# Formal Reasoning 2021
# Test Blocks 1, 2 and 3: Additional Test
**(12/01/22)**

There are six multiple choice questions and two open questions (questions 2 and 8). Each multiple choice question is worth 10 points, and the open questions are worth 15 points each. The mark for this test is the number of points divided by ten, and the first ten points are free. Good luck!

1. Someone wants to write $a \leftrightarrow b$ using only negation and disjunction, and is considering the following two options:

   $$a \leftrightarrow b \equiv \neg(a \vee b) \vee \neg(\neg a \vee \neg b)$$
   $$a \leftrightarrow b \equiv \neg(\neg a \vee b) \vee \neg(a \vee \neg b)$$

   Which of these logical equivalences is/are correct?

   (a) Both are correct.
   (b) The first is correct, but the second is incorrect.
   (c) The second is correct, but the first is incorrect.
   (d) Both are incorrect.

2. Translate the following sentence into a formula of predicate logic with equality as well as possible.

   *There is at most one king in the Netherlands.*

   (Consider for instance the situation when the sovereign is female: in that case there are zero kings.) Use for the dictionary:

   | | |
   |---|---|
   | $N$ | the domain of Dutch people |
   | $K(x)$ | $x$ is king |

3. We are looking for a regular expression $r$ such that

   $$\mathcal{L}(r) = \{w \in \{a,b\}^* \mid w \text{ contains an even number of } a\text{'s}\}$$

   Which of the following regular expressions is a good solution for this?

   (a) $r = ((ab^*a)^*b^*)^*$
   (b) $r = (aa \cup b)^*$
   (c) $r = (ab^*a)^* \cup b^*$
   (d) $r = ((aa)^*b^*)^*$

4. Consider the context-free grammar $G_4$:

   $$S \to aA \mid bS \mid \lambda$$
   $$A \to aS \mid bA$$

   The language of this grammar is:

   $$\mathcal{L}(G_4) = \{w \in \{a,b\}^* \mid w \text{ contains an even number of } a\text{'s}\}$$

Someone wants to prove that $ab \notin \mathcal{L}(G_4)$ using the property

$$P(w) := \big[w \text{ contains an even number of symbols from the set } \{a, A\}\big]$$

Is this a suitable invariant for proving this?

(a) Yes.

(b) No.

And now in more detail: is this a suitable invariant for proving this?

(a) Yes, because $S$ has an even number of symbols from $\{a, A\}$, and every step in a production adds zero or two to the count of symbols from $\{a, A\}$.

(b) Yes, because every word in $\mathcal{L}(G_4)$ has an even number of $a$'s and no $A$'s.

(c) No, because the right hand side $aS$ only contains a single symbol from $\{a, A\}$.

(d) No, because $S$ does not contain a symbol from $\{a, A\}$.

5. How many states has the smallest deterministic finite automaton $M_5$ that recognizes the language:

$$\mathcal{L}(M_5) = \{w \in \{a, b\}^* \mid w \text{ contains } aa, \text{ and } w \text{ contains an even number of } b\text{'s}\}$$

(a) Less than six states.

(b) Six states.

(c) More than six states.

(d) This language cannot be recognized by a deterministic finite automaton.

6. There are six isomorphisms from the graph $K_3$ to itself (an isomorphism of a graph to itself is called an *automorphism* for that graph):

| | | | | | |
|---|---|---|---|---|---|
| $1 \mapsto 1$ | $1 \mapsto 1$ | $1 \mapsto 2$ | $1 \mapsto 2$ | $1 \mapsto 3$ | $1 \mapsto 3$ |
| $2 \mapsto 2$ | $2 \mapsto 3$ | $2 \mapsto 1$ | $2 \mapsto 3$ | $2 \mapsto 1$ | $2 \mapsto 2$ |
| $3 \mapsto 3$ | $3 \mapsto 2$ | $3 \mapsto 3$ | $3 \mapsto 1$ | $3 \mapsto 2$ | $3 \mapsto 1$ |

How many automorphisms are there for the graph $K_n$?

(a) $n!$

(b) $2n$

(c) $n^2 - n$

(d) None of the above is correct.

7. In an induction proof, for all $k \geq 0$ we assume $P(k)$ (the induction hypothesis: IH). So why aren't we done with this, because it seems that we have then already that the property holds for all natural numbers?

(a) Because the assumption $P(k)$ only may be used when proving $P(k{+}1)$ for the same $k$.

(b) Because in the induction step it does not say 'assume that $P(k)$ for all $k \geq 0$, and then prove ...', but 'for all $k \geq 0$, assume that $P(k)$ holds and then prove ...'. In other words, the 'assume' and the 'for all' are in the wrong order for using the induction hypothesis for all $k$ as a general assumption.

(c) Because the induction scheme amounts to the predicate logic formula:

$$P(0) \land \big(\forall k \in \mathbb{N}\,(P(k) \to P(k+1))\big) \to \forall n \in \mathbb{N}\,P(n)$$

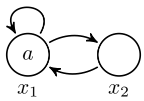
and not to the formula:

$$P(0) \land \big((\forall k \in \mathbb{N}\,P(k)) \to P(k+1)\big) \to \forall n \in \mathbb{N}\,P(n)$$

(which is illegal anyway, because the $k$ is not in scope at the $k+1$.)

(d) These three answers above are all correct, as they are all different ways of saying the same thing.

8. Consider the Kripke model $\mathcal{M}_8$:




We want to know whether the statements

$$\mathcal{M}_8 \models \Box\Diamond a \qquad \text{and} \qquad \mathcal{M}_8 \models \Diamond\Box a$$

hold or not.

In order to get to the right conclusion you have to fill out the following $\Vdash$-table first and then answer the questions about the two statements given. (Sorry for the randomization of 0 and 1, which is a feature in Cirrus that we cannot turn off.)

| model | world $\Vdash$ | $a$ | $\Box a$ | $\Diamond a$ | $\Box\Diamond a$ | $\Diamond\Box a$ |
|---|---|---|---|---|---|---|
| $\mathcal{M}_8$ | $x_1$ | | | | | |
| $\mathcal{M}_8$ | $x_2$ | | | | | |

The statement $\mathcal{M}_8 \models \Box\Diamond a$ [Choose from: holds / does not hold].

And the statement $\mathcal{M}_8 \models \Diamond\Box a$ [Choose from: holds / does not hold].