Formal Reasoning 2021
Solutions Test Block 3: Discrete Mathematics and Modal Logic
(22/12/21)

Discrete Mathematics

1. The *cycle graph* C_n is defined for $n \geq 3$ as a connected graph with n vertices in which each vertex has degree two, or equivalently, as a graph with n vertices in which a cycle exists that is both Eulerian and Hamiltonian at the same time. What is the chromatic number of the graph C_n?

   (a) The chromatic number is always 2.

   (b) is correct — (b) The chromatic number is 2 if n is even, and 3 if n is odd.

   (c) The chromatic number is n.

   (d) None of the above is correct.

   Answer (b) is correct.

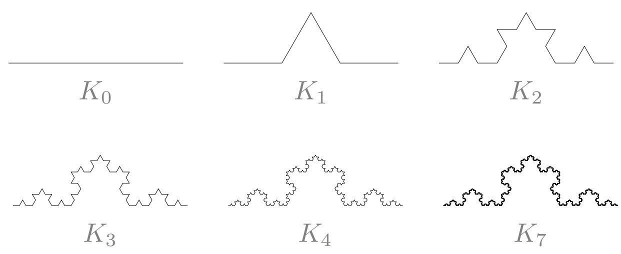
   All cycle graphs defined by the given definition are isomorphic to this one:

   $$C_n = \langle \{1, 2, \dots, n\}, \{(1, 2), (2, 3), (3, 4), \dots, (n-1, n), (n, 1)\} \rangle$$

   As $n \geq 3$ it follows immediately that the chromatic number is at least 2. And now we color all vertices with an odd label blue, and all vertices with an even label red. Note that this is the only coloring with two colors that has a chance of being a proper coloring where neighbors have different colors. By construction, it is clear that this is indeed a proper coloring if we omit the edge $(n, 1)$.

   - Now if we add this edge and n is even, then it is still a proper coloring since n is red and 1 is blue. So if n is even the chromatic number is 2.
   - Now if we add this edge and n is odd, then it is not a proper coloring since n and 1 are both blue. But we can solve this by coloring vertex n green. So if n is odd the chromatic number is 3.

   It is clear that none of the other options can be correct at the same time.

2. 

   The approximations K_n of the *Koch curve* are recursively defined as suggested by the pictures. The distance between the endpoints of each approximation is always 1, so for example the length of K_1 is $\frac{4}{3}$.

   Which of the following statements is true?

(a) is correct

(a) The length of the K_n is unbounded. When n goes to infinity, the length goes to infinity too. In particular, the length grows exponentially with n.
(b) The length of the K_n is unbounded. When n goes to infinity, the length goes to infinity too. However, the length does not grow exponentially with n.
(c) The length of the K_n is bounded. When n goes to infinity, the length approaches some value that can be considered to be 'the length of the Koch curve'.
(d) None of the above is correct.

Answer (a) is correct.

We define our induction predicate as

$$P(n) \quad := \quad |K_n| = \left(\tfrac{4}{3}\right)^n$$

where $|K_n|$ denotes the length of K_n. We prove by (shorthand) induction that $P(n)$ holds for all $n \in \mathbb{Z}$ with $n \geq 0$.

Base Case $P(0)$ holds because by definition $|K_0| = 1 = \left(\tfrac{4}{3}\right)^0$.

Induction Step Assume that $k \in \mathbb{Z}$ and $k \geq 0$ such that $P(k)$ holds. Hence $|K_k| = \left(\tfrac{4}{3}\right)^k$ (IH).

We now prove that $P(k+1)$ also holds. More precisely, we prove that $|K_{k+1}| = \left(\tfrac{4}{3}\right)^{k+1}$.

This holds because K_{k+1} is created from K_k by replacing each line segment by four line segments that are each three times as small. So the length of each line segment in K_k is multiplied by $\tfrac{4}{3}$. Hence

$$|K_{k+1}| = \tfrac{4}{3} \cdot |K_k| \overset{\text{IH}}{=} \tfrac{4}{3} \cdot \left(\tfrac{4}{3}\right)^k = \left(\tfrac{4}{3}\right)^{k+1}$$

So by induction we know that $|K_n| = \left(\tfrac{4}{3}\right)^n$ for all $n \in \mathbb{Z}$ with $n \geq 0$.

Hence in particular the length of K_n is unbounded and grows exponentially with factor $\tfrac{4}{3}$.

It is clear that none of the other options can be correct at the same time.

3. We define the sequence a_n recursively by:

$$\begin{aligned} a_0 &= 0 \\ a_{n+1} &= a_n + 2^{-(n+1)} \qquad \text{for } n \geq 0 \end{aligned}$$

For example, we have $a_0 = 0$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{4}$.

Use induction to prove that

$$a_n + 2^{-n} = 1$$

for all $n \geq 0$. Make sure you make all the elements of the induction proof explicit.

0 **Proposition:**

$$a_n + 2^{-n} = 1$$

for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \quad := \quad a_n + 2^{-n} = 1$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that

$a_0 + 2^{-0} = 1$

4 This indeed holds, because $a_0 = 0$ by definition and by elementary algebra $2^{-0} = \frac{1}{2^0} = \frac{1}{1} = 1$ and hence

$$a_0 + 2^{-0} = 0 + 1 = 1$$

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that $a_k + 2^{-k} = 1$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that $a_{k+1} + 2^{-(k+1)} = 1$

8 This indeed holds, because

$a_{k+1} + 2^{-(k+1)}$	$=$	$a_k + 2^{-(k+1)} + 2^{-(k+1)}$	definition of a_{k+1}
	$=$	$a_k + 2 \cdot 2^{-(k+1)}$	elementary algebra
	$=$	$a_k + 2^{-(k+1)+1}$	elementary algebra
	$=$	$a_k + 2^{-k-1+1}$	elementary algebra
	$=$	$a_k + 2^{-k}$	elementary algebra
	$=$	1	IH

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

4. The *Bell number* B_4 counts the ways in which we can partition four objects in non-empty groups. It can be computed as the sum of Stirling numbers. What is the value of this number B_4?

(a) is correct

(a) $B_4 = 15$

(b) $B_4 = 24$

(c) $B_4 = 52$

(d) None of the above is correct.

By definition

$$\begin{aligned} B_4 &= \left\{ {4 \atop 1} \right\} + \left\{ {4 \atop 2} \right\} + \left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 4} \right\} \\ &= 1 + 7 + 6 + 1 \\ &= 15 \end{aligned}$$

These Stirling numbers of the second kind can be found on the fourth row of this triangle:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 7 & & 6 & & 1 \end{array}$$

It is clear that none of the other options can be correct at the same time.

Modal Logic

5. Consider the two sentences:

   I don't know that it rains.
   I know it doesn't rain.

   We use a dictionary in which R formalizes 'it rains'.

   Can these sentences both be formalized by the same formula in epistemic logic?

   (a) Yes.

   (b) No.

   What are the proper formalizations of these two sentences in epistemic logic?

   (a) Both sentences should be formalized as $\neg\Box R$.

   (b) The first sentence should be formalized as $\neg\Box R$ and the second sentence should be formalized as $\Box\neg R$.

   (b) is correct

   (c) The first sentence should be formalized as $\Box\neg R$ and the second sentence should be formalized as $\neg\Box R$.

   (d) Both sentences should be formalized as $\Box\neg R$.

   Answer (b) is correct.

   The sentence *I don't know that it rains.* is more or less equivalent to the sentence *It is not the case that I know that it rains.* which translates to $\neg\Box R$. One could argue that the sentence should translate to a more precise formula $(\neg\Box R) \wedge R$ to stress the fact that apparently it actually rains, but that is not one of the options.

   The sentence *I know it doesn't rain.* clearly translates to $\Box\neg R$.

   It is clear that none of the other options can be correct at the same time.

6. Consider the three logics:

   - epistemic logic
   - doxastic logic
   - deontic logic

In how many of these three logics do all formulas of the form

$$\neg f \to \Diamond \neg f$$

hold?

Hint: Use logical laws to show this formula logically equivalent to some formula without negations.

(c) is correct

(a) In all three.
(b) In two of the three.
(c) In one of the three.
(d) In none of the three.

Answer (c) is correct.

This formula can be rewritten by:

$$\begin{aligned}
\neg f \to \Diamond\neg f &\equiv \neg\neg f \lor \Diamond\neg f && (\text{use } F \to G \equiv \neg F \lor G)\\
&\equiv f \lor \Diamond\neg f && (\text{use } \neg\neg F \equiv F)\\
&\equiv f \lor \neg\neg\Diamond\neg f && (\text{use } F \equiv \neg\neg F)\\
&\equiv \neg\neg\Diamond\neg f \lor f && (\text{use } F \lor G \equiv G \lor F)\\
&\equiv \neg\Diamond\neg f \to f && (\text{use } \neg F \lor G \equiv F \to G)\\
&\equiv \Box f \to f && (\text{use } \neg\Diamond\neg F \equiv \Box F)
\end{aligned}$$

Or, if you know contraposition, by:

$$\begin{aligned}
\neg f \to \Diamond\neg f &\equiv \neg f \to \neg\Box f && (\text{use } \Diamond\neg F \equiv \neg\Box F)\\
&\equiv \Box f \to f && (\text{use } \neg F \to \neg G \equiv G \to F)
\end{aligned}$$

So the formula is basically the axiom for *reflexivity*. If we interpret this axiom in the three logics we get:

- Epistemic: If I know that f holds, then f holds. This is considered to be true for all formulas f.
- Doxastic: If I believe that f holds, then f holds. This is not considered to be true for all formulas f, since people definitely can believe in things that do not hold.
- Deontic: If f ought to be done, then f is done. This is not considered to be true for all formulas f, since people definitely do not always do what is ought to be done.

So the formula is considered an axiom scheme in exactly one of these three logics.

It is clear that none of the other options can be correct at the same time.

7. We are looking for a Kripke model $\mathcal{M}$ for which:

$$\mathcal{M} \not\models \Diamond a \to \Box a$$

We wonder whether there exists a model that satisfies this that has only a single world. Is there a model like that with a single world?

(a) Yes.

(b) No.

And what is the case more precisely?

(a) There is a model like that with a single world. In fact, the formula holds in no Kripke model at all.

(b) There is a model like that with a single world, but there are models in which the formula holds too.

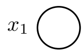
(c) is correct

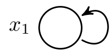
(c) There is no model like that with a single world, but there are models with more than one world in which the formula does *not* hold.

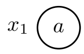
(d) There is no model like that with a single world. In fact, the formula holds in all Kripke models.

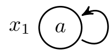
Answer (c) is correct.

If we have a Kripke model with only one world x_1 and a formula with only one atomic proposition a, we have only four possibilities, as we can vary between having no arrows or a single reflexive arrow, and independently, between $V(x_1) = \emptyset$ or $V(x_1) = \{a\}$.

$\mathcal{M}_1 \quad := $ 

$\mathcal{M}_2 \quad := $ 

$\mathcal{M}_3 \quad := $ 

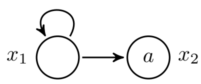
$\mathcal{M}_4 \quad := $ 

For these models we get the following $\Vdash$-table:

model	world $\Vdash$	a	$\Diamond a$	$\Box a$	$\Diamond a \to \Box a$
$\mathcal{M}_1$	x_1	0	0	1	1
$\mathcal{M}_2$	x_1	0	0	0	1
$\mathcal{M}_3$	x_1	1	0	1	1
$\mathcal{M}_4$	x_1	1	1	1	1

So we see that in each of the models i for $i = 1, 2, 3, 4$ that $\mathcal{M}_i, x_1 \Vdash \Diamond a \to \Box a$ holds. So $\mathcal{M}_i \models \Diamond a \to \Box a$ holds also for all i. Hence in all of the possible models with one world that exist, the formula $\Diamond a \to \Box a$ holds. So there is no model $\mathcal{M}$ with a single world such that $\mathcal{M} \not\models \Diamond a \to \Box a$.

However, there does exist a model $\mathcal{M}_5$ with two worlds for which the formula does not hold:

$\mathcal{M}_5 \quad :=$ 

For this model we get the following $\Vdash$-table:

model	world $\Vdash$	a	$\Diamond a$	$\Box a$	$\Diamond a \to \Box a$
$\mathcal{M}_5$	x_1	0	1	0	0
	x_2	1	0	1	1

So we see that $\mathcal{M}_5, x_1 \nVdash \Diamond a \to \Box a$ and hence $\mathcal{M}_5 \not\models \Diamond a \to \Box a$.

It is clear that none of the other options can be correct at the same time.

8. Give a Kripke model for the LTL formula

$$(\mathcal{G}\, a) \wedge ((\neg a)\, \mathcal{U}\, b)$$

(without further explanation) or explain why such a model does not exist. Recall that LTL Kripke models $\langle W, R, V \rangle$ have a fixed Kripke frame

$$W = \{x_i \mid i \in \mathbb{N}\}$$
$$R(x_i) = \{x_j \mid j \geq i\}$$

and therefore, if you think that such a model exists, in this exam an LTL model should be defined by just giving $V(x_i)$ for all worlds $x_i \in W$.

Take the LTL Kripke model defined by this valuation:

$$V(x_i) = \{a, b\} \quad \text{for all } i \in \mathbb{N}$$

We need a model $\mathcal{M}$ such that

$$\mathcal{M} \vDash (\mathcal{G}\, a) \wedge ((\neg a)\, \mathcal{U}\, b)$$

so we need a model $\mathcal{M}$ such that

$$x_i \Vdash (\mathcal{G}\, a) \wedge ((\neg a)\, \mathcal{U}\, b)$$

for all $i \in \mathbb{N}$. This means that we need to have that $x_i \Vdash (\mathcal{G}\, a)$ for all $i \in \mathbb{N}$ and $x_i \Vdash (\neg a)\, \mathcal{U}\, b$ for all $i \in \mathbb{N}$. The first requirement is clearly met as $a \in V(x_i) = \{a, b\}$ for all $i \in \mathbb{N}$. The second requirement is also met, but is a bit more tricky. The definition of

$$x_i \Vdash (\neg a)\, \mathcal{U}\, b$$

states that there exists a $j \geq i$ such that $x_j \Vdash b$ and for all $k \in \{i, i+1, \ldots, j-1\}$ we have $x_k \Vdash \neg a$. The trick is that we can take $j = i$, which is counterintuitive to the natural language. Because $b \in V(x_j) = V(x_i) = \{a, b\}$ for all $i \in \mathbb{N}$, the requirement that $x_j \Vdash b$ is met. And because $j = i$ the second requirement, which looks like a contradiction with the $\mathcal{G}a$ requirement, is actually an empty requirement as there is no $k \in \{i, i+1, \ldots, i-1\} = \emptyset$. So $x_i \Vdash \neg a\, \mathcal{U}\, b$ indeed holds for all $i \in \mathbb{N}$.

Hence the formula holds in all worlds of the model $\mathcal{M}$ and hence the formula holds in the model $\mathcal{M}$ itself.