Formal Reasoning 2021
Test Block 3: Discrete Mathematics and Modal Logic
(22/12/21)

There are six multiple choice questions and two open questions (questions 3 and 8). Each multiple choice question is worth 10 points, and the open questions are worth 15 points each. The mark for this test is the number of points divided by ten, and the first ten points are free. Good luck!

Discrete Mathematics

1. The *cycle graph* C_n is defined for $n \geq 3$ as a connected graph with n vertices in which each vertex has degree two, or equivalently, as a graph with n vertices in which a cycle exists that is both Eulerian and Hamiltonian at the same time. What is the chromatic number of the graph C_n?

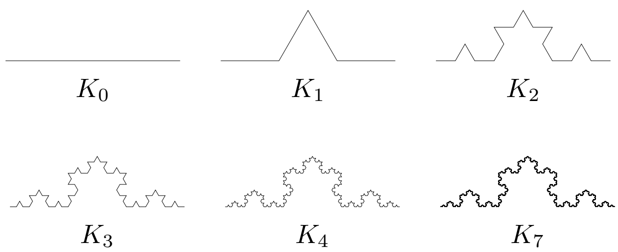
   (a) The chromatic number is always 2.

   (b) The chromatic number is 2 if n is even, and 3 if n is odd.

   (c) The chromatic number is n.

   (d) None of the above is correct.

2.



   The approximations K_n of the *Koch curve* are recursively defined as suggested by the pictures. The distance between the endpoints of each approximation is always 1, so for example the length of K_1 is $\frac{4}{3}$.

   Which of the following statements is true?

   (a) The length of the K_n is unbounded. When n goes to infinity, the length goes to infinity too. In particular, the length grows exponentially with n.

   (b) The length of the K_n is unbounded. When n goes to infinity, the length goes to infinity too. However, the length does not grow exponentially with n.

   (c) The length of the K_n is bounded. When n goes to infinity, the length approaches some value that can be considered to be 'the length of the Koch curve'.

   (d) None of the above is correct.

3. We define the sequence a_n recursively by:

$$a_0 = 0$$
$$a_{n+1} = a_n + 2^{-(n+1)} \qquad \text{for } n \geq 0$$

For example, we have $a_0 = 0$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{3}{4}$.

Use induction to prove that

$$a_n + 2^{-n} = 1$$

for all $n \geq 0$. Make sure you make all the elements of the induction proof explicit.

4. The *Bell number* B_4 counts the ways in which we can partition four objects in non-empty groups. It can be computed as the sum of Stirling numbers.

   What is the value of this number B_4?

   (a) $B_4 = 15$
   (b) $B_4 = 24$
   (c) $B_4 = 52$
   (d) None of the above is correct.

Modal Logic

5. Consider the two sentences:

   I don't know that it rains.
   I know it doesn't rain.

   We use a dictionary in which R formalizes 'it rains'.

   Can these sentences both be formalized by the same formula in epistemic logic?

   (a) Yes.
   (b) No.

   What are the proper formalizations of these two sentences in epistemic logic?

   (a) Both sentences should be formalized as $\neg\Box R$.
   (b) The first sentence should be formalized as $\neg\Box R$ and the second sentence should be formalized as $\Box\neg R$.
   (c) The first sentence should be formalized as $\Box\neg R$ and the second sentence should be formalized as $\neg\Box R$.
   (d) Both sentences should be formalized as $\Box\neg R$.

6. Consider the three logics:

   - epistemic logic
   - doxastic logic
   - deontic logic

In how many of these three logics do all formulas of the form

$$\neg f \to \Diamond \neg f$$

hold?

Hint: Use logical laws to show this formula logically equivalent to some formula without negations.

(a) In all three.
(b) In two of the three.
(c) In one of the three.
(d) In none of the three.

7. We are looking for a Kripke model $\mathcal{M}$ for which:

$$\mathcal{M} \not\models \Diamond a \to \Box a$$

We wonder whether there exists a model that satisfies this that has only a single world. Is there a model like that with a single world?

(a) Yes.
(b) No.

And what is the case more precisely?

(a) There is a model like that with a single world. In fact, the formula holds in no Kripke model at all.
(b) There is a model like that with a single world, but there are models in which the formula holds too.
(c) There is no model like that with a single world, but there are models with more than one world in which the formula does *not* hold.
(d) There is no model like that with a single world. In fact, the formula holds in all Kripke models.

8. Give a Kripke model for the LTL formula

$$(\mathcal{G}\, a) \wedge ((\neg a)\, \mathcal{U}\, b)$$

(without further explanation) or explain why such a model does not exist.

Recall that LTL Kripke models $\langle W, R, V \rangle$ have a fixed Kripke frame

$$W = \{x_i \mid i \in \mathbb{N}\}$$
$$R(x_i) = \{x_j \mid j \geq i\}$$

and therefore, if you think that such a model exists, in this exam an LTL model should be defined by just giving $V(x_i)$ for all worlds $x_i \in W$.