Formal Reasoning 2016
Solutions Test Block 6: Additional Test
(11/01/17)

1. Give a formula f_1 of propositional logic such that:

$$\begin{aligned} \neg a \wedge \neg c &\models f_1 \\ \neg b \wedge c &\models f_1 \\ f_1 &\models \neg a \vee c \\ f_1 &\models \neg b \vee \neg c \end{aligned}$$

Explain your answer using a truth table.

The first two requirements indicate in which situations the formula f_1 has to be true and the last two requirements indicate in which situations the formula f_1 has to be false. More precisely, if the columns for $\neg a \wedge \neg c$ or $\neg b \wedge c$ have 1, then the column for f_1 has to be 1 also and if the columns for $\neg a \vee c$ or $\neg b \vee \neg c$ have 0, then the column for f_1 has to be 0 also. In other situations, we can safely choose f_1 to be 0, but the table below shows that there are no such situations.

a	b	c	$\neg a$	$\neg b$	$\neg c$	$\neg a \wedge \neg c$	$\neg b \wedge c$	$\neg a \vee c$	$\neg b \vee \neg c$	f_1
0	0	0	1	1	1	1	0	1	1	1
0	0	1	1	1	0	0	1	1	1	1
0	1	0	1	0	1	1	0	1	1	1
0	1	1	1	0	0	0	0	1	0	0
1	0	0	0	1	1	0	0	0	1	0
1	0	1	0	1	0	0	1	1	1	1
1	1	0	0	0	1	0	0	0	1	0
1	1	1	0	0	0	0	0	1	0	0

Colors are used to mark the relevant and irrelevant entries.

Now we have to find a formula that gives exactly this column for f_1. By using the so called *disjunctive normal form* we get the formula

$$f_1 = (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c) \vee (a \wedge \neg b \wedge c)$$

However, we can reduce this formula by drawing the Karnaugh diagram[1] for this formula:

	$\neg c$	c	
$\neg a$	1	1	$\neg b$
	1	0	b
a	0	0	
	0	1	$\neg b$

From this diagram it follows that we can reduce it by combining the ones that have the same color to the equivalent formula $f_1' = (\neg a \wedge \neg c) \vee (\neg b \wedge c)$.

2. Use the dictionary:

[1] The theory behind Karnaugh diagrams is not part of this course, but in general these diagrams can be used to find minimal but equivalent formulas.

V	vertices of a graph
$E(x, y)$	there is an edge between x and y

Give a formula f_2 of predicate logic with equality that formalizes the English sentence:

> *The graph is non-empty and each vertex in the graph has degree two.*

Let f_2 be the formula:

$$\exists v \in V\,[v = v]$$
$$\wedge$$
$$\forall v \in V\,[\exists x \in V\,[E(v, x) \wedge \exists y \in V\,[\neg(x = y) \wedge E(v, y) \wedge \forall z \in V\,[E(v, z) \to (z = x) \vee (z = y)]]]]$$

The first part expresses that there exists at least one vertex in V which indicates that the graph is non-empty. The second part expresses that each vertex has exactly two neighbours which indicates that the degree of each vertex is two.

3. Explain why a finite tree cannot be a model in which the formula f_2 from the previous exercise is true.

   A model in which the formula f_2 holds is nothing but a non-empty graph where each vertex has degree two. Such a graph cannot be a finite tree, which is a connected graph without cycles.

   Assume that our graph $\langle V, E\rangle$ is a finite tree with n vertices.

   - If $n = 0$ then V is empty, which is not allowed. So this cannot be the case.
   - If $n = 1$ then $V = \{v_1\}$ and $E = \emptyset$. Hence the degree of v_1 is zero, but acccording to f_2 it should be two. So this cannot be the case.
   - If $n = 2$ then $V = \{v_1, v_2\}$ and $E = \{(v_1, v_2)\}$. Hence the degree of v_1 is one, but acccording to f_2 it should be two. So this cannot be the case.
   - Now assume that $n \geq 3$ and $V = \{v_1, v_2, \ldots, v_n\}$. Because v_1 has degree two, we know that it has at least two neighbours. We may assume that v_2 is one of these neighbours.
   - But v_2 also has degree two, so besides v_1 it should have another neighbour v_3.
   - But v_3 also has degree two, so besides v_2 it should have another neighbour. This neighbour cannot be v_1, because that would lead to a cycle $v_1 \to v_2 \to v_3 \to v_1$. So it must be a 'fresh' neighbour v_4.
   - But v_4 als has degree two, so besides v_3 it should have another neighbour. This neighbour cannot be v_1 because that would lead to a cycle $v_1 \to v_2 \to v_3 \to v_4 \to v_1$. In addition, it cannot be v_2 because that would imply that the degree of v_2 is at least three and not two. So it must be a 'fresh' neighbour v_5.
   - If we continue this construction of our finite tree, we see that for every vertex v_i for $i \geq 3$ which has v_{i-1} as its neighbour, it must also have either v_1 as its neighbour or a 'fresh' neighbour v_{n+1}. However,

the first option leads to a cycle which is not allowed in a tree and the second option leads to an infinite number of vertices which cannot happen within a finite tree.

So the assumption that there is a finite tree $\langle V, E\rangle$ that satisfies formula f_2 always leads to contradictions. So this assumption cannot hold.

4. In this exercise we consider modal logic with Kripke semantics. We want to know whether two properties hold for all Kripke models $\mathcal{M}$ and all formulas f.

   (a) Does $\mathcal{M} \vDash f$ imply $\mathcal{M} \vDash \Box f$? Explain your answer.

   Yes, it does. Let $\mathcal{M} = \langle W, R, V\rangle$. If we assume $\mathcal{M} \vDash f$, it follows by definition that for each world $x \in W$ we have that $x \Vdash f$ holds. If we want to show that $\mathcal{M} \vDash \Box f$ holds, we have to show that for each world $x \in W$ we have that $x \Vdash \Box f$. Now let $x \in W$. Then there are two possibilities:

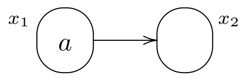
   - Either $R(x) = \emptyset$ and then $x \Vdash \Box f$ holds vacuously,
   - or $R(x) \neq \emptyset$, but then for any $y \in R(x)$ we have that $y \in W$ and hence $y \Vdash f$, so $x \Vdash \Box f$ also holds.

   So for each $x \in W$ it follows that $x \Vdash \Box f$, and hence $\mathcal{M} \vDash f$.

   (b) Does $\mathcal{M} \vDash f \to \Box f$ hold? Explain your answer.

   No, it does not. Take for instance $f = a$ and $\mathcal{M} = \mathcal{M}_4$ where

   $\mathcal{M}_4$:



   Then $x_1 \Vdash a$ because $a \in V(x_1)$. But because $x_2 \in R(x_1)$ and $a \notin V(x_2)$, we have that $x_1 \nVdash \Box a$. So $x_1 \nVdash a \to \Box a$. But then $\mathcal{M}_4 \nvDash a \to \Box a$. So $\mathcal{M} \vDash f \to \Box f$ does not hold for all models $\mathcal{M}$ and all formulas f.

5. We define the language L_5 as

$$L_5 := \{w \in \{a, b\}^* \mid w \text{ contains the substring } aba\}$$

Give a regular expression r_5 such that

$$\mathcal{L}(r_5) = \overline{L_5}$$

Explain your answer.

We need to give a regular expression that defines the complement of L_5. In other words, we need to define a language that describes all words over $\{a, b\}^*$ that do *not* contain the substring aba.

In particular this means that the amount of consecutive b's is always unlimited. The same holds for the amount of consecutive a's. We only have to worry about the transitions from a to b, because we need to prevent that such a transition can be followed by another a. In order to arrange this we simply require that such a transition is immediately followed by another b. So each a is either followed by another a or by at least two b's.

This can be achieved by the regular expression:

$$r_5 = b^*(a(\lambda \cup bbb^*)^*)^*b^*$$