_\mathcal{S})$

and

(1_ρ) $\mathcal{C}_f^0(\overrightarrow{[\ell]_\mathcal{C}}, \overrightarrow{[\ell]_\mathcal{S}}) + \mathcal{C}_f^i(\overrightarrow{[\ell]_\mathcal{C}}, \overrightarrow{[\ell]_\mathcal{S}}) > [\xi_\top(r)]_\mathcal{C}$,

(2_ρ) $\mathcal{C}_f^i(\overrightarrow{[\ell]_\mathcal{C}}, \overrightarrow{[\ell]_\mathcal{S}}) \geqslant \mathcal{C}_{f,i}^1(\overrightarrow{[\ell]_\mathcal{C}}, [\xi_\top(a_1)]_\mathcal{C}, \overrightarrow{[\ell]_\mathcal{S}}, [\xi_\top(a_1)]_\mathcal{S})$,

(3_ρ) $\mathcal{C}_{f,i}^k(\overrightarrow{[\ell]_\mathcal{C}}, [b_1]_\mathcal{C}, \ldots, [b_k]_\mathcal{C}, \overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_k]_\mathcal{S}) + \mathcal{C}_f^0(\overrightarrow{[\ell]_\mathcal{C}}, \overrightarrow{[\ell]_\mathcal{S}}) > [\xi_\top(r)]_\mathcal{C}$,

(4_ρ) $\mathcal{C}_{f,i}^j(\overrightarrow{[\ell]_\mathcal{C}}, [b_1]_\mathcal{C}, \ldots, [b_j]_\mathcal{C}, \overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_j]_\mathcal{S}) \geqslant$
$\qquad \mathcal{C}_{f,i}^{j+1}(\overrightarrow{[\ell]_\mathcal{C}}, [b_1]_\mathcal{C}, \ldots, [b_j]_\mathcal{C}, [\xi_\top(a_{j+1})]_\mathcal{C}, \overrightarrow{[\ell]_\mathcal{S}}, [b_1]_\mathcal{S}, \ldots, [b_j]_\mathcal{S}, [\xi_\top(a_{j+1})]_\mathcal{S})$

for the same cases of k and j as in Definition 5.6. Here $[s]_\mathcal{S} = size([s]_\mathcal{I}^\alpha)$, $[s]_\mathcal{C} = cost([s]_\mathcal{I}^\alpha)$, and $\overrightarrow{[\ell]_\mathcal{S}}$ and $\overrightarrow{[\ell]_\mathcal{C}}$ denotes the sequences $[\ell_1]_\mathcal{S}, \ldots, [\ell_n]_\mathcal{S}$ and $[\ell_1]_\mathcal{C}, \ldots, [\ell_n]_\mathcal{C}$.

Lemma 8.5. *The interpretation $\mathcal{I}$ from Recipe C is a context-sensitive interpretation for $(\mathcal{H}, \mu\upsilon)$. If the corresponding functions $\mathcal{C}$ and $\mathcal{S}$ satisfy the compatibility constraints from Definition 8.4, then*

$$[\xi_\top(f(t_1, \ldots, t_n))]_\mathcal{S} = \mathcal{S}_f([\xi_\top(t_1)]_\mathcal{S}, \ldots, [\xi_\top(t_n)]_\mathcal{S})$$

$$[\xi_\top(f(t_1, \ldots, t_n))]_\mathcal{C} = \sum_{i=0}^{m_f} \mathcal{C}_f^i([\xi_\top(t_1)]_\mathcal{C}, \ldots, [\xi_\top(t_n)]_\mathcal{C}, [\xi_\top(t_1)]_\mathcal{S}, \ldots, [\xi_\top(t_n)]_\mathcal{S})$$

Moreover, $\mathcal{I}$ is compatible with $\mathcal{H}$. Therefore

$$\mathsf{cdc}_\mathcal{R}(n) = \max\{cost([\xi_\top(t)]_\mathcal{I}) \mid t \in \mathcal{T}(\mathcal{F}) \text{ and } |t| \leqslant n\}$$
$$\mathsf{crc}_\mathcal{R}(n) = \max\{cost([\xi_\top(t)]_\mathcal{I}) \mid t \in \mathcal{T}(\mathcal{F}), |t| \leqslant n, \text{ and } t \text{ is basic}\}$$

Proof. For the first part of the claim, it is not hard to see that $\mathcal{I}$ satisfies the monotonicity requirements: Every interpretation function $\mathcal{I}_f$ is strictly monotone in each argument position belonging to $\mu\upsilon(f) = \upsilon(f)$ (or $\{1, \ldots, n\}$ for derivational complexity), and every $\mathcal{I}_{f_i^j}$ is strictly monotone in argument position $n + i + j - 1$. The second part of the claim is obtained by writing out definitions. As for compatibility, minimality of $[\bot]_\mathcal{I}^\alpha$ ensures that all constraints obtained from clause (6_ρ) are satisfied, while those obtained from clause (5_ρ) are oriented because

$$\mathcal{C}_{f,i}^j(\cdots) \geqslant 0 = [\bot]_\mathcal{S} \cdot \mathcal{C}_f^i(\cdots)$$

and

$$\max\{\mathcal{S}_f(size(\vec{x})), \mathcal{S}_{f,i}^j(\cdots)\} \geqslant \mathcal{S}_f(size(\vec{x}))$$

always hold. The requirements for the other rules follow from the compatibility constraints, by expanding the inequalities $[\ell]_{\mathcal{S}} \geqslant [r]_{\mathcal{S}}$ and $[\ell]_{\mathcal{C}} \geqslant [r]_{\mathcal{C}}$ or $[\ell]_{\mathcal{C}} > [r]_{\mathcal{C}}$ depending on the cost of the rule. For instance, the actual size constraint for (3_ρ) is

$$\max(\mathcal{S}_f(\overrightarrow{[\ell]_{\mathcal{S}}}), \mathcal{S}_{f,i}^k(\overrightarrow{[\ell]_{\mathcal{S}}}, [b_1]_{\mathcal{S}}, \ldots, [b_k]_{\mathcal{S}})) > [\xi_\top(r)]_{\mathcal{S}}$$

while for (4_ρ) we obtain

$$\max(\mathcal{S}_f(\overrightarrow{[\ell]_{\mathcal{S}}}), \mathcal{S}_{f,i}^j(\overrightarrow{[\ell]_{\mathcal{S}}}, [b_1]_{\mathcal{S}}, \ldots, [b_j]_{\mathcal{S}})) \geqslant$$
$$\max(\mathcal{S}_f(\overrightarrow{[\ell]_{\mathcal{S}}}), \mathcal{S}_{f,i}^{j+1}(\overrightarrow{[\ell]_{\mathcal{S}}}, [b_1]_{\mathcal{S}}, \ldots, [b_j]_{\mathcal{S}}, [\xi_\top(a_{j+1})]_{\mathcal{S}}))$$

Both constraints are clearly implied by the compatibility constraints of Definition 8.4. The claims on $\mathsf{cdc}_{\mathcal{R}}$ and $\mathsf{crc}_{\mathcal{R}}$ hold because $\mathsf{dh}(s, \to_{\Xi(\mathcal{R}),\mu\upsilon}) \leqslant cost([s]_{\mathcal{I}})$. □

As with Lemma 6.5, we can find bounds on derivation heights without calculating $\xi_\top(t)$.

Example 8.6. We derive an upper bound for the runtime complexity of $\mathcal{R}_{\mathsf{fib}}$, detailing how we arrive at the chosen interpretation. Recall the rules:

$$0 + y \to y \qquad\qquad \mathsf{fib}(0) \to \langle 0, \mathsf{s}(0) \rangle$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{fib}(\mathsf{s}(x)) \to \langle z, w \rangle \;\Leftarrow\; \mathsf{fib}(x) \approx \langle y, z \rangle, \; y + z \approx w$$

We take the same usable replacement map υ as in Example 7.8: $\upsilon(\mathsf{s}) = \{1\}$, $\upsilon(+) = \upsilon(\langle \cdot, \cdot \rangle) = \{1, 2\}$, and $\upsilon(\mathsf{fib}) = \varnothing$. To facilitate understanding of the following constraints, we present the rules in $\Xi(\mathcal{R}_{\mathsf{fib}})$ that derive from the conditional rule (but note that they are not necessary to apply the recipe):

$$\mathsf{fib}(\mathsf{s}(x), c_1, \top) \to \mathsf{fib}_2^1(\mathsf{s}(x), c_1, \mathsf{fib}(x, \top, \top))$$
$$\mathsf{fib}_2^1(\mathsf{s}(x), c_1, \langle y, z \rangle) \to \mathsf{fib}_2^2(\mathsf{s}(x), c_1, \langle y, z \rangle, +(y, z, \top, \top))$$
$$\mathsf{fib}_2^2(\mathsf{s}(x), c_1, \langle y, z \rangle, w) \to \langle z, w \rangle$$

Following the recipe, let $N = \mathcal{S}_0$, $S = \mathcal{S}_{\mathsf{s}}$, $P = \mathcal{S}_{\langle \cdot, \cdot \rangle}$, $A = \mathcal{S}_+$, $F = \mathcal{S}_{\mathsf{fib}}$, $B = \mathcal{S}_{\mathsf{fib},2}^1$ and $C = \mathcal{S}_{\mathsf{fib},2}^2$. The interpretation functions S, P and A must be weakly monotone in all arguments, B and C only in the last argument, and F does not need to be weakly monotone due to υ. The requirements on the size component give rise to the constraints

$$A(N, y) \geqslant y \tag{8.1}$$
$$A(S(x), y) \geqslant S(A(x, y)) \tag{8.2}$$
$$F(N) \geqslant P(N, S(N)) \tag{8.3}$$

for the unconditional rules and

$$F(S(x)) \geqslant B(S(x), F(x)) \tag{8.4}$$
$$B(S(x), P(y, z)) \geqslant C(S(x), P(y, z), A(y, z)) \tag{8.5}$$
$$C(S(x), P(y, z), w) \geqslant P(z, w) \tag{8.6}$$

for the conditional rule of $\mathcal{R}_{\mathsf{fib}}$. For the cost component we will follow the guiding principle that $\mathcal{C}_f^0(x_1, \ldots, x_n, y_1, \ldots, y_n) \leqslant x_1 + \cdots + x_n$ for all constructor symbols $f \in \mathcal{F}_{\mathcal{C}}$, which gives cost 0 for ground constructor terms. As $\mathcal{C}_f^0$ must be strictly monotone in the first n arguments for $f \in \mathcal{F}_{\mathcal{C}}$, we fix $\mathcal{C}_0 = 0$, $\mathcal{C}_{\mathsf{s}}(x, y) = x$ and $\mathcal{C}_{\langle \cdot, \cdot \rangle}(cx, cy, sx, sy) = cx + cy$. We also fix $\mathcal{C}_+^1(cx, cy, sx, sy) = \mathcal{C}_+^2(cx, cy, sx, sy) = \mathcal{C}_{\mathsf{fib}}^1(cx, sx) = 0$ since these are the "conditional

evaluation" components for the unconditional rules. For the remaining interpretation functions, write $Q = \mathcal{C}_+^0$, $G = \mathcal{C}_{\mathsf{fib}}^0$, $H = \mathcal{C}_{\mathsf{fib}}^2$, $D = \mathcal{C}_{\mathsf{fib},2}^1$, and $E = \mathcal{C}_{\mathsf{fib},2}^2$, which yields

$$Q(0, cy, N, sy) > cy \tag{8.7}$$

$$Q(cx, cy, S(sx), sy) > Q(cx, cy, sx, sy) \tag{8.8}$$

$$G(0, N) > 0 \tag{8.9}$$

for the unconditional rules and

$$H(cx, S(sx)) \geqslant D(cx, G(cx, sx) +$$
$$H(cx, sx), S(sx), F(sx)) \tag{8.10}$$

$$D(cx, cy + cz, S(sx), P(sy, sz)) \geqslant$$
$$E(cx, cy + cz, Q(cy, cz, sy, sz), S(sx), P(sy, sz), A(sy, sz)) \tag{8.11}$$

$$G(cx, S(sx)) + E(cx, cy + cz, cw, S(sx), P(sy, sz), sw) > cz + cw \tag{8.12}$$

for the conditional rule. Here, Q is strictly monotone in its first two arguments and weakly in the last two, D is strictly monotone in argument 2 and weakly in 4, while E is strictly monotone in argument 3 and weakly in 6. There is no monotonicity constraint for G or H.

Choosing minimal polynomials to satisfy the constraints deriving from the rules for $+$, we set $N = 0$, $S(x) = x + 1$, $A(x, y) = x + y$, and $Q(cx, cy, sx, sy) = cx + cy + sx + 1$. Since G need not be monotone, we simply take $G(x, y) = 1$ to satisfy (8.9). Further choosing $P(x, y) = x + y$, the constraints simplify to

$$F(0) \geqslant 1 \tag{8.3}$$

$$F(x + 1) \geqslant B(x + 1, F(x)) \tag{8.4}$$

$$B(x + 1, y + z) \geqslant C(x + 1, y + z, y + z) \tag{8.5}$$

$$C(x + 1, y + z, w) \geqslant z + w \tag{8.6}$$

$$H(cx, sx + 1) \geqslant D(cx, H(cx, sx) + 1, sx + 1, F(sx)) \tag{8.10}$$

$$D(cx, cy + cz, sx + 1, sy + sz) \geqslant$$
$$E(cx, cy + cz, cy + cz + sy + 1, sx + 1, sy + sz, sy + sz) \tag{8.11}$$

$$1 + E(cx, cy + cz, cw, sx + 1, sy + sz, sw) > cz + cw \tag{8.12}$$

The size constraints are satisfied if we choose $C(x, y, z) = y + z$, $B(x, y) = 2y$, and $F(x) = 2^x$. Choosing $E(cx, cy, cz, sx, sy, sz) = cy + cz$ and $D(cx, cy, sx, sy) = 2cy + sy + 1$ takes care of (8.11) and (8.12), leaving only

$$H(c, s + 1) \geqslant 2 \cdot (H(c, s) + 1) + 2^s + 1 \tag{8.10}$$

This final constraint is satisfied for $H(c, s) = (s + 1) \cdot (2^{s+1} - 2)$ since

$$H(c, s + 1) = (s + 2) \cdot (2^{s+2} - 2) = s \cdot 2^{s+2} + 8 \cdot 2^s - 2s - 4$$
$$= s \cdot 2^{s+2} + 5 \cdot 2^s - 2s - 4 + 3 \cdot 2^s \geqslant s \cdot 2^{s+2} + 5 \cdot 2^s - 4s - 4 + 3$$
$$= 2 \cdot (s + 1) \cdot 2^{s+1} - 4 \cdot (s + 1) + 2^s + 3 = 2 \cdot (s + 1) \cdot (2^{s+1} - 2) + 2^s + 3$$
$$= 2 \cdot H(c, s) + 2^s + 3 = 2 \cdot (H(c, s) + 1) + 2^s + 1$$

Since all ground constructor terms s have cost 0 and size at most $|s|$, for ground basic terms s with $|s| \leqslant n$, $cost([s]_{\mathcal{I}})$ is bounded by $G(0, n - 1) + H(0, n - 1) = 1 + n \cdot 2^n - 2n$. We conclude a runtime complexity of $\mathcal{O}(n \cdot 2^n)$ by Lemma 8.5.

9. Conclusions

In this paper we have improved and extended the notion of complexity for conditional term rewriting first introduced in [17]. This notion takes failed calculations into account as any automatic rewriting engine would. We have defined a transformation to unconditional left-linear context-sensitive TRSs whose complexity is the same as the conditional complexity of the original system, and shown how this transformation can be used to find bounds for conditional complexity using traditional interpretation-based methods.

9.1. **Implementation and Experiments.** At present, we have not implemented the results of Sections 6, 7, and 8. However, we did implement the transformation from Section 5. The resulting (context-sensitive) TRSs can be used as input to a conventional TRS complexity tool, which by Theorem 5.11 gives an upper bound for conditional complexity. Although existing tools do not take advantage of either information regarding the replacement map, nor of the specific shape of the rules or the fact that only terms of the form $\xi_\top(s)$ need to be considered, the results are often tight bounds.

We have used this approach with $\mathsf{T_CT}$ [4] as the underlying complexity tool, to analyze the runtime complexity of the 57 strong CCTRSs in the current version of the termination problem database (TPDB 10.3),[3] along with 5 examples in this paper. The results are summarized to the right. A full evaluation page is available at

$\mathcal{O}(1)$	16
$\mathcal{O}(n)$	10
$\mathcal{O}(n^2)$	3
MAYBE	33

<center>http://cl-informatik.uibk.ac.at/experiments/2016/cc</center>

About half of the systems in our example set could not be handled. This is largely due to the presence of non-terminating CCTRSs as well as systems with exponential runtime complexity, which existing complexity tools do not support. Many benchmarks of conditional rewriting have rules similar to our Example 3.1, which lead to exponential complexity due to failed evaluations, and consequently cannot be handled. We do, however, obtain a constant upper bound for Example 3.2, a quadratic upper bound for Example 3.6, as well as the tight bound $\mathcal{O}(n)$ for Example 7.9.

9.2. **Related Work.** We are not aware of any other attempt to study the complexity of conditional rewriting, but numerous transformations from CTRSs to TRSs have been proposed in the literature. They can roughly be divided into so-called *unravelings* and *structure-preserving* transformations. The former were coined by Marchiori [23] and have been extensively investigated (e.g. [24, 28, 30, 31, 33]), mainly to establish (operational) termination and confluence of the input CTRS. The latter originate from Viry [37] and improved versions were proposed in [1, 8, 13].

The transformations that are known to transform CTRSs into TRSs such that (simple) termination of the latter implies quasi-decreasingness of the former, are natural candidates for study from a complexity perspective. We observe that unravelings are not suitable in this regard, since they do not take the cost for failed computations into account. For instance,

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[3]See http://termination-portal.org/wiki/TPDB for more details.

the unraveling from [24] transforms the CCTRS $\mathcal{R}_{\mathsf{even}}$ into

$$\mathsf{even}(0) \to \mathsf{true} \qquad \mathsf{even}(\mathsf{s}(x)) \to \mathsf{U}_1(\mathsf{odd}(x), x) \qquad \mathsf{U}_1(\mathsf{true}, x) \to \mathsf{true}$$
$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{U}_2(\mathsf{even}(x), x) \qquad \mathsf{U}_2(\mathsf{true}, x) \to \mathsf{false}$$
$$\mathsf{odd}(0) \to \mathsf{false} \qquad \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{U}_3(\mathsf{odd}(x), x) \qquad \mathsf{U}_3(\mathsf{true}, x) \to \mathsf{false}$$
$$\mathsf{odd}(\mathsf{s}(x)) \to \mathsf{U}_4(\mathsf{even}(x), x) \qquad \mathsf{U}_4(\mathsf{true}, x) \to \mathsf{true}$$

This TRS has a linear runtime complexity, which is readily confirmed by TCT. As the conditional runtime complexity is exponential, the transformation is not suitable for measuring conditional complexity. The same holds for the transformation in [30].

Structure-preserving transformations are better suited for studying conditional complexity since they keep track of the conditions in all applicable rules.

However, existing transformations of this kind are also unsuitable for measuring conditional runtime complexity. For instance, the CCTRS $\mathcal{R}_{\mathsf{even}}$ is transformed into the TRS

$$\mathsf{even}(0, x, y) \to \mathsf{m}(\mathsf{true}) \qquad\qquad\qquad \mathsf{odd}(0, x, y) \to \mathsf{m}(\mathsf{false})$$
$$\mathsf{even}(\mathsf{s}(x), \bot, z) \to \mathsf{even}(\mathsf{s}(x), \mathsf{c}(\mathsf{m}(\mathsf{odd}(x, \bot, \bot))), z) \quad \mathsf{even}(\mathsf{s}(x), \mathsf{c}(\mathsf{m}(\mathsf{true})), z) \to \mathsf{m}(\mathsf{true})$$
$$\mathsf{even}(\mathsf{s}(x), y, \bot) \to \mathsf{even}(\mathsf{s}(x), y, \mathsf{c}(\mathsf{m}(\mathsf{even}(x, \bot, \bot)))) \quad \mathsf{even}(\mathsf{s}(x), y, \mathsf{c}(\mathsf{m}(\mathsf{true}))) \to \mathsf{m}(\mathsf{false})$$
$$\mathsf{odd}(\mathsf{s}(x), \bot, z) \to \mathsf{odd}(\mathsf{s}(x), \mathsf{c}(\mathsf{m}(\mathsf{even}(x, \bot, \bot))), z) \quad \mathsf{odd}(\mathsf{s}(x), \mathsf{c}(\mathsf{m}(\mathsf{true})), z) \to \mathsf{m}(\mathsf{true})$$
$$\mathsf{odd}(\mathsf{s}(x), y, \bot) \to \mathsf{odd}(\mathsf{s}(x), y, \mathsf{c}(\mathsf{m}(\mathsf{odd}(x, \bot, \bot)))) \quad \mathsf{odd}(\mathsf{s}(x), y, \mathsf{c}(\mathsf{m}(\mathsf{true}))) \to \mathsf{m}(\mathsf{false})$$
$$\mathsf{even}(\mathsf{m}(x), y, z) \to \mathsf{m}(\mathsf{even}(x, \bot, \bot)) \qquad\qquad\qquad \mathsf{s}(\mathsf{m}(x)) \to \mathsf{m}(\mathsf{s}(x))$$
$$\mathsf{odd}(\mathsf{m}(x), y, z) \to \mathsf{m}(\mathsf{odd}(x, \bot, \bot)) \qquad\qquad\qquad \mathsf{m}(\mathsf{m}(x)) \to \mathsf{m}(x)$$

by the transformation of Şerbănuţă and Roşu [8]. TCT reports a constant runtime complexity, which is explained by the fact that the symbol s is turned into a defined symbol. Hence a term like $\mathsf{even}(\mathsf{s}(0), \top, \top)$ is *not* basic and thus disregarded for runtime complexity. The derivational complexity of the transformed TRS is harder to confirm automatically, as it is exponential, but likely not to differ much from the conditional derivational complexity of $\mathcal{R}_{\mathsf{even}}$. However, in general, we may well obtain much greater bounds due to the forced reevaluation of conditions when a subterm is reduced. Consider for instance a term $\mathsf{even}(\mathsf{s}(t))$ with $t = \mathsf{s}^{21}(0) + \mathsf{s}^{21}(0)$ in an extension of $\mathcal{R}_{\mathsf{even}}$ with rules for $+$. This term is encoded as $\mathsf{even}(\mathsf{s}(t), \bot, \bot)$, the $\bot$s indicating that no condition has been evaluated yet, and might be reduced as follows:

$$\mathsf{even}(\mathsf{s}(t), \bot, \bot) \to \ \mathsf{even}(\mathsf{s}(t), \mathsf{c}(\mathsf{m}(\mathsf{odd}(t, \bot, \bot))), \bot)$$
$$\to^* \mathsf{even}(\mathsf{s}(t), \mathsf{c}(\mathsf{m}(\mathsf{false})), \bot)$$
$$\to \ \mathsf{even}(\mathsf{s}(t), \mathsf{c}(\mathsf{m}(\mathsf{false})), \mathsf{c}(\mathsf{m}(\mathsf{even}(t, \bot, \bot))))$$
$$\to^* \mathsf{even}(\mathsf{s}(t), \mathsf{c}(\mathsf{m}(\mathsf{false})), \mathsf{c}(\mathsf{m}(\mathsf{true})))$$
$$\to \ \mathsf{even}(\mathsf{s}(\mathsf{m}(\mathsf{s}^{42}(0))), \mathsf{c}(\mathsf{m}(\mathsf{false})), \mathsf{c}(\mathsf{m}(\mathsf{true})))$$
$$\to \ \mathsf{even}(\mathsf{m}(\mathsf{s}^{43}(0)), \mathsf{c}(\mathsf{m}(\mathsf{false})), \mathsf{c}(\mathsf{m}(\mathsf{true})))$$
$$\to \ \mathsf{m}(\mathsf{even}(\mathsf{s}^{43}(0), \bot, \bot))$$

We observe that an evaluation in the instance $\mathsf{s}(t)$ of the pattern $\mathsf{s}(x)$ forces a reevaluation of t when checking the second condition. The fundamental difference with our approach is that we have used Lemma 3.4 to avoid such reevaluations.

Less recent, the transformation of Antoy *et al.* [1] operates in a more restrictive setting: weakly orthogonal constructor-based CTRSs without extra variables in the conditions. Like the transformation in [8], it blocks conditions when their evaluation fails; however, conditions are not reevaluated when arguments are modified. A crucial difference with our transformation Ξ is that different conditions in the same conditional rule are not evaluated from left to right but combined into a single condition, which has a negative impact on complexity. As an extreme example, consider the CCTRS $\mathcal{R}$ consisting of the four rules

$$\mathsf{f}(x) \to \mathsf{a} \Leftarrow \mathsf{c} \approx \mathsf{d}, \ \mathsf{g}(x) \approx \mathsf{a}, \ \mathsf{g}(x) \approx \mathsf{b} \qquad\qquad \mathsf{g}(\mathsf{s}(x)) \to \mathsf{f}(x)$$
$$\mathsf{f}(x) \to \mathsf{b} \Leftarrow \mathsf{c} \approx \mathsf{e} \qquad\qquad\qquad\qquad \mathsf{c} \to \mathsf{e}$$

The conditional runtime complexity of $\mathcal{R}$ is linear, which is confirmed by running $\mathsf{T}\mathsf{C}\mathsf{T}$ on $\Xi(\mathcal{R})$. The transformation of [1] produces the TRS

$$\mathsf{f}(x, \bot, \bot) \to \mathsf{f}(x, \langle \mathsf{c}, \mathsf{g}(x), \mathsf{g}(x)\rangle, \mathsf{c}) \qquad \mathsf{f}(x, \langle \mathsf{d}, \mathsf{a}, \mathsf{b}\rangle, z) \to \mathsf{a} \qquad \mathsf{g}(\mathsf{s}(x)) \to \mathsf{f}(x, \bot, \bot)$$
$$\mathsf{f}(x, y, \mathsf{e}) \to \mathsf{b} \qquad\qquad \mathsf{c} \to \mathsf{e}$$

whose runtime complexity is at least exponential because of the rules $\mathsf{f}(x, \bot, \bot) \to \mathsf{f}(x, \langle \mathsf{c}, \mathsf{g}(x), \mathsf{g}(x)\rangle, \mathsf{c})$ and $\mathsf{g}(\mathsf{s}(x)) \to \mathsf{f}(x, \bot, \bot)$. If the (undecidable) weak orthogonality restriction in [1] is not imposed, the same phenomenon may occur if rules have at most one condition.

However, it is worth noting also the similarities to our method, especially when there is at most one condition. Consider for example the result of transforming our CCTRS $\mathcal{R}_{\mathsf{even}}$:

$$\mathsf{even}(0, y, z) \to \mathsf{true} \qquad \mathsf{even}(\mathsf{s}(x), \mathsf{true}, y) \to \mathsf{true} \qquad \mathsf{even}(\mathsf{s}(x), y, \mathsf{true}) \to \mathsf{false}$$
$$\mathsf{odd}(0, y, z) \to \mathsf{false} \qquad \mathsf{odd}(\mathsf{s}(x), \mathsf{true}, y) \to \mathsf{true} \qquad \mathsf{odd}(\mathsf{s}(x), y, \mathsf{true}) \to \mathsf{false}$$
$$\mathsf{even}(\mathsf{s}(x), \bot, \bot) \to \mathsf{even}(\mathsf{s}(x), \mathsf{odd}(x, \bot, \bot), \mathsf{even}(x, \bot, \bot))$$
$$\mathsf{odd}(\mathsf{s}(x), \bot, \bot) \to \mathsf{odd}(\mathsf{s}(x), \mathsf{even}(x, \bot, \bot), \mathsf{odd}(x, \bot, \bot))$$

This does not look too different from the result of our transformation Ξ if the set AP is not used. In addition, the method used could be generalised with some of the ideas from [8], for instance by evaluating multiple conditions sequentially rather than in parallel.

Even ignoring the issue of multiple conditions—or, for [8], the issue of reevaluation—there are some fundamental differences between our transformation Ξ and the structure-preserving transformations of [1, 8]. In both of these, the conditions for different rules may be evaluated in parallel, which we do not permit. Moreover, neither transformation separates defined symbols (e.g. even) from "active" symbols used to evaluate conditions (e.g. even_2^1). This separation is necessary to impose a context-sensitive replacement map as we have done here, and makes it much easier to use traditional techniques such as polynomial interpretations. Most importantly, neither transformation *defines*—or is based on a formal definition of—conditional complexity; rather, they define upper bounds for a reasonable evaluation strategy.

9.3. **Avenues for Future Work.** There are several possibilities to continue our research.

Weakening restrictions. An obvious direction for future research is to broaden the class of CTRSs we consider. While it would make little sense to consider CTRSs that are not deterministic or of type 3—as the rewrite relation in these systems is undecidable in general—it may be possible to drop the variable and constructor requirements.

The linearity requirements in *strong* CCTRSs are an obvious target for improvement. These requirements were not needed in the definition or justification of our primary complexity

notion, but essential for the correctness of the way we use the anti-pattern set AP. However, if we are willing to lose completeness, we may drop the anti-pattern set, replacing the use of v in $\mathsf{AP}(\ell_i)$ or $\mathsf{AP}(b_j)$ in Definition 5.6 by a fresh variable; doing so, the transformation would not preserve derivation heights, but we would retain the possibility to obtain *upper* bounds. Alternatively, we might consider an infinite set of transformed rules $\Xi'(\mathcal{R})$ instead.

As for the restrictions in general CCTRSs, the proof of the important locality Lemma 3.4 requires only that the left-hand side ℓ of every rule $\ell \to r \Leftarrow c$ is a basic term such that $\mathcal{V}\mathrm{ar}(\ell) \cap \mathcal{V}\mathrm{ar}(c) = \varnothing$. This can always be satisfied by altering the system without changing the rewrite relation in an essential way, replacing for instance $\mathsf{f}(\mathsf{g}(x), y) \to r$ by $\mathsf{f}(z, y) \to r \Leftarrow z \approx \mathsf{g}(x)$. However, in such cases, the definition of conditional complexity needs to be revisited, as the restrictions on the conditions are needed for Lemma 3.5, which is important to justify our complexity notion. For example, if the right-hand sides of conditions were allowed to be arbitrary terms, it would be possible to define a system with rules

$$\mathsf{g}(x) \to x \qquad \mathsf{h}(x) \to \mathsf{g}(x) \qquad \mathsf{h}(x) \to x \qquad \mathsf{f}(z, y) \to \mathsf{a} \Leftarrow z \approx \mathsf{g}(x)$$

In this CTRS, a term $\mathsf{f}(\mathsf{h}(0), 0)$ *can* be reduced by the last rule, but we would only find this out if we reduced $\mathsf{h}(0)$ with the second rule, rather than with the third. Thus, to accurately analyze such a system, we would likely need a backtracking mechanism. To drop the restriction that the right-hand sides of conditions may not repeat variables, we would need the same, or alternatively a strategy which enforces that left-hand sides of conditions must always be reduced to normal form. Similar revisions could be used to extend the definition to take non-confluence into account, as discussed at the end of Section 3.

Alternatively, we could weaken the restrictions only partially, allowing for instance irreducible patterns—terms b such that for no instance $b\gamma$, a reduction step is possible at a position in $\mathcal{P}\mathrm{os}(b)$— as right-hand sides of conditions rather than only constructor terms.

Rules with branching conditions. Consider the following variant of $\mathcal{R}_{\mathsf{even}}$:

$$\mathsf{even}(0) \to \mathsf{true} \qquad\qquad (9.1) \qquad\quad \mathsf{odd}(0) \to \mathsf{false} \qquad\qquad (9.4)$$

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \approx \mathsf{true} \quad (9.2) \quad \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{even}(x) \approx \mathsf{true} \quad (9.5)$$

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{odd}(x) \approx \mathsf{false} \quad (9.3) \quad \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{even}(x) \approx \mathsf{false} \quad (9.6)$$

Unlike Example 3.1, rules (9.2) and (9.3), and rules (9.5) and (9.6) have very similar conditions. Currently, we do not exploit this. Evaluating $\mathsf{even}(\mathsf{s}^9(0))$ with rule (9.2) causes the calculation of the normal form false of $\mathsf{odd}(\mathsf{s}^8(0))$, before concluding that the rule does not apply. In our definitions (of $\rightharpoonup$ and Ξ), and in line with the behavior of Maude, we would dismiss the result and continue trying the next rule. In this case, that means recalculating the normal form of $\mathsf{odd}(\mathsf{s}^8(0))$, but now to verify whether rule (9.3) applies.

This is wasteful, as there is clearly no benefit in recalculating this normal form. The rules are defined in a *branching* manner: If the condition evaluation gives one result, we should apply rule (9.2); if it gives another, we should use rule (9.3). A clever rewriting engine could use this branching, and avoid recalculating obviously unnecessary results. Thus, future extensions of the complexity notion might take such *groupings* of rules into account.

Improving the transformation. With regard to the transformation Ξ, it is would be easy to obtain smaller resulting systems using various optimizations, such as reducing the set AP of anti-patterns using typing considerations, or leaving defined symbols untouched when they are only defined by unconditional rules.

Implementation and further complexity methods. The strength of our implementation—which relies simply on a transformation to unconditional complexity—is necessarily limited by the possibilities of existing complexity tools. Thus, we hope that, in the future, developers of complexity tools will branch out towards context-sensitive rewriting. Moreover, we encourage developers to add support for exponential upper bounds.

To take full advantage of the initial conditional setting, it would be ideal for complexity tools to directly support conditional rewriting. This would enable tools to use methods like Recipe C, which uses a max-interpretation to immediately eliminate a large number of rules—an interpretation which an automatic tool is unlikely to find by itself. It is likely that other, non-interpretation-based methods, can be optimized for the conditional setting as well.

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Appendix A. Proof of Theorem 5.12

Recall the statement of Theorem 5.12:

> Let $\mathcal{R}$ be a strong CCTRS and $s \in \mathcal{T}(\mathcal{G})$. If $\zeta(s)$ is terminating and there exists a context-sensitive reduction $\zeta(s) \to_{\Xi(\mathcal{R}),\mu}^* t$ for some t with cost N, then there exists a complexity-conscious reduction $s \rightharpoonup^* t'$ with cost at least N. If there exists an infinite $(\Xi(\mathcal{R}), \mu)$ reduction starting from $\zeta(s)$ then $s \xrightarrow{\infty}$.

In this appendix we present the proof. We fix a strong CCTRS $\mathcal{R}$, and corresponding signatures $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. In order to relate certain reduction sequences in $(\Xi(\mathcal{R}), \mu)$ to complexity-conscious reductions with $\rightharpoonup$, we start by defining an inverse of ζ.

Definition A.1. A term $s \in \mathcal{T}(\mathcal{H}, \mathcal{V})$ is *proper* if

- s is a variable, or
- $s = f(s_1, \ldots, s_n)$ with f a constructor symbol and proper subterms $s_1, \ldots, s_n$, or
- $s = f(s_1, \ldots, s_n, c_1, \ldots, c_{m_f})$ with f a defined symbol, proper subterms $s_1, \ldots, s_n$, and $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$.

We denote the set of all proper (ground) terms by $\mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ ($\mathcal{T}_{\mathrm{p}}(\mathcal{H})$). For proper terms s we define $\zeta^-(s) \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ as follows. If s is a variable then $\zeta^-(s) = s$, if $s = f(s_1, \ldots, s_n)$ with f a constructor then $\zeta^-(s) = f(\zeta^-(s_1), \ldots, \zeta^-(s_n))$, and if $s = f(s_1, \ldots, s_n, c_1, \ldots, c_{m_f})$ with f a defined symbol then $\zeta^-(s) = f_R(\zeta^-(s_1), \ldots, \zeta^-(s_n))$ for $R = \{\rho_i^f \mid c_i = \top\}$.

Note that $\bot$-patterns (Definition 5.2) are proper. The following lemma collects some easy properties of ζ^-.

Lemma A.2. (1) If $s \in \mathcal{T}(\mathcal{G}, \mathcal{V})$ then $\zeta(s) \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\zeta^-(\zeta(s)) = s$.
  (2) If $t \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ then $\zeta(\zeta^-(t)) = t$.
  (3) If $t \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\tau \colon \mathcal{V} \to \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ then $t\tau \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\zeta^-(t\tau) = \zeta^-(t)\tau_{\zeta^-}$ (where $\tau_{\zeta^-} = \zeta^- \circ \tau$).
  (4) If $u \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and $\tau \colon \mathcal{V} \to \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ then $\xi_\top(u)\tau \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\zeta^-(\xi_\top(u)\tau) = \mathsf{label}(u)\tau_{\zeta^-}$.
  (5) If $v \in \mathsf{AP}(u)$ for some linear constructor term u then $v \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\zeta^-(v)$ is a linear labeled normal form which does not unify with u.

Proof. The first three statements are proved by an obvious induction argument.
  (4) We have $\xi_\top(u) = \zeta(\mathsf{label}(u))$ by Lemma 5.10. From statements (3) and (1) we infer $\zeta(\mathsf{label}(u))\tau \in \mathcal{T}_{\mathrm{p}}(\mathcal{H}, \mathcal{V})$ and $\zeta^-(\zeta(\mathsf{label}(u))\tau) = \zeta^-(\zeta(\mathsf{label}(u)))\tau_{\zeta^-} = \mathsf{label}(u)\tau_{\zeta^-}$.
  (5) From the definition of AP it follows that v is a $\bot$-pattern and thus proper. By structural induction on v we easily obtain that $\zeta^-(v)$ is a linear labeled normal form which does not unify with u. $\square$

An important preliminary result is that terminating proper ground terms have a $\bot$-pattern as normal form. This allows us to eliminate f_i^j symbols in selected (sub)terms, which is crucial for transforming a $(\Xi(\mathcal{R}), \mu)$ reduction into a complexity-conscious reduction.

Lemma A.3. If $s \in \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ then any $(\Xi(\mathcal{R}), \mu)$ normal form of s is a $\bot$-pattern.

Proof. For the purpose of this proof, a ground term u in $\mathcal{T}(\mathcal{H})$ is said to be an *intermediate* term if

- $u = f(u_1, \ldots, u_n)$ with f a constructor symbol and intermediate arguments $u_1, \ldots, u_n$, or
- $u = f(u_1, \ldots, u_n, c_1, \ldots, c_{m_f})$ with f a defined symbol, $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$, and intermediate arguments $u_1, \ldots, u_n$, or
- $u = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_{j-1}, v]_i)\sigma$ with $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$ and intermediate terms v and $\sigma(y)$ for all $y \in \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_{j-1})$, whenever $\rho_i^f \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$ and $1 \leqslant j \leqslant k$. (Note that $v\sigma = v$ since intermediate terms are ground.)

We use $\mathcal{T}_\mathsf{i}(\mathcal{H})$ to denote the set of intermediate terms. The following properties are easily established:

(a) proper ground terms are intermediate terms,
(b) if u is proper and the domain of $\sigma \colon \mathcal{V} \to \mathcal{T}_\mathsf{i}(\mathcal{H})$ includes $\mathcal{V}\mathsf{ar}(u)$ then $u\sigma$ is an intermediate term,
(c) if u is proper and $u\sigma$ an intermediate term then $\sigma(x)$ is an intermediate term for every $x \in \mathcal{V}\mathsf{ar}(u)$.

Next we prove that intermediate terms are closed under $(\Xi(\mathcal{R}), \mu)$ reduction. So let $u \in \mathcal{T}_\mathsf{i}(\mathcal{H})$ and $u \to_{\Xi(\mathcal{R}),\mu} u'$. We use induction on the size of u.

- Suppose $u = f(u_1, \ldots, u_n)$ with f a constructor symbol and intermediate arguments $u_1, \ldots, u_n$. The reduction step from u to u' must take place in one of the arguments, so $u' = f(u_1, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_n)$ for some $1 \leqslant i \leqslant n$ with $u_i \to_{\Xi(\mathcal{R}),\mu} u_i'$. The term u_i' is intermediate according to the induction hypothesis. Hence u' is intermediate by definition.
- Suppose $u = f(u_1, \ldots, u_n, c_1, \ldots, c_{m_f})$ with f a defined symbol, $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$, and intermediate arguments $u_1, \ldots, u_n$. If the reduction step takes place in one of the arguments $u_1, \ldots, u_n$, we reason as in the preceding case. Suppose the step takes place at the root. We distinguish three subcases, depending on which kind of rule of $\Xi(\mathcal{R})$ is used.
  (1) If a rule of type (1_ρ) is used then $u' = \xi_\top(r)\sigma$ for some right-hand side of an unconditional rule $\ell \to r$ in $\mathcal{R}{\restriction}f$ such that $\ell\sigma = f(u_1, \ldots, u_n)$. From property (c) we infer that $\sigma(y)$ is intermediate for all $y \in \mathcal{V}\mathsf{ar}(\ell)$. Since $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$ and $\xi_\top(r)$ is proper by Lemma A.2(4), u' is intermediate by property (b).
  (2) If a rule of type (2_ρ) is used then $u = f(\ell_1\sigma, \ldots, \ell_n\sigma, c_1, \ldots, c_{m_f})$ such that $c_i = \top$ for some $1 \leqslant i \leqslant n$ with $\rho_i \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$ in $\mathcal{R}{\restriction}f$. We have $u' = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\xi_\top(a_1)]_i)\sigma$ and from property (c) we infer that $\sigma(y)$ is intermediate for all $y \in \mathcal{V}\mathsf{ar}(\ell)$. Since $\mathcal{V}\mathsf{ar}(a_i) \subseteq \mathcal{V}\mathsf{ar}(\ell)$, the term $\xi_\top(a_1)\sigma$ is intermediate by property (b) and thus also ground. Hence $u' = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\xi_\top(a_1)\sigma]_i)\sigma$, which is of the required shape to be intermediate.
  (3) The final possibility is that a rule of type (6_ρ) is used. In this case we have $u' = f(u_1, \ldots, u_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)$ for some $1 \leqslant i \leqslant m_f$. Since the arguments $u_1, \ldots, u_n$ are intermediate, u' is intermediate by definition.
- Suppose $u = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_{j-1}, v]_i)\sigma$. If the reduction step from u to u' takes place below the root, it must take place in $v\sigma = v$, due to restrictions on the replacement map μ. Hence the result follows from the induction hypothesis. Suppose the step takes place at the root. Note that the rule $\rho_i^f \colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$

must exist in $\mathcal{R}{\restriction}f$. We again distinguish three subcases, depending on which kind of rule of $\Xi(\mathcal{R})$ is used.

(1) If a rule of type (3_ρ) is used then $u = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_k]_i)\tau$, and $u' = \xi_\top(r)\tau$ for some substitution τ with $\mathsf{dom}(\tau) \subseteq \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_k)$. Hence $\ell_l\tau = \ell_l\sigma$ for all $1 \leqslant l \leqslant n$, $b_l\tau = b_l\sigma$ for all $1 \leqslant l < j = k$, and $v = b_k\tau$. From property (c) we infer that $\sigma(y) = \tau(y)$ is intermediate for all $y \in \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_k) \supseteq \mathcal{V}\mathsf{ar}(r)$. Hence u' is intermediate by property (b) since $\xi_\top(r)$ is proper by Lemma A.2(4).

(2) If a rule of type (4_ρ) is used then $u = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_j]_i)\tau$, $j < k$, and $u' = f_i^{j+1}(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_j, \xi_\top(a_{j+1})]_i)\tau$ for some substitution τ with $\mathsf{dom}(\tau) \subseteq \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_j)$. Hence $\ell_l\tau = \ell_l\sigma$ for all $1 \leqslant l \leqslant n$, $b_l\tau = b_l\sigma$ for all $1 \leqslant l < j$, and $v = b_j\tau$. Therefore, $u' = f_i^{j+1}(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_j, \xi_\top(a_{j+1}))\tau]_i\sigma$ and it suffices if $\xi_\top(a_{j+1})\tau$ is an intermediate term. This follows from $\mathcal{V}\mathsf{ar}(a_{j+1}) \subseteq \mathcal{V}\mathsf{ar}(\ell_1, \ldots, \ell_n, b_1, \ldots, b_j)$ in combination with Lemma A.2(4) and properties (b) and (c).

(3) The final possibility is that a rule of type (5_ρ) is used. In this case we have $u' = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)\sigma$ for some $1 \leqslant i \leqslant m_f$. Since the arguments $\ell_1\sigma, \ldots, \ell_n\sigma$ are intermediate, u' is intermediate by definition.

Now suppose that s has a normal form t in $(\Xi(\mathcal{R}), \mu)$. We already know that t is an intermediate term. So it suffices to show that intermediate terms in normal form are $\bot$-patterns. We show instead that any intermediate term t which is not a $\bot$-pattern is reducible, by induction on its size.

- Suppose $t = f(t_1, \ldots, t_n)$ with f a constructor symbol. One of the arguments, say t_i, is not a $\bot$-pattern. The induction hypothesis yields the reducibility of t_i. Since $i \in \mu(f)$, t is reducible as well.

- Suppose $t = f(t_1, \ldots, t_n, c_1, \ldots, c_{m_f})$ with f a defined symbol and $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$. If one of the terms $t_1, \ldots, t_n$ is not a $\bot$-pattern, we reason as in the previous case. Otherwise, $c_i = \top$ for some $1 \leqslant i \leqslant m_f$. Consider $\rho_i^f\colon f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$. If $f(t_1, \ldots, t_n)$ is an instance of $f(\ell_1, \ldots, \ell_n)$ then t is reducible by rule (1_ρ) or (2_ρ). If $f(t_1, \ldots, t_n)$ is not an instance of $f(\ell_1, \ldots, \ell_n)$ then, using the linearity of $f(l_1, \ldots, l_n)$, there exists an argument position $1 \leqslant j \leqslant n$ such that t_j is not an instance of ℓ_j. According to Lemma 5.5 t_j is an instance of an anti-pattern in $\mathsf{AP}(\ell_j)$. Consequently, t is reducible by rule (6_ρ).

- The final case is $t = f_i^j(\vec{\ell}, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_{j-1}, v]_i)\sigma$ with $c_1, \ldots, c_{m_f} \in \{\bot, \top\}$. Consider the intermediate subterm v. If v is not a $\bot$-pattern we reason as in the first case. If v is an instance of b_j then rule (4_ρ) is applicable. Otherwise, again using Lemma 5.5, v must be an instance of an anti-pattern in $\mathsf{AP}(\ell_j)$ and thus t is reducible by rule (5_ρ). $\qquad\square$

The restriction to proper terms in Lemma A.3 is essential. For instance, $\mathsf{even}(0, \bot, \bot, 0)$ and $\mathsf{even}_2^1(0, \bot, \mathsf{true}, \bot)$ are ground normal forms (w.r.t. Example 5.7) but not $\bot$-patterns.

We have reached the point where we can prove the main result, for terminating proper terms. Since a term whose subterms contain symbols f_i^j has no parallel in the labeled setting, the proof will require a fair bit of reshuffling; some steps must be postponed, while other subterms must be eagerly evaluated. This is all done in Lemma A.4.

In the following, $s \to^* t\,[N]$ or $s \rightharpoonup^* t\,[N]$ indicates a reduction of cost N.

Lemma A.4. *Let $s \in \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ be a terminating term, t a $\perp$-pattern, and $\sigma\colon \mathcal{V}\mathrm{ar}(t) \to \mathcal{T}(\mathcal{H})$. If $s \to^*_{\Xi(\mathcal{R}),\mu} t\sigma\,[N]$ then there exist a substitution $\tau\colon \mathcal{V}\mathrm{ar}(t) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers K and M with $K + M \geqslant N$ such that $\zeta^-(s) \rightharpoonup^* \zeta^-(t\tau)\,[K]$ and $t\tau \to^*_{\Xi(\mathcal{R}),\mu} t\sigma\,[M]$.*

Proof. We use induction on s with respect to $> := (\to_{\Xi(\mathcal{R}),\mu} \cup \rhd_\mu)^+$, which is a well-founded order on terminating terms. (Here $s \rhd_\mu t$ if t is a subterm of s occuring at an active position.) We distinguish a number of cases. First of all, if t is a variable then we can simply take $\tau = \{t \mapsto s\}$, $K = 0$, and $M = N$. Next suppose $s = f(s_1, \ldots, s_n)$ with f a constructor symbol. We have $\zeta^-(s) = f(\zeta^-(s_1), \ldots, \zeta^-(s_n))$ and $t = f(t_1, \ldots, t_n)$ with $s_i \to^*_{\Xi(\mathcal{R}),\mu} t_i\sigma\,[N_i]$ for all $1 \leqslant i \leqslant n$, such that $N = N_1 + \cdots N_n$. Fix i. Since $s \rhd_\mu s_i$, we can apply the induction hypothesis, resulting in a substitution $\tau_i\colon \mathcal{V}\mathrm{ar}(t_i) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers K_i and M_i with $K_i + M_i \geqslant N_i$ such that $\zeta^-(s_i) \rightharpoonup^* \zeta^-(t_i\tau_i)\,[K_i]$ and $t_i\tau_i \to^*_{\Xi(\mathcal{R}),\mu} t_i\sigma\,[M_i]$. Since $\perp$-patterns are linear by definition, the substitution $\tau := \tau_1 \cup \cdots \cup \tau_n$ is well-defined. Let $K = K_1 + \cdots + K_n$ and $M = M_1 + \cdots + M_n$. We clearly have $K + M \geqslant N$. Furthermore, $\zeta^-(s) \rightharpoonup^* f(\zeta^-(t_1\tau), \ldots, \zeta^-(t_n\tau)) = \zeta^-(t\tau)$ with cost K and $t\tau \to^*_{\Xi(\mathcal{R}),\mu} t\sigma\,[M]$.

The remaining case for s is $s = f(s_1, \ldots, s_n, c_1, \ldots, c_{m_f})$ with f a defined symbol. Let $R = \{\rho_i^f \mid c_i = \top\}$. We have $\zeta^-(s) = f_R(\zeta^-(s_1), \ldots, \zeta^-(s_n))$. If there is no root step in the reduction $s \to^*_{\Xi(\mathcal{R}),\mu} t\sigma$ then the result is obtained exactly as in the preceding case. So suppose the reduction contains a root step. We prove the following claim $(*)$:

> There exists a term $u \in \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ different from s and numbers A and B with $A + B \geqslant N$ such that $\zeta^-(s) \rightharpoonup^+ \zeta^-(u)\,[A]$ and $u \to^*_{\Xi(\mathcal{R}),\mu} t\sigma\,[B]$.

The statement of the lemma follows from $(*)$, as can be seen as follows. We have $s = \zeta(\zeta^-(s)) \to^*_{\Xi(\mathcal{R}),\mu} \zeta(\zeta^-(u)) = u$ by Lemma A.2(2) and Theorem 5.11. Since $s \neq u$ we must have $s > u$ and thus we can apply the induction hypothesis to $u \to^*_{\Xi(\mathcal{R}),\mu} t\sigma$. This yields a substitution $\tau\colon \mathcal{V}\mathrm{ar}(t) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers K and M with $K + M \geqslant B$ such that $\zeta^-(u) \rightharpoonup^* \zeta^-(t\tau)\,[K]$ and $t\tau \to^*_{\Xi(\mathcal{R}),\mu} t\sigma\,[M]$. Hence $\zeta^-(s) \rightharpoonup^* \zeta^-(t\tau)\,[A + K]$ and $(A + K) + M = A + (K + M) \geqslant A + B \geqslant N$.

To prove the claim, we distinguish a few subcases depending on which rule of $\Xi(\mathcal{R})$ is applied in the first root step.

(a) Suppose the first root step uses a rule of type (1_ρ) and let $\rho_i\colon \ell = f(\ell_1, \ldots, \ell_n) \to r$ be the originating rule in $\mathcal{R}{\restriction}f$. (So $c_i = \top$ and $i \in R$.) The reduction from s to $t\sigma$ has the shape

$$s \xrightarrow{>\epsilon}^* f(\ell_1\gamma, \ldots, \ell_n\gamma, c_1, \ldots, c_{m_f}) \xrightarrow{\epsilon} \xi_\top(r)\gamma \to^* t\sigma$$

for some substitution γ with $\mathrm{dom}(\gamma) \subseteq \mathcal{V}\mathrm{ar}(\ell)$. Fix $1 \leqslant j \leqslant n$ and let C_j be the cost of $s_j \to^* \ell_j\gamma$. Let $C = C_1 + \cdots + C_n$. From the induction hypothesis we obtain a substitution $\delta_j\colon \mathcal{V}\mathrm{ar}(\ell_j) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers K_j and M_j with $K_j + M_j \geqslant C_j$ such that $\zeta^-(s_j) \rightharpoonup^* \zeta^-(\ell_j\delta_j)\,[K_j]$ and $\ell_j\delta_j \to^* \ell_j\gamma\,[M_j]$. Because $f(\ell_1, \ldots, \ell_n)$ is linear, the substitution $\delta := \delta_1 \cup \cdots \cup \delta_n$ is well-defined. With help of Lemma A.2(3) we obtain $\zeta^-(s) \rightharpoonup^* f_R(\ell_1\delta_{\zeta^-}, \ldots, \ell_n\delta_{\zeta^-})\,[K]$. As $\ell_1, \ldots, \ell_n$ are constructor terms, the reductions $\ell_j\delta_j \to^* \ell_j\gamma\,[M_j]$ take place in the substitution part. Hence for every $x \in \mathcal{V}\mathrm{ar}(\ell)$ we have $x\delta \to^* x\gamma\,[M_x]$ such that $M := M_1 + \cdots + M_n = \sum\{M_x \mid x \in \mathcal{V}\mathrm{ar}(\ell)\}$ and $K + M \geqslant C$, where $K = K_1 + \cdots + K_n$.

After these preliminaries, we proceed as follows. Let $V = \mathcal{V}\mathsf{ar}(\ell) \setminus \mathcal{V}\mathsf{ar}(r)$. For every $x \in V$ we fix a $\bot$-pattern u_x such that $\gamma(x) \to^* u_x$. The existence of u_x is guaranteed by Lemma A.3 and the termination of $\gamma(x)$, which follows because $s \to^* \cdot \rhd_\mu \gamma(x)$. Define the substitution $\eta \colon \mathcal{V}\mathsf{ar}(\ell) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ as follows:

$$\eta(x) = \begin{cases} u_x & \text{if } x \in V \\ \delta(x) & \text{if } x \notin V \end{cases}$$

We divide M into $M_V = \sum\{M_x \mid x \in V\}$ and $M_{\bar{V}} = \sum\{M_x \mid x \notin V\} = M - M_V$. We have $\ell\delta \to^* \ell\eta$. Applying the induction hypothesis to this reduction (with $t = \ell\delta$ and empty substitution σ) yields $\zeta^-(\ell\delta) \rightarrowtail^* \zeta^-(\ell\eta)\,[L]$ for some $L \geqslant M_V$. Let $u = \xi_\top(r)\eta$. Lemma A.2 yields $\zeta^-(u) = \mathsf{label}(r)\eta_{\zeta^-}$. Hence $\zeta^-(s) \rightarrowtail^* \zeta^-(u)\,[A]$ with $A = K + L + 1$. We clearly have $s \neq u$. In order to conclude $(*)$, it remains to show that $u \to^* t\sigma\,[B]$ for some $B \geqslant N - A$. We have $u = \xi_\top(r)\delta$ due to the definitions of V and η. Hence $u \to^* \xi_\top(r)\gamma\,[D]$ for some $D \geqslant M_{\bar{V}}$ and thus $u \to^* t\sigma\,[B]$ with $B := D + N - (C+1) \geqslant M_{\bar{V}} + N - (C+1) \geqslant M_{\bar{V}} + N - (K+M+1) = N - (K + M_V + 1) \geqslant N - (K + L + 1) \geqslant N - A$.

(b) Suppose the first root step uses a rule of type (6_ρ) and let $f(\ell_1, \ldots, \ell_n)$ be the left-hand side of the rule in $\mathcal{R}$ that gave rise to this rule. The reduction from s to $t\sigma$ has the following shape:

$$s \xrightarrow{>\epsilon}{}^* f(u_1, \ldots, u_n, c_1, \ldots, c_{m_f}) \xrightarrow{\epsilon} f(u_1, \ldots, u_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i) \to^* t\sigma$$

with u_j an instance of an anti-pattern $v \in \mathsf{AP}(\ell_j)$, so $u_j = v\gamma$ for some substitution γ and fixed j. We have $s_i \to^* u_i$ for all $1 \leqslant i \leqslant n$. By postponing the steps in arguments different from j, we obtain

$$
\begin{aligned}
s \xrightarrow{\geqslant j}{}^* & \; f(s_1, \ldots, u_j, \ldots, s_n, c_1, \ldots, c_{m_f}) && [A] \\
\longrightarrow & \; f(s_1, \ldots, u_j, \ldots, s_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i) && [0] \\
\xrightarrow{>\epsilon} & \; f(u_1, \ldots, u_j, \ldots, u_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i) \to^* t\sigma && [N - A]
\end{aligned}
$$

Since $s \rhd_\mu s_j \to^* v\gamma$, we can apply the induction hypothesis to obtain a substitution $\delta \colon \mathcal{V}\mathsf{ar}(v) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers K and M with $K + M \geqslant A$ such that $\zeta^-(s_j) \rightarrowtail^* \zeta^-(v\delta)\,[K]$ and $v\delta \to^* v\gamma\,[M]$. Lemma A.2(5) yields $\zeta^-(v\delta) = \zeta^-(v)\delta_{\zeta^-}$ and from Lemma 5.5 we know that $\zeta^-(v)$ is a linear labeled normal form which does not unify with ℓ_j. Therefore

$$
\begin{aligned}
\zeta^-(s) \rightarrowtail^* & \; f_R(\zeta^-(s_1), \ldots, \zeta^-(v)\delta_{\zeta^-}, \ldots, \zeta^-(s_n)) && [K] \\
\xrightarrow{\bot} & \; f_{R \setminus \{\rho_i\}}(\zeta^-(s_1), \ldots, \zeta^-(v)\delta_{\zeta^-}, \ldots, \zeta^-(s_n)) && [0]
\end{aligned}
$$

The latter term equals $\zeta^-(u)$ where $u = f(s_1, \ldots, v\delta, \ldots, s_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)$. Furthermore,

$$
\begin{aligned}
u \to^* & \; f(s_1, \ldots, v\gamma, \ldots, s_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i) && [M] \\
\to^* & \; t\sigma && [N - A]
\end{aligned}
$$

Hence $\zeta^-(s) \rightarrowtail^+ \zeta^-(u)\,[K]$ and $u \to^* t\sigma\,[M + N - A]$ with $M + N - A \geqslant M + N - (K + M) = N - K$. Since $s \neq u$, this proves $(*)$.

(c) In the remaining case, the first root step in reduction from s to $t\sigma$ uses a rule of type (2_ρ). Let $\rho = \rho_i \colon \ell = f(\ell_1, \ldots, \ell_n) \to r \Leftarrow c$ Since t is a non-variable $\bot$-pattern, $t\sigma$ cannot have some f_j^i as root symbol. Hence the application of (2_ρ) will be followed

by (possibly zero) root steps of type (4_ρ), for $j = 1, \ldots, m - 1$, until either a step of type (3_ρ) with cost $Q = 1$ (when $m = k$) or a step of type (5_ρ) with cost $Q = 0$ is used at the root position. We have

$$[C] \qquad s \xrightarrow{>\epsilon}{}^* f(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [\top]_i) \gamma$$

$$[0] \qquad \xrightarrow{\epsilon} f_i^1(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [\xi_\top(a_1)]_i) \gamma \qquad\qquad (2_\rho)$$

$$[D_1] \qquad \xrightarrow{>\epsilon}{}^* f_i^1(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [b_1]_i) \gamma$$

$$[0] \qquad \xrightarrow{\epsilon} f_i^2(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \xi_\top(a_2)]_i) \gamma \qquad\qquad (4_\rho)$$

$$\qquad\qquad \xrightarrow{>\epsilon}{}^* \ldots$$

$$[0] \qquad \xrightarrow{\epsilon} f_i^m(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_{m-1}, \xi_\top(a_m)]_i) \gamma \qquad (4_\rho)$$

$$[D_m] \qquad \xrightarrow{>\epsilon}{}^* f_i^m(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f} \rangle [b_1, \ldots, b_{m-1}, v]_i) \gamma$$

$$[Q] \qquad \xrightarrow{\epsilon} w \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3_\rho) \text{ or } (5_\rho)$$

$$[E] \qquad \longrightarrow{}^* t\sigma$$

for some substitution γ, $\bot$-pattern v, ground term w, and numbers $C, D_1, \ldots, D_m, E$ such that $N = C + D_1 + \cdots + D_m + E + Q$. (Here we use the fact that b_j does not share variables with $\ell_1, \ldots, \ell_n, b_1, \ldots, b_{j-1}$, for $1 \leqslant j < m$. Moreover, b_m as well as members of $\mathsf{AP}(b_m)$ are $\bot$-patterns.) Like in case (a), we obtain a substitution $\delta \colon \mathcal{V}\mathsf{ar}(\ell) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$ and numbers K_j and M_j such that $\zeta^-(s_j) \rightharpoonup^* \zeta^-(\ell_j\delta) \, [K_j]$ and $\ell_j\delta \to^* \ell_j\gamma \, [M_j]$. Moreover, $K + M \geqslant C$ where $K = K_1 + \cdots + K_n$ and $M = M_1 + \cdots + M_n$. We now distinguish two cases, depending on whether (3_ρ) or (5_ρ) is used in the step to w.

- Suppose the step to w uses (3_ρ). In this case we have $Q = 1$, $v = b_m$ and $w = \xi_\top(r)\gamma$. Let $V = \mathcal{V}\mathsf{ar}(b_0, \ldots, b_m) \setminus \mathcal{V}\mathsf{ar}(a_1, \ldots, a_{m+1})$. (Recall that $b_0 = \ell$ and $a_{k+1} = r$). For every $x \in V$ we fix a $\bot$-pattern u_x such that $\gamma(x) \to^* u_x$. The existence of u_x is guaranteed by Lemma A.3 and the termination of $\gamma(x)$, which follows from $s \to^* \cdot \rhd_\mu \xi_\top(a_j)\gamma$ for all $1 \leqslant j \leqslant m$. We inductively define substitutions $\eta_0, \ldots, \eta_m$ with $\eta_j \colon \mathcal{V}\mathsf{ar}(b_0, \ldots, b_j) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$ as well as numbers $L_0, \ldots, L_m$ and G_x for all $x \in \mathcal{V}\mathsf{ar}(b_0, \ldots, b_m) \setminus V$ such that

  (a) $\eta_j(x) \to^* \gamma(x) \, [G_x]$ for all $0 \leqslant j \leqslant m$ and $x \in \mathcal{V}\mathsf{ar}(b_j) \setminus V$,

  (b) $\zeta^-(\ell\delta) \rightharpoonup^* \zeta^-(\ell\eta_0) \, [L_0]$ with $L_0 \geqslant M - \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_0) \setminus V\}$, and

  (c) $\zeta^-(\xi_\top(a_j)\eta_{j-1}) \rightharpoonup^* \zeta^-(b_j\eta_j) \, [L_j]$ with $L_j \geqslant D_j + \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_j)\} - \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_j) \setminus V\}$ for all $0 < j \leqslant m$.

  – Let $j = 0$. We define

  $$\eta_0(x) = \begin{cases} u_x & \text{if } x \in \mathcal{V}\mathsf{ar}(b_0) \cap V \\ \delta(x) & \text{if } x \in \mathcal{V}\mathsf{ar}(b_0) \setminus V \end{cases}$$

  We obtain $\eta_0(x) \to^* \gamma(x)$ for all $x \in \mathcal{V}\mathsf{ar}(b_0) \setminus V$ from $\ell\delta \to^* \ell\gamma$, and define G_x as the cost of this reduction. This establishes property (a). Applying the induction hypothesis to the reduction $\ell\delta \to^* \ell\eta_0$ (with $t = \ell\eta_0$ and σ the empty substitution) yields $\zeta^-(\ell\delta) \rightharpoonup^* \zeta^-(\ell\eta_0) \, [L_0]$ for some $L_0 \geqslant \sum\{\mathsf{cost}(\delta(x) \to^* \eta_0(x)) \mid x \in \mathcal{V}\mathsf{ar}(b_0) \cap V\}$. Note that $L_0 + \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_0) \setminus V\} \geqslant M$. Hence property (b) holds. Property (c) holds vacuously.

    – Consider $0 < j \leqslant m$. Because $\mathcal{V}\mathrm{ar}(a_j) \subset \mathcal{V}\mathrm{ar}(b_0, \ldots, b_{j-1}) \setminus V$ we obtain $\xi_\top(a_j)\eta_{j-1} \to^* \xi_\top(a_j)\gamma\,[G_j]$ for some $G_j \geqslant \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(a_j)\}$. (Equality need not hold if a_j is a non-linear term.) We apply the induction hypothesis to $\xi_\top(a_j)\eta_{j-1} \to^* \xi_\top(a_j)\gamma \to^* b_j\gamma[G_j + D_j]$, yielding a substitution $\delta_j \colon \mathcal{V}\mathrm{ar}(b_j) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$ and numbers L' and N' with $L' + N' \geqslant G_j + D_j$ such that $\zeta^-(\xi_\top(a_j)\eta_{j-1}) \rightharpoonup^* \zeta^-(b_j\delta_j)\,[L']$ and $b_j\delta_j \to^* b_j\gamma\,[N']$. We divide N' into $X + Y$ where

$$X = \sum\{\mathsf{cost}(\delta_j(x) \to^* \gamma(x)) \mid x \in \mathcal{V}\mathrm{ar}(b_j) \cap V\}$$

$$Y = \sum\{\mathsf{cost}(\delta_j(x) \to^* \gamma(x)) \mid x \in \mathcal{V}\mathrm{ar}(b_j) \setminus V\}$$

and define the substitution η_j as follows:

$$\eta_j(x) = \begin{cases} \eta_{j-1}(x) & \text{if } x \in \mathcal{V}\mathrm{ar}(b_0, \ldots, b_{j-1}) \\ u_x & \text{if } x \in \mathcal{V}\mathrm{ar}(b_j) \cap V \\ \delta_j(x) & \text{if } x \in \mathcal{V}\mathrm{ar}(b_j) \setminus V \end{cases}$$

Since b_j is a constructor term, from $b_j\delta_j \to^* b_j\gamma$ we infer $\eta_j(x) \to^* \gamma(x)$ for all $x \in \mathcal{V}\mathrm{ar}(b_j) \setminus V$, at a cost we can safely define as G_x. Hence property (a) holds. Property (b) holds vacuously. Note that $Y = \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(b_j) \setminus V\}$. Applying the induction hypothesis to $b_j\delta_j \to^* b_j\eta_j$ (with $t = b_j\eta_j$ and σ the empty substitution) yields $\zeta^-(b_j\delta_j) \rightharpoonup^* \zeta^-(b_j\eta_j)\,[Z]$ for some number

$$Z \geqslant \sum\{\mathsf{cost}(\delta_j(x) \to^* \gamma(x) \to^* u_x) \mid x \in \mathcal{V}\mathrm{ar}(b_j) \cap V\} \geqslant X$$

Let $L_j = L' + Z$. So $\zeta^-(\xi_\top(a_j)\eta_{j-1}) \rightharpoonup^* \zeta^-(b_j\eta_j)\,[L_j]$. We have

$$L_j \geqslant L' + X = L' + N' - Y \geqslant G_j + D_j - Y$$

$$\geqslant D_j + \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(a_j)\} - \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(b_j) \setminus V\}$$

    establishing property (c).

Let $\eta = \eta_m$. Since η coincides with η_j on $\mathcal{V}\mathrm{ar}(b_0, \ldots, b_j)$ for all $0 \leqslant j \leqslant m$, we obtain

$$\mathsf{label}(a_j)\eta_{\zeta^-} = \zeta^-(\xi_\top(a_j)\eta) \rightharpoonup^* \zeta^-(b_j\eta) = b_j\eta_{\zeta^-}\,[L_j]$$

for $1 \leqslant j \leqslant m$. Hence

$$\zeta^-(s) \rightharpoonup^* f_R(\ell_1\eta_{\zeta^-}, \ldots, \ell_n\eta_{\zeta^-}) \rightharpoonup \mathsf{label}(r)\eta_{\zeta^-}\,[A]$$

with $A = (K + L_0) + L_1 + \cdots + L_m + 1$. Let $u = \xi_\top(r)\eta$. Lemma A.2 yields $\zeta^-(u) = \mathsf{label}(r)\eta_{\zeta^-}$. To establish the claim $(*)$, it remains to show $u \to^* t\sigma\,[B]$ for some B such that $A + B \geqslant N$. Because $\mathcal{V}\mathrm{ar}(r) \subseteq \mathcal{V}\mathrm{ar}(b_0, \ldots, b_m) \setminus V$, we obtain

$$u = \xi_\top(r)\eta \to^* \xi_\top(r)\gamma = w \to^* t\sigma\,[B]$$

with $B \geqslant \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(r)\} + E$. We have

$$A + B \geqslant K + L_0 + L_1 + \cdots + L_m + \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(r)\} + E + 1$$
$$\geqslant (C - M) + (M - \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_0) \setminus V\}) + D_1 + \cdots + D_m$$
$$+ \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_1, \ldots, a_{m+1})\}$$
$$- \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_1, \ldots, b_m) \setminus V\} + E + 1$$
$$\geqslant C + D_1 + \cdots + D_m + \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_1, \ldots, a_{m+1})\}$$
$$- \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_0, \ldots, b_m) \setminus V\} + E + 1$$
$$\geqslant C + D_1 + \cdots + D_m + E + 1 = N$$

where the last inequality follows from the inclusion $(\mathcal{V}\mathsf{ar}(b_0, \ldots, b_m) \setminus V) \subseteq \mathcal{V}\mathsf{ar}(a_1, \ldots, a_{m+1})$.

- Suppose the step to w uses (5_ρ). In this case we have $Q = 0$, $v \in \mathsf{AP}(b_m)$ and $w = f(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)\gamma$. Let $V = \mathcal{V}\mathsf{ar}(b_1, \ldots, b_{m-1}, v) \setminus \mathcal{V}\mathsf{ar}(a_1, \ldots, a_m)$. For every $x \in V$ we fix a $\bot$-pattern u_x such that $\gamma(x) \to^* u_x$. The existence of u_x is guaranteed by Lemma A.3 and the termination of $\gamma(x)$, which follows from $s \to^* \cdot \rhd_\mu \xi_\top(a_j)\gamma$ for all $1 \leqslant j \leqslant m$. We inductively define substitutions $\eta_0, \ldots, \eta_m$ with $\eta_j \colon \mathcal{V}\mathsf{ar}(b_0, \ldots, b_j) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ for $1 \leqslant j < m$ and $\eta_m \colon \mathcal{V}\mathsf{ar}(v) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ as well as numbers $L_1, \ldots, L_m$ and G_x for all $x \in \mathcal{V}\mathsf{ar}(b_0, \ldots, b_{m-1}) \setminus V$ such that

  (a) $\eta_j(x) \to^* \gamma(x) [G_x]$ for all $0 \leqslant j \leqslant m$ and $x \in \mathcal{V}\mathsf{ar}(b_j) \setminus V$,

  (b) $\zeta^-(\xi_\top(a_j)\eta_{j-1}) \rightharpoonup^* \zeta^-(b_j\eta_j) [L_j]$ with $L_j \geqslant D_j + \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_j)\} - \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_j) \setminus V\}$ for all $0 < j < m$.

  (c) $\zeta^-(\xi_\top(a_m)\eta_{m-1}) \rightharpoonup^* \zeta^-(v\eta_m) [L_m]$ with $L_m \geqslant \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_m)\} + D_m$.

    – We define $\eta_0 = \delta$. We obtain $\eta_0(x) \to^* \gamma(x)$ for all $x \in \mathcal{V}\mathsf{ar}(\ell) = \mathcal{V}\mathsf{ar}(b_0) \setminus V$ from $\ell\delta \to^* \ell\gamma$, and define G_x as the cost of this reduction. This establishes property (a). Note that $\sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(b_0)\} = M$.

    – The case $0 < j < m$ is exactly the same as for (3_ρ), establishing properties (a) and (b).

    – For $j = m$ we have $\xi_\top(a_m)\eta_{m-1} \to^* \xi_\top(a_m)\gamma \to^* v\gamma$. Let G_m be the cost of $\xi_\top(a_m)\eta_{m-1} \to^* \xi_\top(a_m)\gamma$, so $G_m \geqslant \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_m)\}$. The induction hypothesis yields a substitution $\delta_m \colon \mathcal{V}\mathsf{ar}(v) \to \mathcal{T}_{\mathrm{p}}(\mathcal{H})$ and numbers L' and N' with $L' + N' \geqslant G_m + D_m$ such that $\zeta^-(\xi_\top(a_m)\eta_{m-1}) \rightharpoonup^* \zeta^-(v\delta_m) [L']$ and $v\delta_m \to^* v\gamma [N']$. We define the substitution η_m as follows:

$$\eta_m(x) = \begin{cases} \eta_{m-1}(x) & \text{if } x \in \mathcal{V}\mathsf{ar}(b_0, \ldots, b_{m-1}) \\ u_x & \text{if } x \in \mathcal{V}\mathsf{ar}(v) \end{cases}$$

    Applying the induction hypothesis to $v\delta_m \to^* v\eta_m$ (with $t = v\eta_m$ and σ the empty substitution) yields $\zeta^-(v\delta_m) \rightharpoonup^* \zeta^-(v\eta_m) [Z]$ for some number $Z \geqslant N'$. Let $L_m = L' + Z$. Thus, $\zeta^-(\xi_\top(a_m)\eta_m) \rightharpoonup^* \zeta^-(v\eta_m) [L_m]$. We have $L_m \geqslant L' + N' \geqslant G_m + D_m \geqslant \sum\{G_x \mid x \in \mathcal{V}\mathsf{ar}(a_m)\} + D_m$. Hence property (c) holds.

Let $\eta = \eta_m$. Since η coincides with η_j on $\mathcal{V}\mathrm{ar}(b_0, \ldots, b_j)$ for all $0 \leqslant j < m$, we obtain

- $\zeta^-(s) = f_R(\zeta^-(s_1), \ldots, \zeta^-(s_n)) \rightharpoonup^* f_R(\ell_1, \ldots, \ell_n)\eta_{\zeta^-}$ $[K]$
- $\mathsf{label}(a_j)\eta_{\zeta^-} = \zeta^-(\xi_\top(a_j)\eta) \rightharpoonup^* \zeta^-(b_j\eta) = \zeta^-(b_j)\eta_{\zeta^-}$ $[L_j]$ for $1 \leqslant j < m$
- $\mathsf{label}(a_m)\eta_{\zeta^-} = \zeta^-(\xi_\top(a_m)\eta) \rightharpoonup^* \zeta^-(v\eta) = \zeta^-(v)\eta_{\zeta^-}$ $[L_m]$, with $\zeta^-(v)$ a $\bot$-pattern that does not unify with v according to Lemma 5.5.

Let $u = f(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\bot]_i)\eta$. We have

$$\zeta^-(s) \rightharpoonup \zeta^-(f_{R\setminus\{\rho_i\}}(\zeta^-(s_1), \ldots, \zeta^-(s_n))) = \zeta^-(u) \ [K + L]$$

for $L = L_1 + \cdots + L_m$. Furthermore, $u \rightarrow^* w \rightarrow^* t\sigma \ [M + E]$. It remains to show that $K + L + M + E \geqslant N$. Since $K + M \geqslant C$, this amounts to showing $L \geqslant D_1 + \cdots + D_m$. We have

$$L \geqslant \sum_{j=1}^{m-1}\left(D_j + \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(a_j)\} - \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(b_j)\setminus V\}\right) + L_m$$

$$\geqslant \sum_{j=1}^{m} D_m + \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(a_1, \ldots, a_m)\}$$

$$- \sum\{G_x \mid x \in \mathcal{V}\mathrm{ar}(b_1, \ldots, b_{m-1})\setminus V\}$$

$$\geqslant \sum_{j=1}^{m} D_m$$

where the last inequality follows from the inclusion $\mathcal{V}\mathrm{ar}(b_1, \ldots, b_{m-1})\setminus V \subseteq \mathcal{V}\mathrm{ar}(a_1, \ldots, a_m)$. Since $s \neq u$, we established $(*)$. $\square$

Thus, we proved the main part of Theorem 5.12 for *terminating* terms. For non-terminating terms, we can use this result, as we will see in the proof of Lemma A.5. The following lemma handles the main step.

Lemma A.5. *For every minimal non-terminating term $s \in \mathcal{T}_\mathrm{p}(\mathcal{H})$ there exists a non-terminating term $t \in \mathcal{T}_\mathrm{p}(\mathcal{H})$ such that $\zeta^-(s) \rightharpoonup^+ \zeta^-(t)$ or $\zeta^-(s) \rightharpoonup^* \cdot \stackrel{\rhd}{\rightharpoonup} \zeta^-(t)$.*

Here a minimal non-terminating term is a non-terminating term with the property that every proper subterm at an active position is terminating.

Proof. We must have $s = f(s_1, \ldots, s_n, c_1, \ldots, c_{m_f})$ for some defined function symbol f. Let $R = \{\rho_i^f \mid c_i = \top\}$. We have $\zeta^-(s) = f_R(\zeta^-(s_1), \ldots, \zeta^-(s_n))$. Since the terms $s_1, \ldots, s_n$ are terminating by minimality, any infinite reduction starting at s must contain a root step. So

$$s \xrightarrow{>\epsilon}{}^* u\gamma \xrightarrow{\epsilon} v\gamma$$

for some rule $u \rightarrow v$ of $\Xi(\mathcal{R})$ and substitution γ such that $v\gamma$ is non-terminating. Inspecting the applicable rules in $\Xi(\mathcal{R})$, it follows that u is a linear basic term of the form $u = f(u_1, \ldots, u_n, \langle y_1, \ldots, y_{m_f}\rangle[\top]_i)$. Let δ be the restriction of γ to $\{y_1, \ldots, y_{m_f}\}$ We have $\delta(y_j) = c_j$ for all $1 \leqslant j \leqslant m_f$. Let $u' = u\delta$ and $v' = v\delta$. Clearly $u'\gamma = u\gamma$ and $v'\gamma = v\gamma$, while u' is a proper linear term. Because the terms $s_1, \ldots, s_n$ are terminating by minimality, Lemma A.4 provides substitutions $\tau_1, \ldots, \tau_n$ with $\tau_j \colon \mathcal{V}\mathrm{ar}(u_i) \rightarrow \mathcal{T}_\mathrm{p}(\mathcal{H})$ such that

$\zeta^-(s_j) \to^* \zeta^-(u_j\tau_j)$ and $u_j\tau_j \to^* u_j\gamma$. Since u is linear, the substitution $\tau = \tau_1 \cup \cdots \cup \tau_n$ is well-defined. We obtain

$$\zeta^-(s) \rightharpoonup^* \zeta^-(u'\tau) = \zeta^-(u')\tau_{\zeta^-} = f_R(\zeta^-(u_1), \ldots, \zeta^-(u_n))\tau_{\zeta^-}$$

with $\tau(x) \to^* \gamma(x)$ for all $x \in \mathcal{V}\mathrm{ar}(u')$. We now distinguish three cases, depending on the nature of the rule $u \to v$. Let $\rho_i\colon f(\ell_1, \ldots, \ell_n) \to r$ be the rule in $\mathcal{R}$ that give rise to $u \to v$.

(1) Suppose $u \to v$ is a rule of type (6_ρ). There exists $1 \leqslant j \leqslant n$ such that $u_j \in \mathsf{AP}(\ell_j)$. We have $v = f(u_1, \ldots, u_n, \langle x_1, \ldots, x_{m_f}\rangle[\bot]_i)$. According to Lemma A.2(5) $\zeta^-(u_j)$ is a linear labeled normal form which does not unify with ℓ_j. Hence

$$\zeta^-(u'\tau) \xrightarrow{\bot} f_{R\setminus\{\rho_i\}}(\zeta^-(u_1\tau), \ldots, \zeta^-(u_n\tau)) = \zeta^-(v'\tau)$$

Since all variables in v' are at active positions, we have $v'\tau \to^* v'\gamma = v\gamma$. It follows that $v'\tau$ is non-terminating and thus we can take $v'\tau$ for t to satisfy the first possibility of the statement of the lemma.

(2) Suppose $u \to v$ is a rule of type (1_ρ). So $u_j = \ell_j$ for all $1 \leqslant j \leqslant n$ and $v' = \xi_\top(r)$. Using Lemma A.2 we obtain $\zeta^-(u_j\tau) = u_j\tau_{\zeta^-}$ for $1 \leqslant j \leqslant n$ as well as $\zeta^-(v'\tau) = \mathsf{label}(r)\tau_{\zeta^-}$. Hence $\zeta^-(u'\tau) = f_R(u_1\tau_{\zeta^-}, \ldots, u_n\tau_{\zeta^-}) \rightharpoonup \zeta^-(v'\tau)$ and we conclude as in the preceding case.

(3) Suppose $u \to v$ is a rule of type (2_ρ). So $u_j = \ell_j$ for all $1 \leqslant j \leqslant n$ and $v' = f_i^1(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[\xi_\top(a_1)]_i)$. We have $\zeta^-(s) \rightharpoonup^* f_R(\ell_1, \ldots, \ell_n)\tau_{\zeta^-}$. We will define a number $1 \leqslant m \leqslant k$, substitutions $\tau_1, \gamma_1, \ldots, \tau_m, \gamma_m$, and terms $r_1, \ldots, r_m$ such that
  (a) $\tau_j\colon \mathcal{V}\mathrm{ar}(b_0, \ldots, b_{j-1}) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$,
  (b) $r_j = f_i^j(\ell_1, \ldots, \ell_n, \langle c_1, \ldots, c_{m_f}\rangle[b_1, \ldots, b_{j-1}, \xi_\top(a_j)]_i)$,
  (c) $\mathsf{label}(a_l)(\tau_j)_{\zeta^-} \rightharpoonup^* b_l(\tau_j)_{\zeta^-}$ for all $1 \leqslant l < j$,
  (d) $\tau_j(x) \to^* \gamma_j(x)$ for all $x \in \mathcal{V}\mathrm{ar}(b_0, \ldots, b_{j-1})$,
  (e) $r_j\gamma_j$ is non-terminating, and
  (f) $\zeta^-(s) \rightharpoonup^* f_R(\ell_1, \ldots, \ell_n)(\tau_j)_{\zeta^-}$

for all $1 \leqslant j \leqslant m$. By defining $\tau_1 = \tau$, $\gamma_1 = \gamma$, and $r_1 = v'$, the above properties are clearly satisfied for $j = 1$. Consider $\xi_\top(a_j)\tau_j$, which is a ground proper term by Lemma A.2(4). If $\xi_\top(a_j)\tau_j$ is non-terminating then we let $m = j$ and define $t = \xi_\top(a_j)\tau_j$. In this case we have $\zeta^-(t) = \mathsf{label}(a_j)(\tau_j)_{\zeta^-}$ by the same lemma and thus $f_R(\ell_1, \ldots, \ell_n)(\tau_j)_{\zeta^-} \xrightarrow{\rhd} \zeta^-(t)$ by property (c), establishing the second possibility of the statement of the lemma.

So assume that $\xi_\top(a_j)\tau_j$ is terminating. We have $\xi_\top(a_j)\tau_j \to^* \xi_\top(a_j)\gamma_j$, so the latter term is terminating as well. Since $\xi_\top(a_j)\gamma_j$ is the only active argument in $r_j\gamma_j$, the infinite reduction starting from the latter term must contain a root step. So $r_j\gamma_j \xrightarrow{\geqslant\epsilon} \ell'\gamma_{j+1} \xrightarrow{\epsilon} r'\gamma_{j+1}$ for some rule $\ell' \to r' \in \Xi(\mathcal{R})$ and substitution γ_{j+1} with $\mathsf{dom}(\gamma_{j+1}) = \mathcal{V}\mathrm{ar}(\ell')$ such that $r'\gamma_{j+1}$ is non-terminating. Since $\mathsf{root}(r_j\gamma_j) = f_i^j$, $\ell' = f_i^j(\ell_1, \ldots, \ell_n, \langle x_1, \ldots, x_{m_f}\rangle[b_1, \ldots, b_{j-1}, w]_i)$ for some $\bot$-pattern w ($w = b_j$ when $\ell' \to r'$ is a rule of type (3_ρ) or (4_ρ) and $w \in \mathsf{AP}(b_j)$ when $\ell' \to r'$ is a rule of type (5_ρ)) which has no variables in common with $\ell_1, \ldots, \ell_n, b_1, \ldots, b_{j-1}$. We have $\xi_\top(a_j)\tau_j \to^* \xi_\top(a_j)\gamma_j \to^* w\gamma_{j+1}$. From Lemma A.4 we obtain a substitution $\tau\colon \mathcal{V}\mathrm{ar}(w) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$ such that $\zeta^-(\xi_\top(a_j)\tau_j) \rightharpoonup^* \zeta^-(w\tau)$ and $w\tau \to^* w\gamma_{j+1}$. Let $\tau_{j+1} = \tau_j \cup \tau$. We have $\tau_{j+1}\colon \mathcal{V}\mathrm{ar}(b_0, \ldots, b_{j-1}, w) \to \mathcal{T}_\mathrm{p}(\mathcal{H})$ and $\zeta^-(\xi_\top(a_j)\tau_j) = \zeta^-(\xi_\top(a_j)\tau_{j+1}) = \mathsf{label}(a_j)(\tau_{j+1})_{\zeta^-}$ by Lemma A.2(4). Furthermore, $\tau_{j+1}(x) \to^*$

$\gamma_{j+1}(x)$ for all $x \in \mathcal{V}\mathsf{ar}(b_0, \ldots, b_{j-1}, w)$. We distinguish three subcases, depending of the type of the rule $\ell' \to r'$. In the first and third case, we obtain the statement of the lemma. In the second case, we establish the properties (a)–(f) for $j + 1$. Since rules of type (4_ρ) can be used only finitely many times, this concludes the proof.

(3_ρ) In this case we have $j = k$, $w = b_j$, and $r' = \xi_\top(r)$. So $\mathsf{label}(a_l)(\tau_{j+1})_{\zeta^-} \rightharpoonup^*$ $b_l(\tau_{j+1})_{\zeta^-}$ for all $1 \leqslant l \leqslant k$. Since all variables in $\xi_\top(r)$ occur at active positions, $r'\tau_{j+1} \to^* r'\gamma_{j+1}$ and thus $r'\tau_{j+1}$ is non-terminating. According to Lemma A.2(4) $r'\tau_{j+1}$ is proper and $\zeta^-(r'\tau_{j+1}) = \mathsf{label}(r)(\tau_{j+1})_{\zeta^-}$. So we choose $t = r'\tau_{j+1}$ to obtain a successful reduction step $f_R(\ell_1, \ldots, \ell_n)(\tau_{j+1})_{\zeta^-} \rightharpoonup \zeta^-(t)$. Hence $\zeta^-(s) \rightharpoonup^+ \zeta^-(t)$ and thus the first possibility of the statement of the lemma holds.

(4_ρ) In this case, $w = b_j$ and $r' = f_i^{j+1}(\ell_1, \ldots, \ell_n, \langle y_1, \ldots, y_{m_f} \rangle [b_1, \ldots, b_j, \xi_\top(a_{j+1})]_i)$ with $j < k$. We have $\mathsf{label}(a_l)(\tau_{j+1})_{\zeta^-} \rightharpoonup^* b_l(\tau_{j+1})_{\zeta^-}$ for all $1 \leqslant l < j + 1$. Let $r_{j+1} = r'\delta$. One easily checks that the properties (a)–(f) are satisfied for $j + 1$.

(5_ρ) In this case, $w \in \mathsf{AP}(b_j)$ and $r' = f(\ell_1, \ldots, \ell_n, \langle y_1, \ldots, y_{m_f} \rangle [\bot]_i)$. According to Lemma A.2, $\zeta^-(w)$ is a $\bot$-pattern which does not unify with b_j and $\zeta^-(w)(\tau_{j+1})_{\zeta^-} = \zeta^-(w\tau_{j+1})$. Since $\mathsf{label}(a_j)(\tau_{j+1})_{\zeta^-} \rightharpoonup^* \zeta^-(w)(\tau_{j+1})_{\zeta^-}$ and $\mathsf{label}(a_l)(\tau_{j+1})_{\zeta^-} \rightharpoonup^* b_l(\tau_{j+1})_{\zeta^-}$ for all $1 \leqslant l < j$, the conditions for a failing step are satisfied and thus $f_R(\ell_1, \ldots, \ell_n)(\tau_{j+1})_{\zeta^-} \rightharpoonup f_{R \setminus \{\rho_i\}}(\ell_1, \ldots, \ell_n)(\tau_{j+1})_{\zeta^-}$. Let $t = r'\delta\tau_{j+1}$. The term t is proper and since all variables in $r'\delta$ occur at active positions, $t \to^* r'\delta\gamma_{j+1}$ and thus t is non-terminating. Since $\zeta^-(t) = \zeta^-(r'\delta)(\tau_{j+1})_{\zeta^-} = f_{R \setminus \{\rho_i\}}(\ell_1, \ldots, \ell_n)(\tau_{j+1})_{\zeta^-}$, we obtain $\zeta^-(s) \rightharpoonup^+ \zeta^-(t)$ to satisfy the first possibility of the statement of the lemma. $\qquad\square$

Lemma A.6. *If* $s \in \mathcal{T}_\mathrm{p}(\mathcal{H})$ *is non-terminating then* $\zeta^-(s) \xrightarrow{\infty}$.

Proof. We construct an infinite sequence of non-terminating proper ground terms $s_0, s_1, s_2, \ldots$ with $s_0 = s$ such that $\zeta^-(s_i) \; (\rightharpoonup \cup \xrightarrow{\rhd})^+ \; \zeta^-(s_{i+1})$ for all $i \geqslant 0$. Suppose s_j has been defined. Since s_j is non-terminating, it contains a minimal non-terminating subterm u, say at position $p \in \mathcal{P}\mathsf{os}_\mu(s_j)$. According to Lemma A.5 there exists a non-terminating term $v \in \mathcal{T}_\mathrm{p}(\mathcal{H})$ such that $\zeta^-(u) \rightharpoonup^+ \zeta^-(v)$ or $\zeta^-(u) \rightharpoonup^* \cdot \xrightarrow{\rhd} \zeta^-(v)$. We distinguish three cases.

- If $\zeta^-(u) \rightharpoonup^+ \zeta^-(v)$ then $\zeta^-(s_i) = \zeta^-(s_i[u]_p) = \zeta^-(s_i)[\zeta^-(u)]_p$ by Lemma A.2(3) and thus $\zeta^-(s_i) \rightharpoonup^+ \zeta^-(s_i)[\zeta^-(v)]_p = \zeta^-(s_i[v]_p)$. Note that $s_i[v]_p$ is non-terminating. Hence we can take $s_{i+1} = s_i[v]_p$.
- Suppose $\zeta^-(u) \xrightarrow{\rhd} \zeta^-(v)$. We have $\zeta^-(s_i) = \zeta^-(s_i[u]_p) = \zeta^-(s_i)[\zeta^-(u)]_p$ and thus $\zeta^-(s_i) \xrightarrow{\rhd} \zeta^-(v)$ by the definition of $\xrightarrow{\rhd}$. Hence we define $s_{i+1} = v$.
- Suppose $\zeta^-(u) \rightharpoonup^+ w \xrightarrow{\rhd} \zeta^-(v)$. We have $\zeta^-(s_i) \rightharpoonup^+ \zeta^-(s_i)[w]_p \xrightarrow{\rhd} \zeta^-(v)$ and hence also in this case we take $s_{i+1} = v$. $\qquad\square$

Proof of Theorem 5.12. Let $\mathcal{R}$ be a strong CCTRS and $s \in \mathcal{T}(\mathcal{G})$. We have $\zeta(s) \in \mathcal{T}_\mathrm{p}(\mathcal{H})$ by Lemma A.2(1). First suppose that $\zeta(s)$ is terminating and there exists a context-sensitive reduction $\zeta(s) \to^*_{\Xi(\mathcal{R}),\mu} t \, [N]$. Let u be a normal form of t. Obviously, $\zeta(s) \to^*_{\Xi(\mathcal{R}),\mu} u \, [M]$ for some $M \geqslant N$. According to Lemma A.3 the term u is a $\bot$-pattern. Lemma A.4 yields $s = \zeta^-(\zeta(s)) \rightharpoonup^* \zeta^-(u) \, [K]$ with $K \geqslant M$. Next suppose the existence of an infinite $(\Xi(\mathcal{R}), \mu)$ reduction starting from $\zeta(s)$. In this case $s = \zeta^-(\zeta(s)) \xrightarrow{\infty}$ by Lemma A.6. $\qquad\square$