Formal Reasoning 2022
Solutions Exam
(18/01/23)

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The first 10 points are free. Good luck!

Propositional logic

1. Give three formulas of propositional logic, which formalize the following three sentences:

   (a) *I get wet when it rains.*
   (b) *I get wet because it rains.*
   (c) *It rains and therefore I get wet.*

   Use for the dictionary:

R	it rains
W	I get wet

   Write the formulas according to the official grammar from the course notes, and indicate clearly which formula corresponds to which sentence!

   These are correct formulas, but there may be others as well.

   (a) *I get wet when it rains.*
   $(R \to W)$.
   (b) *I get wet because it rains.*
   $((R \to W) \land R)$, which is equivalent to $(W \land R)$.
   (c) *It rains and therefore I get wet.*
   $(R \land (R \to W))$, which is equivalent to $(R \land W)$.

2. How many ones are there in the final column of the truth table of the following propositional formula?

   $$\neg b \leftrightarrow a \to b$$

   (a) is correct

   (a) 1
   (b) 2
   (c) 3
   (d) None of the above.

   Answer (a) is correct.

   Let us have a look at the truth table:

a	b	$\neg b$	$a \to b$	$\neg b \leftrightarrow a \to b$
0	0	1	1	1
0	1	0	1	0
1	0	1	0	0
1	1	0	1	0

   So there is one one in the final column.

3. Which of the following is correct?

   (a) $a \to \neg b \equiv \neg a \to b$ and $a \to \neg b \equiv \neg b \to a$

   (b) $a \to \neg b \equiv \neg a \to b$ and $a \to \neg b \equiv b \to \neg a$

   (c) $a \to \neg b \equiv \neg b \to a$ and $a \to \neg b \equiv b \to \neg a$

(d) is correct (d) None of the above.

   Answer (d) is correct.

   Let us have a look at the truth table:

a	b	$\neg a$	$\neg b$	$a \to \neg b$	$\neg a \to b$	$\neg b \to a$	$b \to \neg a$
0	0	1	1	1	0	0	1
0	1	1	0	1	1	1	1
1	0	0	1	1	1	1	1
1	1	0	0	0	1	1	0

   So we see that:

   - $a \to \neg b \not\equiv \neg a \to b$
   - $a \to \neg b \not\equiv \neg b \to a$
   - $a \to \neg b \equiv b \to \neg a$

   So the first two options are wrong because of the first observation. And the third option is wrong because of the second observation.

Predicate logic

4. Consider the following English sentence:

   Every nice man loves an intelligent woman.

   This can be read in different ways. A first interpretation is:

   For every nice man, there is some *intelligent woman that he loves.*

   And a second interpretation is:

   Every nice man loves a woman if she is intelligent. I.e., nice men love all *intelligent women.*

   Give two formulas in predicate logic that formalize these two interpretations. Use for the dictionary:

M	the domain of men
W	the domain of women
$N(x)$	x is nice
$I(x)$	x is intelligent
$L(x, y)$	x loves y

   Indicate clearly which formula corresponds to which interpretation!

   Let us start with the first interpretation:

For every nice man, there is some *intelligent woman that he loves.*

$$\forall m \in M\,[N(m) \to \exists w \in W\,[I(w) \wedge L(m,w)]]$$

And now the second interpretation:

Every nice man loves a woman if she is intelligent. I.e., nice men love all *intelligent women.*

$$\forall m \in M\,[N(m) \to \forall w \in W\,[I(w) \to L(m,w)]]$$

5. Does the following hold?

$$\vDash \forall x \in D\,[R(x,x) \to \exists y \in D\,R(x,y)]$$

(a) is correct

(a) Yes, this holds because if $R(x,x)$ then one can take y to be x.

(b) Yes, this holds because if $R(x,x)$ holds for all x, that means that $R(x,y)$ also holds for all x and y.

(c) No, this does not hold and a counterexample is a structure in which D is interpreted as an empty domain, i.e., a domain with no elements.

(d) No, this does not hold and a counterexample is a structure in which D is interpreted as a domain with a single element.

Answer (a) is correct.

This formula holds independent of the chosen structure and interpretation. We make a case distinction to explain this:

- If D is interpreted as an empty domain, then the whole $\forall x \in D\,[\dots]$ vacuously holds. So the fact that somewhere inside these dots it is stated that there exists an element in the empty set, is not relevant anymore.
- If D is not interpreted as an empty domain, then we can choose an arbitrary element $x \in D$. Again we make a case distinction:
  - If $R(x,x)$ holds, we can indeed choose $y \in D$ to be x and then $R(x,y)$ holds.
  - If $R(x,x)$ does not hold, the implication vacuously holds.

  So in both cases, the formula holds.

So, again, in both cases the formula holds.

Hence, the answer should be a 'yes' option. And although the first option does not explain all the details we described above, it is clearly the best solution as the second option is clearly wrong. For instance, if $D = \mathbb{N}$ and $R(x,y)$ is interpreted as $x = y$, then clearly $R(x,x)$ holds for all natural numbers, but for instance $R(3,7)$ clearly does not hold.

The third and the fourth answer make no sense because the answer is 'yes'.

6. What is a proper formalization in predicate logic of the following English sentence?

Sharon loves at most two men.

For the dictionary we use:

M	the domain of men
s	Sharon

You may use that in this interpretation there are men, whether Sharon loves them or not.

(a) $\forall x, y, z \in M\,[L(s,x) \wedge L(s,y) \wedge L(s,z) \rightarrow x = y \vee x = z \vee y = z]$

(b) $\exists x_1, x_2 \in M\,[(L(s,x_1) \vee \neg L(s,x_1)) \wedge (L(s,x_2) \vee \neg L(s,x_2)) \wedge \forall y \in M\,(L(s,y) \rightarrow y = x_1 \vee y = x_2)]$

(c) $\exists x_1, x_2 \in M\,\forall y \in M\,[L(s,y) \rightarrow y = x_1 \vee y = x_2]$

(d) is correct

(d) All of the above.

Answer (d) is correct.

Let us look at the formulas:

(a) $\forall x, y, z \in M\,[L(s,x) \wedge L(s,y) \wedge L(s,z) \rightarrow x = y \vee x = z \vee y = z]$
This translates to

> *For all combinations of the three men x, y, and z, if all three of them are loved by Sharon, then at least two of them must be the same.*

(b) $\exists x_1, x_2 \in M\,[(L(s,x_1) \vee \neg L(s,x_1)) \wedge (L(s,x_2) \vee \neg L(s,x_2)) \wedge \forall y \in M\,(L(s,y) \rightarrow y = x_1 \vee y = x_2)]$
This translates to

> *There exist two men for which it holds that if at least one of them is loved (or not) by Sharon, then for any man who is loved by Sharon, it holds that he must be one of the first two men.*

(c) $\exists x_1, x_2 \in M\,\forall y \in M\,[L(s,y) \rightarrow y = x_1 \vee y = x_2]$
This translates to

> *There exist two men for which it holds that for any man who is loved by Sharon, it holds that he must be one of the first two men.*

All of these translations match the claim that Sharon loves at most two men.

Languages

7. We define:

$$L_7 := \{awb \mid w \in \{a,b\}^*\}$$

The words in this language are precisely the words that begin with an a and end with a b. Consider the following three statements:

- $(L_7)^* = L_7 \cup \{\lambda\}$
- $L_7 \cup (L_7)^R = \{a,b\}^*$
- $L_7 \cap \overline{(L_7)} = \varnothing$

What is true and what is not?

(b) is correct

(a) The first two are true, but the last one is false.

(b) The first and last are true, but the middle one is false.

(c) The last two are true, but the first one is false.

(d) All three are true.

Answer (b) is correct.

Let us look at all three statements.

- $(L_7)^* = L_7 \cup \{\lambda\}$
  If we take two words $w_1, w_2 \in L_7$ and concatenate them then $w_1w_2 \in L_7$ as w_1 starts with an a and w_2 ends with a b, so w_1w_2 starts with an a and w_1w_2 ends with a b. This holds for any concatenation of two or more words in L_7. So $(L_7)^*$ is almost L_7. The only word in $(L_7)^*$ that is not in L_7 is λ. So the statement is true.
- $L_7 \cup (L_7)^R = \{a,b\}^*$
  Note that $(L_7)^R$ are the words of L_7 in reverse, which means that these are the words that start with a b and end with an a. If we take the union we get all words that either start with an a and end with a b or start with a b and end with an a. However, $a \in \{a,b\}^*$ but $a \notin L_7 \cup (L_7)^R$. So the statement is not true.
- $L_7 \cap \overline{(L_7)} = \varnothing$
  For all languages L it holds that its complement $\overline{L}$ only contains words that are not in L, so in particular for each language L, $L \cap \overline{L} = \varnothing$, hence also for L_7. So the statement is true.

So the first and last statements are true, but the middle one is false.

8. Give a regular expression for the language

$$L_8 := \{w \in \{a,b\}^* \mid w \text{ contains an } \textit{odd} \text{ number of } a\text{'s}\}$$

We can take

$$(ab^*a \cup b)^*a(ab^*a \cup b)^*$$

The a in the middle ensures that there is at least one a in a word in the language described by this expression. And the block $(ab^*a \cup b)$ represents either a single b or exactly two a's with possibly some number of b's in between them. So every application of this block will keep the number of a's (together with the 'middle' a) odd. By adding the Kleene star around these blocks we can get all words with an odd number of a's.

For reasons of simplicity, we used the same block before and after the obligatory a, however, we can also write a regular expression where the obligatory a is not in the 'middle' of the word, but is the first a in the word. This is the strategy behind this (simpler) regular expression:

$$b^*a(ab^*a \cup b)^*$$

9. Consider the language

$$L_9 := \{a^n b a^n b a^m \mid n, m \in \mathbb{N}\}$$

For instance, we have $aabaab \in L_9$ by taking $n = 2$ and $m = 0$, and $bba \in L_9$ by taking $n = 0$ and $m = 1$.

One can show that this language is not regular, but is it context-free?

(a) is correct

(a) Yes, L_9 is context-free as it has a context-free grammar. However, it does not have a right linear context-free grammar.

(b) Yes, L_9 is context-free and it even has a right linear context-free grammar.

(c) No, L_9 is not context-free but $\{a^n ba^m ba^n \mid n, m \in \mathbb{N}\}$ is context-free.

(d) No, L_9 is not context-free and $\{a^n ba^m ba^n \mid n, m \in \mathbb{N}\}$ is not context-free either.

Answer (a) is correct.

It is context free as it can be produced by this context-free grammar:

$$S \to AbB$$
$$A \to aAa \mid b$$
$$B \to aB \mid \lambda$$

However, as it is given that the language is not regular, it is impossible that there is also a right-linear context-free grammar.

The third and the fourth answer make no sense because the answer is 'yes'.

10. Consider the following context-free grammar G_{10}:

$$S \to abS \mid aSb \mid \lambda$$

We want to show that $baba \notin \mathcal{L}(G_{10})$ and consider the following property as an invariant for this:

$$P(w) := [w \text{ begins with } S \text{ or } a, \text{ or } w = \lambda]$$

Does this work?

(a) Yes, this works because all words in $\mathcal{L}(G_{10})$ begin with a or are equal to λ.

(b) Yes, this works because each word in a production of G_{10} satisfies this property.

(c) is correct

(c) No, this does not work because $Sb \to b$ shows that this is not an invariant.

(d) No, this does not work because $bS \to b$ shows that this is not an invariant.

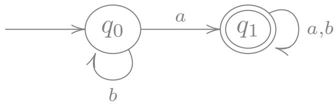
Answer (c) is correct.

If the property P was an invariant, then for all words v and v' such that $P(v)$ holds and $v \to v'$, $P(v')$ should hold as well. However, if we take $v = Sb$ and v' is b then we do have that $P(Sb)$ and $Sb \to b$ hold, but $P(b)$ clearly doesn't hold. So this method doesn't work as P is not an invariant.

The first and the second answer make no sense because the answer is 'no'.

Automata

11. Consider the deterministic finite automaton M_{11}:



Which of the following regular expressions does *not* describe the language of this automaton?

(a) $b^*a(a \cup b)^*$

(b) $(a \cup b)^*a(a \cup b)^*$

(c) $(a^*b^*)^*ab^*$

(d) is correct (d) All of the above describe the language of the automaton.

Answer (d) is correct.

The language accepted by this automaton is the language of all words that contain at least one a. Before we look at the specific regular expressions, note that $(a \cup b)^*$ and $(a^*b^*)^*$ are two ways of expressing any sequence of zero or more a's and b's.

Now let us look at the regular expressions:

(a) $b^*a(a \cup b)^*$
This expression states that any word starts with a possibly empty series of b's, then the obligatory a, followed by any sequence of zero or more a's and b's.

(b) $(a \cup b)^*a(a \cup b)^*$
This expression states that there is an arbitrary a somewhere in the word. And there is a series of zero or more a's and b's before this obligatory a, as well as after it.

(c) $(a^*b^*)^*ab^*$
This expression states that any word ends with a possibly empty series of b's, and just in front of that the obligatory a. And there is a sequence of zero or more a's and b's in front of this obligatory a.

So all expressions indeed describe $\mathcal{L}(M_{11})$.

12. Consider the following context-free grammar G_{12}:

$$S \to AS \mid aB$$
$$A \to aA \mid bA \mid \lambda$$
$$B \to Ba \mid Bb \mid \lambda$$

Is $\mathcal{L}(G_{12})$ regular?

(a) Yes, this language is regular as the language $\mathcal{L}(G_{12})$ can be recognized
(a) is correct by a deterministic automaton of at most two states.

(b) Yes, this language is regular as the language $\mathcal{L}(G_{12})$ can be recognized by a deterministic automaton, but only by one that has more than two states.

(c) No, this language is not regular although the language $\mathcal{L}(G_{12})$ can be recognized by a non-deterministic automaton.

(d) No, this language is not regular and the language $\mathcal{L}(G_{12})$ can not be recognized by a non-deterministic automaton.

Answer (a) is correct.

The language is the same language as in the previous exercise, namely the language of words that contain at least one a. Therefore, this language can be recognized by automaton M_{11}, which has two states.

Not relevant for determining the correct 'yes' option, but we have already seen three possible regular expressions that describe this language.

The third and the fourth answer make no sense because the answer is 'yes'.

13. Give a non-deterministic finite automaton with at most three states for the language:

$$L_{13} := \{wax \mid w \in \{a,b\}^*, x \in \{a,b\}\}$$

This language consists of the words that have the symbol a in their next-to-last place. For example $ab\underline{a}b \in L_{13}$ but $ab\underline{b}a \notin L_{13}$.

Write the automaton as a tuple $\langle \Sigma, Q, q_0, F, \delta \rangle$. If you do not remember how to write an automaton as a tuple, you may describe the automaton in words for partial points.

Take for instance the automaton $M_{13} := \langle \Sigma, Q, q_0, F, \delta \rangle$.

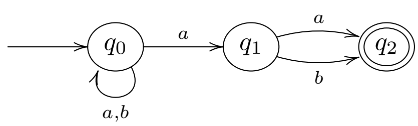
Where

$$\begin{aligned} \Sigma &:= \{a, b\} \\ Q &:= \{q_0, q_1, q_2\} \\ F &:= \{q_2\} \end{aligned}$$

and the transition function δ is given by:

$$\begin{array}{lllllllll} \delta(q_0,\lambda) & := & \varnothing & \delta(q_1,\lambda) & := & \varnothing & \delta(q_2,\lambda) & := & \varnothing \\ \delta(q_0,a) & := & \{q_0,q_1\} & \delta(q_1,a) & := & \{q_2\} & \delta(q_2,a) & := & \varnothing \\ \delta(q_0,b) & := & \{q_0\} & \delta(q_1,b) & := & \{q_2\} & \delta(q_2,b) & := & \varnothing \end{array}$$

This automaton is represented by the following diagram:



Discrete mathematics

14. Which of the following statements holds? *Hint:* Consider what happens when a tree has a vertex with degree greater than two.

(a) All trees with an Eulerian path also have a Hamiltonian path, but there are trees with a Hamiltonian path that do not have an Eulerian path.

(b) All trees with a Hamiltonian path also have an Eulerian path, but there are trees with an Eulerian path that do not have a Hamiltonian path.

(c) A tree has an Eulerian path if and only if it has a Hamiltonian path.

(c) is correct

(d) There are trees with a Hamiltonian path that do not have an Eulerian path, and there are trees with an Eulerian path that do not have a Hamiltonian path.

Answer (c) is correct.

We make a case distinction on the degrees of the vertices.

- The tree has no vertices with degree greater than two. Because a tree is a connected graph without cycles, this means that the tree has two endpoints with degree one, and the remaining vertices have degree two. So the tree is just a chain of vertices. Obviously, if we take the path starting in one endpoint and ending in the other endpoint, this path is both an Eulerian path and a Hamiltonian path.
- The tree has at least one vertex with degree greater than two. Let v be such a vertex with degree greater than two. Then there are vertices a, b, and c and edges (v, a), (v, b), and (v, c). As a tree has no cycles, the only way that a path can visit all vertices a, b, and c is by using the three edges mentioned above. But this implies that vertex v is visited at least twice in this path, which is prohibited for a Hamiltonian path. Hence, such a tree has no Hamiltonian path. However, such a tree also doesn't have an Eulerian path as each of the branches going through a, b, and c will have a different endpoint with degree one. So there are at least three vertices with odd degree and hence an Eulerian path is not possible.

So a tree has an Eulerian path if and only if it has a Hamiltonian path.

15. An automorphism is an isomorphism from a graph to itself. How many automorphisms does the complete graph K_3 have?

(a) 1

(b) 3

(c) is correct

(c) 6

(d) None of the above.

Answer (c) is correct.

As each vertex in K_3 is connected to each other vertex, for an automorphism we can just map the vertices $\{1, 2, 3\}$ onto $\{1, 2, 3\}$ in any order.

If we create such a mapping, we first choose the image for vertex 1. There are three options for that. Then we choose the image for vertex 2. There are only two options for that. Finally, we choose the image for vertex 3. There is only one option for that. So in total there are $3 \cdot 2 \cdot 1 = 6$ ways to do that.

Of course, we can also simply list all these automorphisms:

$$\begin{array}{lll} 1 \mapsto 1 & 1 \mapsto 2 & 1 \mapsto 3 \\ 2 \mapsto 2 & 2 \mapsto 1 & 2 \mapsto 1 \\ 3 \mapsto 3 & 3 \mapsto 3 & 3 \mapsto 2 \end{array}$$

$$\begin{array}{lll} 1 \mapsto 1 & 1 \mapsto 2 & 1 \mapsto 3 \\ 2 \mapsto 3 & 2 \mapsto 3 & 2 \mapsto 2 \\ 3 \mapsto 2 & 3 \mapsto 1 & 3 \mapsto 1 \end{array}$$

16. We write h_n for the number of moves in the recursive solution of the puzzle of the towers of Hanoi with n disks. Which of the following recursive equations correctly defines this sequence?

(a)

$$h_1 = 0$$
$$h_{n+1} = 2h_n + 1 \quad \text{for } n \geq 1$$

(b)

$$h_1 = 0$$
$$h_{n+1} = 2h_n - 1 \quad \text{for } n \geq 1$$

(c)

$$h_1 = 1$$
$$h_{n+1} = 2h_n + 1 \quad \text{for } n \geq 1$$

(c) is correct

(d)

$$h_1 = 1$$
$$h_{n+1} = 2h_n - 1 \quad \text{for } n \geq 1$$

Answer (c) is correct.

If we have only one disk, we can move it in one step to the final position. So $h_1 = 1$.

And if we have $n + 1$ disks, where $n \geq 1$, we first move n disks to the second peg, which takes h_n moves, then we move the largest disk in one move to the third peg, and finally, we move n disks from the second peg to the third peg, which takes again h_n moves. So we get that $h_{n+1} = 2h_n + 1$.

17. We define a sequence a_n for $n \geq 2$ using the recursive definition:

$$a_2 = 2$$
$$a_{n+1} = a_n + 2n + 1 \quad \text{for } n \geq 2$$

Now answer the following two questions:

(a) Prove with induction that for $n \geq 2$ the following equation holds:

$$a_n = n^2 - 2$$

Write your proof according to the induction scheme from the course.

(b) Give the value of a_{45}.

0 This is the proof by induction. **Proposition:**

$a_n = n^2 - 2$ for all $n \geq 2$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \quad := \quad a_n = n^2 - 2$$

3 **Base Case.** We show that $P(2)$ holds, i.e. we show that

$a_2 = 2^2 - 2$

4 This indeed holds, because

$$a_2 = 2 = 4 - 2 = 2^2 - 2$$

5 **Induction Step.** Let k be any natural number such that $k \geq 2$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that

$a_k = k^2 - 2$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that

$a_{k+1} = (k+1)^2 - 2$

8 This indeed holds, because

$$\begin{aligned} a_{k+1} &= a_k + 2k + 1 \\ &\overset{\text{IH}}{=} k^2 - 2 + 2k + 1 \\ &= k^2 + 2k + 1 - 2 \\ &= (k+1)^2 - 2 \end{aligned}$$

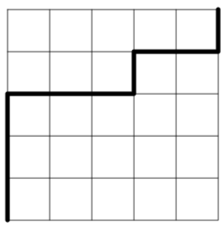
9 Hence it follows by induction that $P(n)$ holds for all $n \geq 2$.

And now we compute a_{45} using the direct formula that we have just derived:

$$a_{45} = 45^2 - 2 = 2025 - 2 = 2023$$

18. Let be given a square grid with $n \times n$ squares, where $n \geq 1$. Now consider all paths from the lower-left corner to the upper-right corner, in which each step is either up or to the right. Such a path always consists of $2n$ steps, of which n are horizontal and n are vertical.

Here is an example of such a path in the grid for $n = 5$:



Now how many paths like this are there?

(a) is correct

(a) $\binom{2n}{n}$

(b) 2^n

(c) 2^{2n}

(d) None of the above.

Answer (a) is correct.

As every path has $2n$ steps, where n of them are horizontal and n of them are vertical, a path is determined by indicating which of the $2n$ steps are horizontal and which are vertical. So we can create such a path by putting the numbers $1, 2, \ldots, 2n$ in a bag, and picking n numbers out of this bag and let these numbers represent the vertical steps. So the path in the example is represented by the numbers $1-2-3-7-10$. Note that the order of the numbers chosen does not matter. From the construction, it follows that the number of ways this can be done is $\binom{2n}{n}$.

19. Consider the number triangle that starts like:

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 3 & & 1 \end{array}$$

What is the following row in this triangle?

(a)

$$1 \quad 4 \quad 4 \quad 1$$

(b)

$$1 \quad 6 \quad 7 \quad 1$$

(c)

$$6 \quad 11 \quad 6 \quad 1$$

(d) is correct

(d) None of the above.

Answer (d) is correct.

These are the Stirling numbers of the second kind. The corresponding triangle starts with:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 3 & & 1 & \\ 1 & & 7 & & 6 & & 1 \end{array}$$

So none of the proposed rows is correct.

Modal logic

20. We use the dictionary:

R it rains

Now we want to give the meaning of the following formula of doxastic logic:

$$R \to \Diamond R$$

Which of the following English sentences does this best? *Hint:* It might be useful to use logical laws.

(b) is correct

(a) If I know that it does not rain, then it does not rain.
(b) If I believe that it does not rain, then it does not rain.
(c) If it rains, then I know it rains.
(d) If it rains, then I believe it rains.

Answer (b) is correct.

As doxastic logic is about belief, the options about knowledge are certainly wrong.

Now let us rewrite the formula into an equivalent formula:

$$\begin{aligned} R \to \Diamond R &\equiv R \to \neg\Box\neg R \\ &\equiv \neg R \vee \neg\Box\neg R \\ &\equiv \neg\Box\neg R \vee \neg R \\ &\equiv \Box\neg R \to \neg R \end{aligned}$$

And the last formula clearly translates to 'If I believe that it does not rain, then it does not rain.'

Note that the other option with belief translates into the formula $R \to \Box R$, which is certainly not equivalent to $R \to \Diamond R$.

21. It is the case that if a Kripke model $\mathcal{M}$ satisfies $\mathcal{M} \vDash a$, then also $\mathcal{M} \vDash \Box a$. But does this mean that $\Box a \to \Box\Box a$ holds in all Kripke models?

Explain your answer. If your explanation contains a Kripke model, you can write it as a tuple $\langle W, R, V\rangle$. If you do not remember how to write a Kripke model as a tuple, you may describe it in words for partial points.

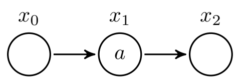
This property doesn't hold for all Kripke models. We are talking about an instance of axiom 4, which only holds for all formulas, in case the accessibility relation is transitive.

Take for instance the model $\mathcal{M}_{21} := \langle W, R, V\rangle$, where

$$\begin{aligned} W &:= \{x_0, x_1, x_2\} \\ R(x_0) &:= \{x_1\} \\ R(x_1) &:= \{x_2\} \\ R(x_2) &:= \varnothing \\ V(x_0) &:= \varnothing \\ V(x_1) &:= \{a\} \\ V(x_2) &:= \varnothing \end{aligned}$$

As a diagram this looks like:

$\mathcal{M}_{21}$:=



In this model we have that $x_0 \Vdash \Box a$, but $x_1 \Vdash \Box a$ does not hold and therefore $x_0 \Vdash \Box\Box a$ does not hold. And hence $x_0 \Vdash \Box a \to \Box\Box a$ does not hold and hence $\mathcal{M}_{21} \vDash \Box a \to \Box\Box a$ does not hold.

22. In an LTL Kripke model, do we always have $x_i \Vdash \mathcal{F}a \to \mathcal{F}\mathcal{F}a$ in each world x_i?

(a) is correct

 (a) Yes, because for any formula f, if f is true in a world, then $\mathcal{F}f$ is true in that world too.
 (b) Yes, because for any formula f, if $\mathcal{F}f$ is true in a world, then f is true in that world too.
 (c) No, in a world where a is only true in just one of its accessible worlds, this does not hold, because then $\mathcal{F}a$ will be true, but $\mathcal{F}\mathcal{F}a$ will not be true.
 (d) No, in a model in which a is never true, this does not hold.

Answer (a) is correct.

As 'the future includes now' it is indeed true that if formula f holds in a world x_i then $\mathcal{F}f$ also holds in world x_i. So in particular this holds for the formula $f := \mathcal{F}a$.

Note that the second option is wrong. If a is only true in world x_1, then $\mathcal{F}a$ holds in world x_0, but a does not hold in world x_0.

The third and the fourth answer make no sense because the answer is 'yes'.