Formal Reasoning 2021
Solutions Test Block 1: Propositional and Predicate Logic
(27/09/21)

Propositional logic

1. Which of the following formulas correctly formalizes the following English sentence?

   I'm only free on Sunday

   Use for a dictionary:

F	I am free
S	it is Sunday

   (a) $F \to S$
   (b) $\neg S \to \neg F$
   (c) $\neg(F \land \neg S)$
   (d) is correct — (d) all of the above

   Answer (d) is correct. If we translate the formulas back into English and stress the actual situation a bit, we get:

   (a) $F \to S$: *If I am free, then it must be Sunday.*
   (b) $\neg S \to \neg F$: *If it is not Sunday, then I cannot be free.*
   (c) $\neg(F \land \neg S)$: *It is not the case that I can be free while it is not Sunday.*

   So they all coincide with *I'm only free on Sunday.* By using a truth table it is also easy to see that these three formulas are indeed logically equivalent:

F	S	$F \to S$	$\neg S$	$\neg F$	$\neg S \to \neg F$	$F \land \neg S$	$\neg(F \land \neg S)$
0	0	1	1	1	1	0	1
0	1	1	0	1	1	0	1
1	0	0	1	0	0	1	0
1	1	1	0	0	1	0	1

   Note that the three corresponding columns are exactly the same.

2. Which of the following statements holds?

   (a) $a \to b \to c \equiv (a \to b) \to c$
   (b) $a \to b \to c \vDash (a \to b) \to c$
   (c) is correct — (c) $(a \to b) \to c \vDash a \to b \to c$
   (d) all of the above

Answer (c) is correct. The solution follows from this truth table and from the fact that the implication is right associative:

a	b	c	$b \to c$	$a \to b \to c$	$a \to b$	$(a \to b) \to c$
0	0	0	1	1	1	0
0	0	1	1	1	1	1
0	1	0	0	1	1	0
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	0	1
1	1	0	0	0	1	0
1	1	1	1	1	1	1

Now it is clear that

(a) $a \to b \to c \equiv (a \to b) \to c$ does not hold since the columns for $a \to b \to c$ and $(a \to b) \to c$ are different, for instance in the first row, so these formulas are not logically equivalent.

(b) $a \to b \to c \models (a \to b) \to c$ does not hold since there are rows where the column of $a \to b \to c$ has a 1 and the column of $(a \to b) \to c$ has a 0. So the second formula is not a logical consequence of the first one.

(c) $(a \to b) \to c \models a \to b \to c$ does hold since, on every row where the first formula has a 1, the second formula also has a 1. So the second formula is indeed a logical consequence of the first one.

(d) all of the above: since only one of the above is correct, this cannot be right.

3. In which model is the formula $\neg(a \to b)$ true?

(a) $v(a) = 1 \quad v(b) = 1$

(b) $v(a) = 0 \quad v(b) = 1$

(c) $v(a) = 0 \quad v(b) = 0$

(d) is correct

(d) none of the above

Answer (d) is correct. We compute the value of $v(\neg(a \to b))$ for each of the models v:

(a) If $v(a) = 1$ and $v(b) = 1$, then $v(a \to b) = 1$, and hence $v(\neg(a \to b)) = 0$. So $\neg(a \to b)$ does not hold in this model.

(b) If $v(a) = 0$ and $v(b) = 1$, then $v(a \to b) = 1$, and hence $v(\neg(a \to b)) = 0$. So $\neg(a \to b)$ does not hold in this model.

(c) If $v(a) = 0$ and $v(b) = 0$, then $v(a \to b) = 1$, and hence $v(\neg(a \to b)) = 0$. So $\neg(a \to b)$ does not hold in this model.

(d) none of the above: this is correct because we have seen that the formula doesn't hold in the three models above.

4. Give a formula f that only uses the atomic propositions a and b and the connectives $\neg$ and $\wedge$, for which

$$f \equiv a \leftrightarrow b$$

Write your solution according to the official grammar from the course notes. We have the following series of logical equivalences, where we already write everything according to the grammar in the course notes:

$$\begin{array}{rcll} (a \leftrightarrow b) & \equiv & ((a \rightarrow b) \wedge (b \rightarrow a)) & \text{see for instance exercise 1.B} \\ & \equiv & ((\neg a \vee b) \wedge (\neg b \vee a)) & \text{applying } f \rightarrow g \equiv \neg f \vee g \text{ twice} \\ & \equiv & ((\neg a \vee \neg\neg b) \wedge (\neg b \vee \neg\neg a)) & \text{applying } f \equiv \neg\neg f \text{ twice} \\ & \equiv & (\neg (a \wedge \neg b) \wedge \neg (b \wedge \neg a)) & \text{applying De Morgan twice} \end{array}$$

So we can take

$$f := \underbrace{\left(\neg \underbrace{(a \wedge \neg b)}_{\wedge} \wedge \neg \underbrace{(b \wedge \neg a)}_{\wedge} \right)}_{\wedge}$$

which is the same formula as in the list of equivalences, but now with an explicit connection between the parentheses and the operators.

Predicate logic

5. Translate the following English sentence into a formula of predicate logic:

   Apart from humans, the only other animal to get sunburnt is the pig.

   Use for a dictionary:

A	domain of animals
$H(x)$	x is human
$P(x)$	x is a pig
$B(x)$	x is sunburnt

   Try to include all information from this sentence in your solution. For example, also represent the word ‘other’. You do not need to write your solution according to the official grammar from the course notes.

   This sentence contains a lot of information:

   - If an animal is sunburnt then it is a human or a pig.
   - Humans and pigs both actually *can* be sunburnt.
   - Humans are not pigs (‘the only *other* animal’; this was the hint in the exercise).
   - There are other animals besides humans and pigs (‘the *only* other animal’).

   How one translates the second of these properties depends on how one interprets the meaning of $B(x)$. If it means that the animal is sunburnt *right now*, then it just says that there are *some* sunburnt humans or pigs. If it means that the animal will be sunburnt when exposed to sufficient sunlight, then it says that *all* humans and pigs are sunburnt.

   A solution with the first interpretation of $B(x)$ is:

$$\begin{array}{l} \forall x \in A\,[B(x) \rightarrow H(x) \vee P(x)] \wedge \\ \exists x \in A\,[H(x) \wedge B(x)] \wedge \exists x \in A\,[P(x) \wedge B(x)] \wedge \\ \neg\exists x \in\ [H(x) \wedge P(x)] \wedge \exists x \in A\,[\neg H(x) \wedge \neg P(x)] \end{array}$$

The second interpretation allows for a bit simpler translation:

$$\begin{array}{l}\forall x \in A\,[B(x) \leftrightarrow H(x) \vee P(x)] \wedge \\ \neg\exists x \in\ [H(x) \wedge P(x)] \wedge \exists x \in A\,[\neg H(x) \wedge \neg P(x)]\end{array}$$

A solution that was given surprisingly often (probably because it matches the linguistic elements in the sentence well) was:

$$\forall x \in A\,[\neg H(x) \wedge B(x) \rightarrow P(x)]$$

But this is logically equivalent to:

$$\forall x \in A\,[B(x) \rightarrow H(x) \vee P(x)]$$

Therefore it only formalizes the first of the four parts of the sentence. (It might seem that the '$\neg H(x)$' part of the condition encodes the fact that humans and pigs do not overlap, but actually it does not.)

6. Someone wants to formalize the sentence

   There is an intelligent man that loves Sharon.

   using the dictionary from the course notes and gives the wrong answer

$$\exists x \in M\,\big(I(x) \rightarrow L(x,s)\big)$$

   accidentally using an implication instead of a conjunction. Which of the following formulas is logically equivalent to this wrong formula?

   (a) $\big(\exists x \in M\, I(x)\big) \rightarrow \big(\exists x \in M\, L(x,s)\big)$

(b) is correct

   (b) $\big(\exists x \in M\, \neg I(x)\big) \vee \big(\exists x \in M\, L(x,s)\big)$

   (c) $\neg\big(\exists x \in M\, I(x)\big) \vee \big(\exists x \in M\, L(x,s)\big)$

   (d) none of the above

Answer (b) is correct. Logically equivalent in predicate logic means that the truth values are the same, independent of the chosen structure and interpretation. So the fact that we know what M, s, $I(x)$, and $L(x,s)$ mean, should not be used in this answer.

We use 'logical laws' to rewrite the wrong formula into one of the options.

$$\begin{array}{ll}\multicolumn{2}{l}{\exists x \in M\,\big(I(x) \rightarrow L(x,s)\big)} \\ & \text{applying } f \rightarrow g \equiv \neg f \vee g \\ \equiv & \exists x \in M\,\big(\neg I(x) \vee L(x,s)\big) \\ & \text{applying } \exists x \in D\,(f \vee g) \equiv (\exists x \in D\, f) \vee (\exists x \in D\, g) \\ \equiv & \big(\exists x \in M\, \neg I(x)\big) \vee \big(\exists x \in M\, L(x,s)\big)\end{array}$$

Which is one of the options listed.

Note that the other two options listed, are actually logically equivalent to each other, by applying the equivalence $f \rightarrow g \equiv \neg f \vee g$. So it cannot be the case that one of these is correct. Either they are bot incorrect, or all three formulas are correct.

However $\exists x \in M\,\big(I(x) \rightarrow L(x,s)\big)$ is clearly not equivalent to $\big(\exists x \in M\, I(x)\big) \rightarrow \big(\exists x \in M\, L(x,s)\big)$. We can take as structure $(\mathbb{N}, 0, <)$ and as interpretation

$$\begin{aligned} M &\mapsto \mathbb{N} \\ s &\mapsto 0 \\ L(x,s) &\mapsto x < 0 \\ I(x) &\mapsto x \text{ is even} \end{aligned}$$

Then the first formula is true, because we can take $x = 3$. Then $I(3) \rightarrow L(3, s)$ means that if 3 is even, then $3 < 0$, which is true, because 3 is not even.

The second formula is false, because it means that if there is a natural number that is even, which is the case, take $x = 2$ for instance, then there exists a natural number that is smaller than 0, but those numbers do not exist of course.

7. Consider the structure $M := (\mathbb{N}, <)$ with the interpretation I that maps:

$$\begin{aligned} N &\mapsto \mathbb{N} \\ L(x,y) &\mapsto x < y \end{aligned}$$

Which of the following holds?

(a) is correct

(a) $(M, I) \models \forall x \in N\, \exists y \in N\, L(x, y)$
(b) $(M, I) \models \forall x \in N\, \exists y \in N\, L(y, x)$
(c) $(M, I) \models \exists x \in N\, \forall y \in N\, L(x, y)$
(d) $(M, I) \models \exists x \in N\, \forall y \in N\, L(y, x)$

Answer (a) is correct. The meaning with respect to this structure and this interpretation is:

(a) $(M, I) \models \forall x \in N\, \exists y \in N\, L(x, y)$: *For every natural number x, there exists a natural number y which is larger than x.* This is true, because we can take $y = x + 1$, which is a natural number and larger than x.
(b) $(M, I) \models \forall x \in N\, \exists y \in N\, L(y, x)$: *For every natural number x there exists a natural number y that is smaller than x.* This is not true for $x = 0$, because there are no natural numbers smaller than 0.
(c) $(M, I) \models \exists x \in N\, \forall y \in N\, L(x, y)$: *There exists a natural number x such that x is smaller than all natural numbers.* This is not true, because whatever natural number x we take, x is not smaller than itself, which should have been the case.
(d) $(M, I) \models \exists x \in N\, \forall y \in N\, L(y, x)$: *There exists a natural number x, such that all natural numbers y are smaller than x.* This is not true, because whatever natural number x we take, x is not larger than itself, which should have been the case.

8. Consider the formula

$$\forall x \in D\, (\exists y_1 \in D\; y_1 \neq x \wedge \exists y_2 \in D\; y_2 \neq x)$$

What is the number of elements that the interpretation of D in a model of this formula can contain?

(c) is correct

(a) two or more
(b) three or more
(c) zero, two, or more
(d) zero, three, or more

Answer (c) is correct. First, note that 'in a model of this formula' means that we only consider structures and interpretations where this formula holds. Note also that the quantifiers bind strongly, so if we make the invisible parentheses visible we get:

$$\big(\forall x \in D\, \big((\exists y_1 \in D\; y_1 \neq x) \wedge (\exists y_2 \in D\; y_2 \neq x)\big)\big)$$

Now we make a case distinction:

- If D is interpreted as an empty set, then each formula $\forall x \in D\, f$ holds, so also this one holds.
- If D is interpreted as a singleton set with, say, element a, then we would have to check whether

  $$(\exists y_1 \in D\; y_1 \neq a) \wedge (\exists y_2 \in D\; y_2 \neq a)$$

  holds or not. However, since it is a conjunction it would mean that for it to hold, both parts should hold, so in particular

  $$\exists y_1 \in D\; y_1 \neq a$$

  should hold. However, since D is interpreted as a singleton set containing only a there is no $y_1 \in D$ which is not equal to a. So this does not hold.
- If D is interpreted as a set with two elements, say a and b, then we would have to check whether

  $$(\exists y_1 \in D\; y_1 \neq a) \wedge (\exists y_2 \in D\; y_2 \neq a)$$

  and

  $$(\exists y_1 \in D\; y_1 \neq b) \wedge (\exists y_2 \in D\; y_2 \neq b)$$

  both hold simultaneously. But they do: for the first condition we can take $y_1 = b$ and $y_2 = b$, and for the second one $y_1 = a$ and $y_2 = a$. Note that there is no requirement that y_1 and y_2 are not the same!
- If D is interpreted as a set with more than two elements, say a, b, and $c_1, c_2, \ldots$, then we would have to check whether

  $$(\exists y_1 \in D\; y_1 \neq a) \wedge (\exists y_2 \in D\; y_2 \neq a)$$

  and

  $$(\exists y_1 \in D\; y_1 \neq b) \wedge (\exists y_2 \in D\; y_2 \neq b)$$

  and

  $$(\exists y_1 \in D\; y_1 \neq c_1) \wedge (\exists y_2 \in D\; y_2 \neq c_1)$$

  and

  $$(\exists y_1 \in D\; y_1 \neq c_2) \wedge (\exists y_2 \in D\; y_2 \neq c_2)$$

  and ... all simultaneously hold. Again, for the first requirement we can take $y_1 = y_2 = b$, for the second requirement we can take $y_1 = y_2 = a$. And for all other requirements, we can take again $y_1 = y_2 = a$.

So the formula holds for interpretations where D is mapped to a set with zero, two, or more elements.