Formal Reasoning 2022
Exam
(18/01/23)

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The first 10 points are free. Good luck!

Propositional logic

1. Give three formulas of propositional logic, which formalize the following three sentences:

   (a) *I get wet when it rains.*

   (b) *I get wet because it rains.*

   (c) *It rains and therefore I get wet.*

   Use for the dictionary:

R	it rains
W	I get wet

   Write the formulas according to the official grammar from the course notes, and indicate clearly which formula corresponds to which sentence!

2. How many ones are there in the final column of the truth table of the following propositional formula?

   $$\neg b \leftrightarrow a \to b$$

   (a) 1

   (b) 2

   (c) 3

   (d) None of the above.

3. Which of the following is correct?

   (a) $a \to \neg b \equiv \neg a \to b$ and $a \to \neg b \equiv \neg b \to a$

   (b) $a \to \neg b \equiv \neg a \to b$ and $a \to \neg b \equiv b \to \neg a$

   (c) $a \to \neg b \equiv \neg b \to a$ and $a \to \neg b \equiv b \to \neg a$

   (d) None of the above.

Predicate logic

4. Consider the following English sentence:

   Every nice man loves an intelligent woman.

   This can be read in different ways. A first interpretation is:

For every nice man, there is some *intelligent woman that he loves.*

And a second interpretation is:

Every nice man loves a woman if she is intelligent. I.e., nice men love all *intelligent women.*

Give two formulas in predicate logic that formalize these two interpretations. Use for the dictionary:

M	the domain of men
W	the domain of women
$N(x)$	x is nice
$I(x)$	x is intelligent
$L(x, y)$	x loves y

Indicate clearly which formula corresponds to which interpretation!

5. Does the following hold?

$$\models \forall x \in D\, [R(x, x) \to \exists y \in D\, R(x, y)]$$

(a) Yes, this holds because if $R(x, x)$ then one can take y to be x.

(b) Yes, this holds because if $R(x, x)$ holds for all x, that means that $R(x, y)$ also holds for all x and y.

(c) No, this does not hold and a counterexample is a structure in which D is interpreted as an empty domain, i.e., a domain with no elements.

(d) No, this does not hold and a counterexample is a structure in which D is interpreted as a domain with a single element.

6. What is a proper formalization in predicate logic of the following English sentence?

Sharon loves at most two men.

For the dictionary we use:

M	the domain of men
s	Sharon

You may use that in this interpretation there are men, whether Sharon loves them or not.

(a) $\forall x, y, z \in M\, [L(s, x) \land L(s, y) \land L(s, z) \to x = y \lor x = z \lor y = z]$

(b) $\exists x_1, x_2 \in M\, [(L(s, x_1) \lor \neg L(s, x_1)) \land (L(s, x_2) \lor \neg L(s, x_2)) \land \forall y \in M\, (L(s, y) \to y = x_1 \lor y = x_2)]$

(c) $\exists x_1, x_2 \in M\, \forall y \in M\, [L(s, y) \to y = x_1 \lor y = x_2]$

(d) All of the above.

Languages

7. We define:

$$L_7 := \{awb \mid w \in \{a,b\}^*\}$$

The words in this language are precisely the words that begin with an a and end with a b. Consider the following three statements:

- $(L_7)^* = L_7 \cup \{\lambda\}$
- $L_7 \cup (L_7)^R = \{a,b\}^*$
- $L_7 \cap \overline{(L_7)} = \varnothing$

What is true and what is not?

(a) The first two are true, but the last one is false.
(b) The first and last are true, but the middle one is false.
(c) The last two are true, but the first one is false.
(d) All three are true.

8. Give a regular expression for the language

$$L_8 := \{w \in \{a,b\}^* \mid w \text{ contains an } \textit{odd} \text{ number of } a\text{'s}\}$$

9. Consider the language

$$L_9 := \{a^n b a^n b a^m \mid n, m \in \mathbb{N}\}$$

For instance, we have $aabaab \in L_9$ by taking $n = 2$ and $m = 0$, and $bba \in L_9$ by taking $n = 0$ and $m = 1$.

One can show that this language is not regular, but is it context-free?

(a) Yes, L_9 is context-free as it has a context-free grammar. However, it does not have a right linear context-free grammar.
(b) Yes, L_9 is context-free and it even has a right linear context-free grammar.
(c) No, L_9 is not context-free but $\{a^n b a^m b a^n \mid n, m \in \mathbb{N}\}$ is context-free.
(d) No, L_9 is not context-free and $\{a^n b a^m b a^n \mid n, m \in \mathbb{N}\}$ is not context-free either.

10. Consider the following context-free grammar G_{10}:

$$S \to abS \mid aSb \mid \lambda$$

We want to show that $baba \notin \mathcal{L}(G_{10})$ and consider the following property as an invariant for this:

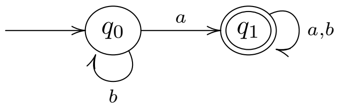
$$P(w) := [w \text{ begins with } S \text{ or } a, \text{ or } w = \lambda]$$

Does this work?

(a) Yes, this works because all words in $\mathcal{L}(G_{10})$ begin with a or are equal to λ.
(b) Yes, this works because each word in a production of G_{10} satisfies this property.
(c) No, this does not work because $Sb \to b$ shows that this is not an invariant.
(d) No, this does not work because $bS \to b$ shows that this is not an invariant.

Automata

11. Consider the deterministic finite automaton M_{11}:



Which of the following regular expressions does *not* describe the language of this automaton?

(a) $b^*a(a \cup b)^*$

(b) $(a \cup b)^*a(a \cup b)^*$

(c) $(a^*b^*)^*ab^*$

(d) All of the above describe the language of the automaton.

12. Consider the following context-free grammar G_{12}:

$$\begin{aligned} S &\to AS \mid aB \\ A &\to aA \mid bA \mid \lambda \\ B &\to Ba \mid Bb \mid \lambda \end{aligned}$$

Is $\mathcal{L}(G_{12})$ regular?

(a) Yes, this language is regular as the language $\mathcal{L}(G_{12})$ can be recognized by a deterministic automaton of at most two states.

(b) Yes, this language is regular as the language $\mathcal{L}(G_{12})$ can be recognized by a deterministic automaton, but only by one that has more than two states.

(c) No, this language is not regular although the language $\mathcal{L}(G_{12})$ can be recognized by a non-deterministic automaton.

(d) No, this language is not regular and the language $\mathcal{L}(G_{12})$ can not be recognized by a non-deterministic automaton.

13. Give a non-deterministic finite automaton with at most three states for the language:

$$L_{13} := \{wax \mid w \in \{a,b\}^*, x \in \{a,b\}\}$$

This language consists of the words that have the symbol a in their next-to-last place. For example $ab\underline{a}b \in L_{13}$ but $ab\underline{b}a \notin L_{13}$.

Write the automaton as a tuple $\langle \Sigma, Q, q_0, F, \delta \rangle$. If you do not remember how to write an automaton as a tuple, you may describe the automaton in words for partial points.

Discrete mathematics

14. Which of the following statements holds? *Hint:* Consider what happens when a tree has a vertex with degree greater than two.

(a) All trees with an Eulerian path also have a Hamiltonian path, but there are trees with a Hamiltonian path that do not have an Eulerian path.

(b) All trees with a Hamiltonian path also have an Eulerian path, but there are trees with an Eulerian path that do not have a Hamiltonian path.

(c) A tree has an Eulerian path if and only if it has a Hamiltonian path.

(d) There are trees with a Hamiltonian path that do not have an Eulerian path, and there are trees with an Eulerian path that do not have a Hamiltonian path.

15. An automorphism is an isomorphism from a graph to itself. How many automorphisms does the complete graph K_3 have?

(a) 1

(b) 3

(c) 6

(d) None of the above.

16. We write h_n for the number of moves in the recursive solution of the puzzle of the towers of Hanoi with n disks. Which of the following recursive equations correctly defines this sequence?

(a)

$$\begin{aligned} h_1 &= 0 \\ h_{n+1} &= 2h_n + 1 \quad \text{for } n \geq 1 \end{aligned}$$

(b)

$$\begin{aligned} h_1 &= 0 \\ h_{n+1} &= 2h_n - 1 \quad \text{for } n \geq 1 \end{aligned}$$

(c)

$$\begin{aligned} h_1 &= 1 \\ h_{n+1} &= 2h_n + 1 \quad \text{for } n \geq 1 \end{aligned}$$

(d)

$$\begin{aligned} h_1 &= 1 \\ h_{n+1} &= 2h_n - 1 \quad \text{for } n \geq 1 \end{aligned}$$

17. We define a sequence a_n for $n \geq 2$ using the recursive definition:

$$\begin{aligned} a_2 &= 2 \\ a_{n+1} &= a_n + 2n + 1 \quad \text{for } n \geq 2 \end{aligned}$$

Now answer the following two questions:

(a) Prove with induction that for $n \geq 2$ the following equation holds:

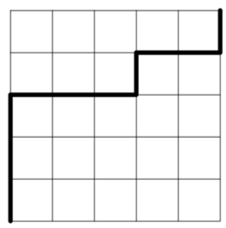
$$a_n = n^2 - 2$$

Write your proof according to the induction scheme from the course.

(b) Give the value of a_{45}.

18. Let be given a square grid with $n \times n$ squares, where $n \geq 1$. Now consider all paths from the lower-left corner to the upper-right corner, in which each step is either up or to the right. Such a path always consists of $2n$ steps, of which n are horizontal and n are vertical.

    Here is an example of such a path in the grid for $n = 5$:



    Now how many paths like this are there?

    (a) $\binom{2n}{n}$
    (b) 2^n
    (c) 2^{2n}
    (d) None of the above.

19. Consider the number triangle that starts like:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 3 & & 1 \end{array}$$

    What is the following row in this triangle?

    (a)
    $$1 \quad 4 \quad 4 \quad 1$$
    (b)
    $$1 \quad 6 \quad 7 \quad 1$$
    (c)
    $$6 \quad 11 \quad 6 \quad 1$$
    (d) None of the above.

Modal logic

20. We use the dictionary:

    R it rains

    Now we want to give the meaning of the following formula of doxastic logic:

$$R \to \Diamond R$$

    Which of the following English sentences does this best? *Hint:* It might be useful to use logical laws.

(a) If I know that it does not rain, then it does not rain.

(b) If I believe that it does not rain, then it does not rain.

(c) If it rains, then I know it rains.

(d) If it rains, then I believe it rains.

21. It is the case that if a Kripke model $\mathcal{M}$ satisfies $\mathcal{M} \vDash a$, then also $\mathcal{M} \vDash \Box a$. But does this mean that $\Box a \to \Box\Box a$ holds in all Kripke models?

    Explain your answer. If your explanation contains a Kripke model, you can write it as a tuple $\langle W, R, V\rangle$. If you do not remember how to write a Kripke model as a tuple, you may describe it in words for partial points.

22. In an LTL Kripke model, do we always have $x_i \Vdash \mathcal{F}a \to \mathcal{F}\mathcal{F}a$ in each world x_i?

    (a) Yes, because for any formula f, if f is true in a world, then $\mathcal{F}f$ is true in that world too.

    (b) Yes, because for any formula f, if $\mathcal{F}f$ is true in a world, then f is true in that world too.

    (c) No, in a world where a is only true in just one of its accessible worlds, this does not hold, because then $\mathcal{F}a$ will be true, but $\mathcal{F}\mathcal{F}a$ will not be true.

    (d) No, in a model in which a is never true, this does not hold.