Formal Reasoning 2021
Solutions Exam
(20/01/22)

There are six sections, with each three multiple choice questions and one open question. Each multiple choice question is worth 3 points, and the open questions are worth 6 points. The first ten points are free. Good luck!

Propositional logic: multiple choice questions

1. Which of the following formulas of propositional logic does not correctly formalize the meaning of the following English sentence

   It rains or it snows, but not both.

   The dictionary we use for this is

R	it rains
S	it snows

   (a) $(R \vee S) \wedge (\neg R \vee \neg S)$

   (b) $R \wedge \neg S \vee \neg R \wedge S$

   (c) $\neg(R \leftrightarrow S)$

   (d) is correct (d) $R \vee S \wedge \neg(R \wedge S)$

   Answer (d) is correct.

   Let us have a look at the formulas in detail

   (a) $(R \vee S) \wedge (\neg R \vee \neg S)$ means *It rains or it snows, and it doesn't rain or it doesn't snow.*, which implies via $R \vee S$ that at least one is true, but via $\neg R \vee \neg S$ it also implies that at least one of R and S is false, which is equivalent to saying that one of them must hold, but not both.

   (b) $R \wedge \neg S \vee \neg R \wedge S$ means *Either it rains and it doesn't snow, or it doesn't rain and it snows.*, so again at least one of R and S holds, but not both. Recall that if we make the implicit parentheses explicit, we get the formula $((R \wedge \neg S) \vee (\neg R \wedge S))$.

   (c) $\neg(R \leftrightarrow S)$ means *It is not the case that it rains if and only if it snows.*, which means again that at least one of R and S holds, but not both.

   (d) $R \vee S \wedge \neg(R \wedge S)$ means *It rains, or it snows and it is not the case that it is raining and snowing.*, which allows for R and S to be true at the same time, due to the fact that the part that states that they cannot be true at the same time is only part of the right side of the disjunction at the top level. Note that this option is wrong because the implicit parentheses are in the wrong place. The formula should be parsed as $(R \vee (S \wedge \neg(R \wedge S)))$. However, by adding two parentheses to the original formula like this $(R \vee S) \wedge \neg(R \wedge S)$, it would have been a correct formalization of the sentence.

2. Is the following a formula of propositional logic?

$$(a \to a) \to a$$

(a) Yes, and it does have the brackets according to the official grammar from the course notes.

(b) is correct

(b) Yes, but it does not have the brackets according to the official grammar from the course notes.

(c) No, a formula has to have the atom b as well.

(d) No, implication in propositional logic is right-associative.

Answer (b) is correct.

As each binary operator should get exactly two parentheses, it is clear that this formula lacks two parentheses. In particular the outer parentheses are missing, it should have been $((a \to a) \to a)$.

Note that there are no restrictions on the usage of atomic propositions, or the order of operators, as long as binary operators are between two subformulas and unary operators are in front of a single subformula.

3. Which formula should be put on the dots to get a true statement?

$$a \to b \equiv \ldots$$

(a) $\neg(\neg a \vee b)$

(b) $\neg(a \vee \neg b)$

(c) $\neg(\neg a \wedge b)$

(d) is correct

(d) $\neg(a \wedge \neg b)$

Answer (d) is correct.

Let us rewrite the formula using logical laws:

$$\begin{aligned} a \to b &\equiv \neg a \vee b \\ &\equiv \neg a \vee \neg\neg b \\ &\equiv \neg(a \wedge \neg b) \end{aligned}$$

Of course, it is also possible to create a truth table for all these options (note that we do not include all intermediate columns this time):

a	b	$a \to b$	$\neg(\neg a \vee b)$	$\neg(a \vee \neg b)$	$\neg(\neg a \wedge b)$	$\neg(a \wedge \neg b)$
0	0	1	0	0	1	1
0	1	1	0	1	0	1
1	0	0	1	0	1	0
1	1	1	0	0	1	1

As the column for $\neg(a \wedge \neg b)$ is the only column that is exactly the same as the column for $a \to b$, it follows that these two are logically equivalent, and the other formulas are not.

Propositional logic: open question

4. The truth table of a formula that contains the atomic propositions a, b and c has eight rows.

   Give a formula with atomic propositions a, b and c, of which the truth table has three ones and five zeroes, and give three different models v_1, v_2 and v_3 that correspond to these three ones.

   We can solve this in an ad-hoc way or we can solve this in a systematic way.

   Let us first give the systematic way, using the so-called disjunctive normal form, where each part coincides exactly with one of the models that make the formula true, by explicitly stating for each atomic proposition whether it should be true or false. So we could use for instance:

$$(\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge \neg c)$$

   to get a formula for which exactly the first three rows have a 1 and all other rows have a 0. This formula is true in the models v_1, v_2, and v_3, where

   - $v_1(a) = 0$, $v_1(b) = 0$, and $v_1(c) = 0$.
   - $v_2(a) = 0$, $v_2(b) = 0$, and $v_2(c) = 1$.
   - $v_3(a) = 0$, $v_3(b) = 1$, and $v_3(c) = 0$.

   Note that this formula is equivalent to $\neg a \wedge \neg(b \wedge c)$.

   A more ad-hoc approach would be by trying simple formulas, seeing whether they work, and adjusting them as needed. For instance, we could start with the formula a, which is true in exactly four models. So how can we get rid of this one model too much? By taking a conjunction with an independent formula that is true in exactly three models, for instance $b \rightarrow c$. Due to the conjunction and the usage of different atomic propositions we get that the formula $a \wedge (b \rightarrow c)$ is true in exactly three models v_1, v_2, and v_3, namely the models defined by

   - $v_1(a) = 1$, $v_1(b) = 0$, and $v_1(c) = 0$.
   - $v_2(a) = 1$, $v_2(b) = 0$, and $v_2(c) = 1$.
   - $v_3(a) = 1$, $v_3(b) = 1$, and $v_3(c) = 1$.

   With the systematic approach, you basically get the models for free, but the formulas are more complex.

Predicate logic: multiple choice questions

5. Consider the English sentence

   All swans are white.

   Using the dictionary

A	domain of animals
$S(x)$	x is a swan
$W(x)$	x is white

this can be formalized by the formula

$$\forall x \in A\,(S(x) \to W(x))$$

Note that the sentence given above means the same as

There are no non-white swans.

Which formula formalizes this second statement?

(a) $\neg\exists x \in A\,(S(x) \to \neg W(x))$

(b) is correct

(b) $\neg\exists x \in A\,(S(x) \wedge \neg W(x))$

(c) $\neg\exists x \in A\,(\neg S(x) \to \neg W(x))$

(d) $\neg\exists x \in A\,(\neg S(x) \wedge \neg W(x))$

Answer (b) is correct.

We derive the proper formula by rewriting using logical laws:

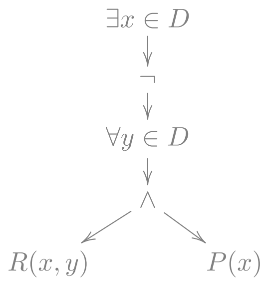
$$\begin{aligned}\forall x \in A\,(S(x) \to W(x)) &\equiv \forall x \in A\,(\neg S(x) \vee W(x))\\ &\equiv \forall x \in A\,(\neg S(x) \vee \neg\neg W(x))\\ &\equiv \forall x \in A\,\neg(S(x) \wedge \neg W(x))\\ &\equiv \neg\exists x \in A\,(S(x) \wedge \neg W(x))\end{aligned}$$

6. Which of the following trees correctly represents the structure of the following predicate logic formula:

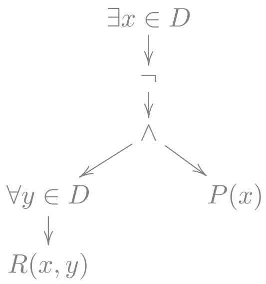
$$\exists x \in D\,(\neg\,\forall y \in D\,R(x,y) \wedge P(x))$$

Hint: The correct answer depends on the relative binding strength of the universal quantifier versus the conjunction.

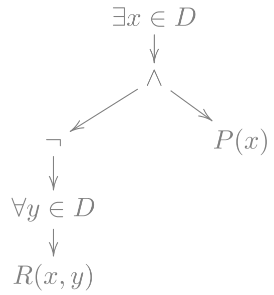
(a)



(b)



(c)



(c) is correct

(d) None of the above.

Answer (c) is correct.

Because the negation $\neg$ and the quantifier $\forall$ both bind stronger than the conjunction $\wedge$, the conjunction $\wedge$ needs to be above its left operand $\neg\forall y \in D\, R(x, y)$ and its right operand $P(x)$.

7. Consider the structure

$$\langle \mathbb{N}, 0, +, \cdot \rangle$$

and the formula

$$\forall x, y \in D\, (R(x, y) \to R(y, x))$$

(this is the property of R called *symmetry*). Under which interpretation is this formula not true in this structure?

(a)

D	$\mathbb{N}$
$R(x, y)$	$x + 0 = y$

(b)

D	$\mathbb{N}$
$R(x, y)$	$x \cdot 0 = y$

(b) is correct

(c)

D	$\mathbb{N}$
$R(x, y)$	$x + y = 0$

(d)

D	$\mathbb{N}$
$R(x, y)$	$x \cdot y = 0$

Answer (b) is correct.

The interpretation of $R(x, y)$ as $x \cdot 0 = y$ is not symmetric. Note that $R(1, 0)$ holds as $1 \cdot 0 = 0$, but $R(0, 1)$ does not hold as $0 \cdot 0 \neq 1$.

It is easy to see that all other interpretations are indeed symmetric.

Predicate logic: open question

8. Using the dictionary

H	domain of people
s	Sharon
$L(x, y)$	x loves y

formalize the meaning of the English sentence

Only Sharon is loved by everyone.

as a formula of predicate logic with equality.

Let us start by defining an abbreviation $LbE(x)$ for 'x is loved by everyone':

$$LbE(x) := \forall u \in H\, L(u, x)$$

Now we can apply the default pattern for 'only' and get the formula:

$$(LbE(s) \wedge (\forall y \in H\ (LbE(y) \to y = s)))$$

This default pattern coincides with the sentence *Sharon is loved by everyone and for each person who is loved by everyone it has to be Sharon.*

Of course, it is also possible to use the short form with the $\leftrightarrow$:

$$\forall y \in H\ (LbE(y) \leftrightarrow y = s)$$

If we expand the abbreviation, these two formulas become

$$((\forall u \in H\, L(u, s)) \wedge (\forall y \in H\ ((\forall u \in H\, L(u, y)) \to y = s)))$$

respectively

$$\forall y \in H\ ((\forall u \in H\, L(u, y)) \leftrightarrow y = s)$$

Languages: multiple choice questions

9. Is the statement

$$\lambda \in \varnothing$$

meaningful? To be more precise, can this be interpreted in such a way that it is indeed a statement within language theory that is either true or false?

(a) Yes, and this statement is true.

(b) is correct (b) Yes, but this statement is not true.

(c) No, λ and $\varnothing$ are regular expressions.

(d) No, $\varnothing$ does not have elements.

Answer (b) is correct.

If we interpret λ as the empty word and $\varnothing$ as the empty language, then it is indeed a statement: *The empty word is in the empty language.*

Obviously, as the empty language does not contain any words, it doesn't contain the empty word and hence the statement is not true.

10. Does the following equality hold?

$$\mathcal{L}(a^*b^*) = \mathcal{L}((ab)^*)$$

(a) Yes.

(b) No, but $\mathcal{L}(a^*b^*) \subset \mathcal{L}((ab)^*)$.

(c) No, but $\mathcal{L}((ab)^*) \subset \mathcal{L}(a^*b^*)$.

(d) is correct (d) No, and neither language is a subset of the other.

Answer (d) is correct.

Note that $a \in \mathcal{L}(a^*b^*)$, but $a \notin \mathcal{L}((ab)^*)$. So $\mathcal{L}(a^*b^*) \subset \mathcal{L}((ab)^*)$ does not hold.

Note also that $abab \in \mathcal{L}((ab)^*)$, but $abab \notin \mathcal{L}(a^*b^*)$. So the statement $\mathcal{L}((ab)^*) \subset \mathcal{L}(a^*b^*)$ also does not hold.

11. Consider the context-free grammar G_{11}:

$$S \to aA \mid \lambda$$
$$A \to bS$$

Someone wants to show that

$$aba \notin \mathcal{L}(G_{11})$$

using the property

$$P(w) := w \text{ has the same number of } a\text{'s and } b\text{'s}$$

in the hope that it is an invariant for this grammar. Does that work?

(a) Yes, as the property holds for all elements in $\mathcal{L}(G_{11})$.

(b) Yes, $S \to aA \to abS$, so in each production the number of a's and b's stays the same.

(c) No, this is not an invariant, as the production steps $aAa \to abSa \to aba$ show.

(d) is correct (d) No, this is not an invariant, as the production step $S \to aA$ shows.

Answer (d) is correct.

Note that $P(S)$ holds as S has zero a's and zero b's. However, from S we can produce aA in a single step. But $P(aA)$ does not hold as aA has one a and zero b's. So the invariance is broken by the rule $S \to aA$.

The third answer makes no sense because we should only look at single step productions and besides that, the property should hold before doing this step, but in this case, the property doesn't hold before any of the two production steps.

The first and the second answer make no sense because the answer is 'no'.

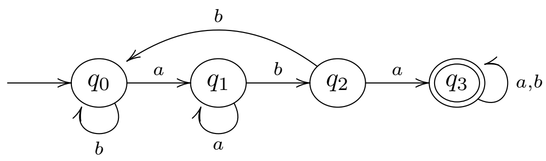
Languages: open question

12. Give a right-linear context-free grammar for the language

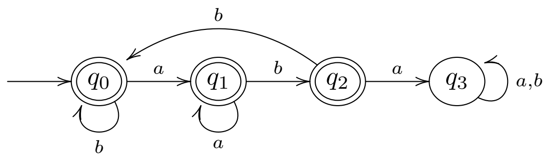
$$L_{12} := \{w \in \{a,b\}^* \mid w \text{ does not contain } aba\}$$

Hint: It may help to make a deterministic finite automaton on scratch paper for this language first and then derive a right-linear grammar for it. You only have to hand in the grammar!

First, we make a DFA that accepts only words containing aba:



And by inverting the final states and non-final states we get a DFA that accepts only words not containing aba:



By identifying state q_0 with non-terminal S, q_1 with A, q_2 with B, and q_3 with C, we can derive the following grammar that accepts the language L_{12}:

$$\begin{aligned} S &\to aA \mid bS \mid \lambda \\ A &\to aA \mid bB \mid \lambda \\ B &\to aC \mid bS \mid \lambda \\ C &\to aC \mid bC \end{aligned}$$

However, as C represents the sink, we can remove it completely and still get a grammar that accepts the same language L_{12}:

$$\begin{aligned} S &\to aA \mid bS \mid \lambda \\ A &\to aA \mid bB \mid \lambda \\ B &\to bS \mid \lambda \end{aligned}$$

And by substituting B in the second line we get an even shorter but still equivalent grammar:

$$\begin{aligned} S &\to aA \mid bS \mid \lambda \\ A &\to aA \mid bbS \mid b \mid \lambda \end{aligned}$$

Automata: multiple choice questions

13. Let $M = \langle \Sigma, Q, q_0, F, \delta \rangle$ be a non-deterministic finite automaton. Which statement is correct for each $q_i \in Q$ and $x \in \Sigma$?

    (a) $\delta(q_i, x) \in \Sigma$
    (b) $\delta(q_i, x) \subseteq \Sigma$
    (c) $\delta(q_i, x) \in Q$
    (d) $\delta(q_i, x) \subseteq Q$

    (d) is correct

    Answer (d) is correct.

    Recall that the transition function δ gives a set of states in an NFA. Hence it must be one of the options with the set of states Q and not one of the options with the alphabet Σ.

    In addition, because the result of δ is a set, the relation with Q has to be about being a subset ($\subseteq$) and not about being an element ($\in$).

14. Let be given an arbitrary deterministic finite automaton M. We want to make another deterministic finite automaton M', such that

    $$\mathcal{L}(M') = \overline{\mathcal{L}(M)}$$

    (so M' should recognize the complement of the language that M recognizes.) How can we do this?

    (a) M' is like M, but we change all the final states to a non-final state, and all non-final states to a final state.
    (b) M' is like M, but we reverse all transitions.
    (c) M' is like M, but we add a sink state.
    (d) M' is like M, but with the sink state removed.

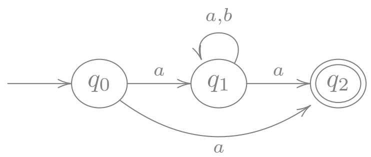
    (a) is correct

    Answer (a) is correct.

    The language $\overline{\mathcal{L}(M)}$ is the complement of the language $\mathcal{L}(M)$. And for a word w it holds that $w \in \overline{\mathcal{L}(M)}$ if and only $w \notin \mathcal{L}(M)$.

    However, in terms of a DFA M, it holds that $w \in \mathcal{L}(M)$ if and only if M ends in a final state after reading w. So if we combine these two things we get that $w \in \overline{\mathcal{L}(M)}$ if and only if w ends in a non-final state in M. So if we create M' by swapping the final and non-final states, we get that $w \in \overline{\mathcal{L}(M)}$ if and only if w ends in a final state in M', which means that $\overline{\mathcal{L}(M)} = \mathcal{L}(M')$.

15. Let be given the following non-deterministic finite automaton M_{15}:

    This recognizes the words that both start and end with the symbol a. How many states has a deterministic finite automaton that recognizes the same language as M_{15} and has a minimal number of states?

(a) Less than three states.

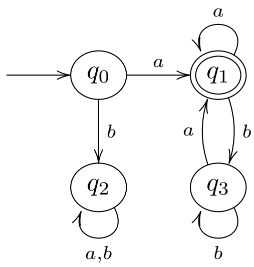
(b) Exactly three states.

(c) is correct (c) Exactly four states.

(d) More than four states.

Answer (c) is correct.

This automaton shows that it can be done with four states:

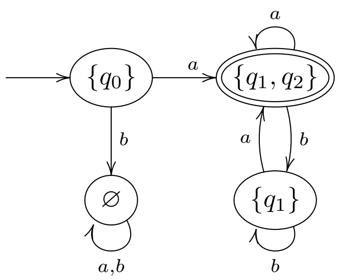


Now if we try to create an automaton that accepts this language with three states, we see that:

- The initial state q_0 is not a final state because λ is not in the language.
- If we read an a in q_0, we need to go to a final state because a is in the language. So we need a final state q_1.
- As words starting with b are not in the language, we need to go from q_0 to a sink q_2 when reading a b.
- State q_1 still needs an a arrow and a b arrow. However, if we read a b here, we must go to a non-final state, so either q_0, q_2, or a new fourth state q_3.
- If we go to q_0 or q_2, then the word *abba* ends up in the sink q_2, whereas it should have been accepted.
- So we need a fourth state q_3.

Hence it is not possible to have a DFA with less than four states that accepts this language.

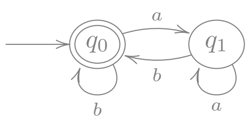
Note that applying the powerset algorithm would lead to the following automaton:



This automaton is apart from the names of the states, the same as the one we saw before.

Automata: open question

16. Let be given the deterministic finite automaton M_{16}:



List all words in $\mathcal{L}(M_{16})$ with length ≤ 3.

$$\lambda$$
$$b$$
$$ab, bb$$
$$aab, abb, bab, bbb$$

Note that besides λ these are all words ending on a b.

Discrete mathematics: multiple choice questions

17. The complete bipartite graphs $K_{m,n}$ with $m, n \geq 2$ have a Hamiltonian circuit if and only if ...

(a) is correct

(a) $m = n$.
(b) m and n are both even.
(c) m is even, n is even, or both.
(d) None of the above.

Answer (a) is correct.

In a Hamiltonian circuit, one has to visit all vertices exactly once and finish in the starting vertex.

Recall that $K_{m,n}$ is a complete bipartite graph with m red vertices and n blue vertices, where there are edges between all combinations of red and blue vertices, but there are never edges between vertices of the same color.

So if we have a Hamiltonian circuit, we may assume that we start in a red vertex, followed by a blue vertex, followed by a red vertex, and so on, ending in the same red vertex that we started from.

So this means that $m + n$ must be even, but also that $m = n$ since the vertices in the circuit change color every step.

Hence if such a Hamiltonian circuit exists, then $m = n$ must hold.

And if we assume that $m = n$, we have m red vertices and m blue vertices. So we may give them the names $r_1, r_2, \ldots, r_m$ and $b_1, b_2, \ldots, b_m$. And we can construct the following Hamiltonian circuit:

$$r_1 \to b_1 \to r_2 \to b_2 \to r_3 \to \cdots b_{m-1} \to r_m \to b_m \to r_1$$

The second answer makes no sense because if $m = 2$ and $n = 4$ you cannot put them all in a cycle where the color changes each step.

The third answer makes no sense because having one of m and n or even both of them to be even is not strong enough: ($m = 2$ and $n = 3$), ($m = 2$

and $n = 4$), ($m = 3$ and $n = 2$), and ($m = 4$ and $n = 2$) are all examples where you cannot put all vertices in a cycle where the color changes each step.

The fourth answer makes no sense because one of the answers above is correct.

18. We recursively define a function f with two natural number arguments, using the following recursion equations:

$$\begin{aligned} f(m, 0) &= m \\ f(m, n+1) &= m + f(m, n) \qquad\qquad \text{for all } n \geq 0 \end{aligned}$$

Which of the following is true?

(a) $f(m, n) = m + n$

(b) $f(m, n) = m \cdot n$

(c) is correct

(c) $f(m, n) = m \cdot (n + 1)$

(d) None of the above.

Answer (c) is correct.

We prove it by induction on n. Let m be an arbitrary natural number. We define our induction predicate as

$$P(n) \quad := \quad f(m, n) = m \cdot (n + 1)$$

We prove by (shorthand) induction that $P(n)$ holds for all $n \in \mathbb{N}$ with $n \geq 0$.

Base Case We have to prove that $P(0)$ holds, or more precisely, that $f(m, 0) = m \cdot (0 + 1)$. This holds because

$$f(m, 0) = m = m \cdot 1 = m \cdot (0 + 1)$$

Induction Step Now let k be an arbitrary natural number such that $k \geq 0$. We assume that $P(k)$ holds, or more precisely that $f(m, k) = m \cdot (k + 1)$ (IH). We now have to prove that $P(k + 1)$ holds, or more precisely that $f(m, k + 1) = m \cdot ((k + 1) + 1)$. This holds because

$$\begin{aligned} f(m, k+1) &= m + f(m, k) \\ &\overset{\text{IH}}{=} m + m \cdot (k + 1) \\ &= m \cdot ((k + 1) + 1) \end{aligned}$$

And hence $P(n)$ indeed holds for all $n \in \mathbb{N}$ such that $n \geq 0$.

19. The sum of the numbers in any row of Pascal's triangle is ...

(a) is correct

(a) A power of two.

(b) The factorial of a number.

(c) A Bell number.

(d) None of the above.

Answer (a) is correct.

On the $(n+1)^{\text{th}}$ row of Pascal's triangle, we have the binomial coefficients

$$\binom{n}{0}, \quad \binom{n}{1}, \quad \binom{n}{2}, \quad \dots, \quad \binom{n}{n}$$

Recall that Newton's Binomial Theorem gives

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

Now if we substitute $x = 1$ we get

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

And hence the sum of the $(n+1)^{\text{th}}$ a row is $(1+1)^n = 2^n$, a power of two.

The second answer makes no sense because on the third row of the triangle, the sum is $1+2+1 = 4$, which is not a factorial: 1, 2, 6, 24,.... You would get the factorials in the triangle for the Stirling numbers of the first kind.

The third answer makes no sense because on the third row of the triangle, the sum is $1+2+1 = 4$, which is not a Bell number: 1, 2, 5, 15, ... You would get the Bell numbers in the triangle for the Stirling numbers of the second kind.

The fourth answer makes no sense because one of the answers above is correct.

Discrete mathematics: open question

20. We define recursively:

$$\begin{aligned} a_0 &= 4 \\ a_{n+1} &= 2a_n + 1 \qquad \text{for all } n \geq 0 \end{aligned}$$

Prove by induction that $a_n = 5 \cdot 2^n - 1$ for all $n \geq 0$. Make sure to follow the template.

0 **Proposition:**

$$a_n = 5 \cdot 2^n - 1$$

for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \quad := \quad a_n = 5 \cdot 2^n - 1$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that
$a_0 = 5 \cdot 2^0 - 1$

4 This indeed holds, because $a_0 = 4 = 5 - 1 = 5 \cdot 1 - 1 = 5 \cdot 2^0 - 1$

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that $a_k = 5 \cdot 2^k - 1$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that $a_{k+1} = 5 \cdot 2^{k+1} - 1$

8 This indeed holds, because

$$\begin{array}{rcll} a_{k+1} & = & 2a_k + 1 & \text{definition of } a_{k+1} \\ & = & 2 \cdot (5 \cdot 2^k - 1) + 1 & \text{IH} \\ & = & (5 \cdot 2 \cdot 2^k - 2) + 1 & \text{elementary algebra} \\ & = & 5 \cdot 2^{k+1} - 1 & \text{elementary algebra} \end{array}$$

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

Modal logic: multiple choice questions

21. We use the dictionary

E	I exist

What is the meaning of the following formula of doxastic logic?

$$\Box E \wedge (\Box E \to E)$$

(a) is correct

(a) I believe that I exist, and therefore I exist.
(b) I know that I exist, and therefore I exist.
(c) I am obliged to exist, and therefore I exist.
(d) None of the above.

Answer (a) is correct.

Recall that doxastic logic is about belief.

If we translate the formula

$$\Box E \wedge (\Box E \to E)$$

to English with a doxastic interpretation, this becomes

I believe that I exist and if I believe that I exist, then I exist.

Note that the left part of the conjunction (*I believe that I exist*) states the premise of the implication (*I believe that I exist*) in the right part of the conjunction. So if we combine this, we know that the conclusion of the implication (*I exist*) also holds. And this means that the meaning of the formula is actually the same as the meaning of the sentence

I believe that I exist, and therefore I exist.

The second answer makes no sense because this is an epistemic interpretation.

The third answer makes no sense because this is a deontic interpretation.

The fourth answer makes no sense because one of the other answers is correct.

Note that this exercise is inspired on Descartes' *Cogito, ergo sum.*

22. Are all formulas of the form

$$\Box f \to \Diamond f$$

true in all reflexive Kripke models?

(a) Yes, as this formula is true in *all* Kripke models.

(b) is correct

(b) Yes: every world x will be its own successor, so $x \Vdash \Box f$ implies $x \Vdash \Diamond f$, as f will be true in x.

(c) No, reflexive models correspond to axiom scheme T, but this is axiom scheme D, which corresponds to serial Kripke models.

(d) No, there is a reflexive model in which $\Box f \to \Diamond f$ does not hold.

Answer (b) is correct.

So the question is whether $\vDash \Box f \to \Diamond f$ holds for all reflexive Kripke models or not. We prove this by showing that it holds for an arbitrary reflexive model $\mathcal{M}$, so we show that $\mathcal{M} \vDash \Box f \to \Diamond f$. However, this holds if for all worlds x_i in model $\mathcal{M}$ it holds that $x_i \Vdash \Box f \to \Diamond f$. We prove this by showing that it holds for an arbitrary world x in $\mathcal{M}$.

If we have to prove that $x \Vdash \Box f \to \Diamond f$, we may assume that $x \Vdash \Box f$, because otherwise the implication trivially holds in world x. Now $x \Vdash \Box f$ means that $y \Vdash f$ holds in all worlds that are accessible from x. However, because $\mathcal{M}$ is a reflexive model, we know that x is accessible from x, so we get that $x \Vdash f$. Now this implies that also $x \Vdash \Diamond f$ holds because there exists at least one world which is accessible from x where f holds, namely x itself. And combining $x \Vdash \Box f$ and $x \Vdash \Diamond f$ leads to $x \Vdash \Box f \to \Diamond f$.

So the formula holds in an arbitrary world x in model $\mathcal{M}$, which implies that the formula holds in all worlds of model $\mathcal{M}$. However, since $\mathcal{M}$ was an arbitrary reflexive Kripke model, this implies that the formula indeed holds in all reflexive Kripke models, which was what we had to prove.

The line of reasoning we presented here is essentially the argument given in answer (b), but with some more details.

The first answer makes no sense because the instance $\Box a \to \Diamond a$ does not hold in the non-reflexive Kripke model $\langle \{x_0\}, R, V \rangle$, where $R(x_0) = \varnothing$ and $V(x_0) = \varnothing$.

The third and the fourth answer make no sense because the answer is 'yes'.

23. Consider the LTL Kripke model $\langle W, R, V \rangle$ with

$$\begin{aligned} V(x_0) &= \{a\} \\ V(x_1) &= \varnothing \\ V(x_2) &= \{b\} \\ V(x_i) &= \varnothing && \text{for all } i \geq 3 \end{aligned}$$

In which worlds x_i does the following hold?

$$x_i \Vdash (\neg a)\, \mathcal{U}\, b$$

(a) In all worlds.

(b) In worlds x_0, x_1 and x_2.

(c) In world x_1.

(d) is correct (d) None of the above.

Answer (d) is correct.

The definition of $x_i \Vdash (\neg a)\,\mathcal{U}\,b$ is

> There is a $j \geq i$ such that $x_j \Vdash b$ and for all $k \in \{i, i{+}1, \ldots, j{-}1\}$ we have $x_k \Vdash \neg a$

Let us have a look at the first states:

- If $x_0 \Vdash (\neg a)\,\mathcal{U}\,b$ holds, there should be a $j \geq 0$ such that $x_j \Vdash b$. There is only one j with that property, namely $j = 2$. However, it should also be the case that $x_0 \Vdash \neg a$ and $x_1 \Vdash \neg a$. But $x_0 \Vdash \neg a$ does not hold because $a \in V(x_0)$. So $x_0 \Vdash (\neg a)\,\mathcal{U}\,b$ does not hold.
- If $x_1 \Vdash (\neg a)\,\mathcal{U}\,b$ holds, there should be a $j \geq 1$ such that $x_j \Vdash b$. There is only one j with that property, namely $j = 2$. However, it should also be the case that $x_1 \Vdash \neg a$. And that is indeed the case as $a \notin V(x_1)$. So $x_1 \Vdash (\neg a)\,\mathcal{U}\,b$ holds.
- If $x_2 \Vdash (\neg a)\,\mathcal{U}\,b$ holds, there should be a $j \geq 2$ such that $x_j \Vdash b$. There is only one j with that property, namely $j = 2$. However, there are no additional restrictions as the set $\{2, 3, \ldots, 1\}$ is empty! So $x_2 \Vdash (\neg a)\,\mathcal{U}\,b$ holds.
- If $x_3 \Vdash (\neg a)\,\mathcal{U}\,b$ holds, there should be a $j \geq 3$ such that $x_j \Vdash b$. However, there is no such j. So $x_3 \Vdash (\neg a)\,\mathcal{U}\,b$ does not hold.

Combining this leads to:

- The first answer makes no sense because the formula doesn't hold in world x_3.
- The second answer makes no sense because the formula doesn't hold in world x_0.
- The third answer makes no sense because although the formula indeed holds in world x_1, it is not stated that it also holds in world x_2.

So the fourth answer is correct.

Modal logic: open question

24. Give an LTL formula that expresses the requirements:

    - At each time a or b is true, but not both.
    - Whenever a is true, then b is true for the next two time instants, and after that a becomes true again.

The first requirement can be expressed by the formula

$$\mathcal{G}\,((a \vee b) \wedge \neg\,(a \wedge b))$$

The second requirement can be expressed by the formula

$$\mathcal{G}\,(a \rightarrow \mathcal{X}b \wedge \mathcal{X}\mathcal{X}b \wedge \mathcal{X}\mathcal{X}\mathcal{X}a)$$

So combined this gives

$$\mathcal{G}\,((a \vee b) \wedge \neg\,(a \wedge b)) \wedge \mathcal{G}\,(a \rightarrow \mathcal{X}b \wedge \mathcal{X}\mathcal{X}b \wedge \mathcal{X}\mathcal{X}\mathcal{X}a)$$

or even

$$\mathcal{G}\,((a \vee b) \wedge \neg\,(a \wedge b) \wedge (a \rightarrow \mathcal{X}b \wedge \mathcal{X}\mathcal{X}b \wedge \mathcal{X}\mathcal{X}\mathcal{X}a))$$

Of course, one could stress the fact that a and b are never true at the same time in the second requirement by the formula

$$\mathcal{G}((a \wedge \neg b) \rightarrow \mathcal{X}(b \wedge \neg a) \wedge \mathcal{X}\mathcal{X}(b \wedge \neg a) \wedge \mathcal{X}\mathcal{X}\mathcal{X}(a \wedge \neg b))$$

but that is not needed as it follows from the first requirement.