Formal Reasoning 2015
Uitwerkingen Test 2: Predicate logic
(23/09/15)

In the first two exercises we use the following interpretation:

M	the domain of men
W	the domain of women
k	Koos
j	Joris
$T(x, y)$	x is taller than y

1. Give for each of these sentences a formula of the predicate logic with equality:

   (a) *Koos and Joris are equally tall.* (10 points)

   $$(\neg T(k, j) \land \neg T(j, k))$$

   (b) *Men are not women.* (10 points)

   $$(\forall m \in M \neg (\exists w \in W \ (m = w)))$$

   (c) *There is exactly one man who is taller than Koos.* (10 points)

   $$(\exists m \in M \ (T(m, k) \land (\forall n \in M \ (T(n, k) \to (m = n)))))$$

   (d) *The tallest man is taller than the tallest woman.* (10 points)

   $$(\forall m \in M \ (\neg (\exists n \in M \, T(n, m)) \to (\forall w \in W \ (\neg (\exists v \in W \, T(W, v)) \to T(m, w)))))$$

2. Consider the following formula of the predicate logic with equality:

   $$\forall x, y \in W \, [x \neq y \to T(x, y) \lor T(y, x)]$$

   (a) Write this formula according to the official grammar in the course notes.
   (10 points)

   $$(\forall x \in W \ (\forall y \in W \ (\neg (x = y) \to (T(x, y) \lor T(y, x)))))$$

(b) Give an English sentence that resembles the meaning if this formula as well as possible. (15 points)

For each two arbitrary women it holds that either the first one is taller than the second one, or the second one is taller than the first one.

(Officially there should have been an 'or both' in this translation, but because both situations are mutually exclusive it makes no sense to add it.)

3. Within the propositional logic the concepts $\models f$, $f \models g$ and $f \equiv g$ are defined. In the course notes only the first two are defined within the predicate logic. Give a definition $f \equiv g$ within the predicate logic that is in line with the other two already known definitions. (15 points)

The statement $f \equiv g$ means that f and g are logically equivalent. There are at least three correct solutions possible:

- The statement can easily be defined using the known definition for logically true:

$$f \equiv g \text{ of and only if } \models f \leftrightarrow g$$

  This corresponds to exercise 1.H in the course notes for propositional logic.

- The statement can also be defined in terms of logical consequences:

$$f \equiv g \text{ if and only id } f \vDash g \text{ and } g \vDash f$$

- Finally, the definition can also be given directly in terms of models and interpretations: $f \equiv g$ if and only if f and g are true in exactly the same models under the same interpretations.

4. Give an interpretation I_4 in a model M_4 making the following formula true:

$$(\forall x \in D\, ((\exists y \in D\, \neg R(x,y)) \wedge ((\exists y \in D\, R(x,y)) \rightarrow (\forall y \in D\, R(x,y)))))$$

Explain your answer. (10 points)

This formula means:

For each x in D it holds that
there exists y in D such that $R(x,y)$ does not hold
and
if there exists some y in D for which $R(x,y)$ does hold,
then $R(x,y)$ holds for all y in D.

There are two ways to make this formula true:

- If D is the empty set, then we have a for-all-statement over an empty set which is true by definition.
- If R is a relation that is never true.

We give an explicit example of the second type of solution.

Let x be an arbitrary element in D. If the statement holds for this x both parts before and after $\wedge$ have to hold.

The first part can only be made true in one way, namely because D really contains an element y such that $R(x, y)$ does not hold.

The second part is an if-then-statement which is true if the if-part is not true or the then-part is true. However, if the then-part is true, this contradicts the part before the $\wedge$. So we have to make the part after the $\wedge$ true by making the if-part not true. This can be done by picking a model and interpretation such that there is no y in D such that $R(x, y)$.

Now take as model M_4

Domain(s)	Subsets of natural numbers
Relation(s)	smaller than (<)

and as interpretation I_4

D	$\{1\}$
$R(x, y)$	$x < y$

From this it follows that

$$(M_4, I_4) \models (\exists y \in D \, \neg R(1, y))$$

because our only choice is $y = 1$ and $1 < 1$ indeed does not hold, so $\neg(1 < 1)$ does hold. Furthermore we know:

$$(M_4, I_4) \not\models (\exists y \in D \, R(1, y))$$

because again our only choice is $y = 1$ and $1 < 1$ indeed does not hold. So

$$(M_4, I_4) \models ((\exists y \in D \, R(1, y)) \rightarrow (\forall y \in D \, R(1, y)))$$

holds automatically because of the truth table for the implication. And hence

$$(M_4, I_4) \models ((\exists y \in D \, \neg R(1, y)) \wedge ((\exists y \in D \, R(1, y)) \rightarrow (\forall y \in D \, R(1, y))))$$

But because 1 is the only element in D, we can derive from this that:

$$(M_4, I_4) \models (\forall x \in D \, ((\exists y \in D \, \neg R(x, y)) \wedge ((\exists y \in D \, R(x, y)) \rightarrow (\forall y \in D \, R(x, y)))))$$