Formal Reasoning 2022
Solutions Test Block 1: Propositional and Predicate Logic
(26/09/22)

Propositional logic

1. How many parentheses occur in the official form of the following formula of propositional logic?

   $$a \land \neg b \to c \to d \lor e$$

   You should count opening and closing parentheses separately. For example, the formula $(a \to b)$ contains two parentheses.

   (a) 6

   (b) is correct — (b) 8

   (c) 10

   (d) a count different from the three options above

   Answer (b) is correct.

   If we write the formula according to the official grammar it would be

   $$\underbrace{\Bigg(\underbrace{(a \land \neg b)}_{\land} \to \underbrace{\Big(c \to \underbrace{(d \lor e)}_{\lor}\Big)}_{\to}\Bigg)}_{\to}$$

   So we have eight parentheses. In fact, this already follows from the fact that all binary operators get exactly two parentheses and unary operators get none. This formula has four binary operators and one unary, so it gets indeed eight parentheses.

2. We want to formalize the following English sentence:

   I exercise, but only when it is not Sunday.

   For the dictionary we use:

E	I exercise
S	it is Sunday

   This sentence can be read in different ways. Which of the following four formulas of propositional logic is the best formalization?

   (a) is correct — (a) $E \to \neg S$

   (b) $E \leftrightarrow \neg S$

   (c) $\neg S \to E$

   (d) $\neg S \leftrightarrow E$

   Answer (a) is correct.

   Let us see what the translations of the four formulas are:

(a) $E \to \neg S$

This means *If I exercise then it is not Sunday.* So this is a proper interpretation of the given sentence.

(b) $E \leftrightarrow \neg S$

This means *I exercise if and only if it is not Sunday.* or with the 'if and only if' expanded, *If I exercise then it is not Sunday and if it is not Sunday I exercise.* As the given sentence does not state that for instance on Monday the person is doing exercises, this is not a proper interpretation.

(c) $\neg S \to E$

This means *If it is not Sunday then I exercise.* As the given sentence does not state that for instance on Tuesday the person is doing exercises, this is not a proper interpretation.

(d) $\neg S \leftrightarrow E$

This means *It is not Sunday if and only if I exercise.* or with the 'if and only if' expanded, *If it is not Sunday then I exercise and if I exercise then it is not Sunday.* As the given sentence does not state that for instance on Thursday the person is doing exercises, this is not a proper interpretation. Note that it is equivalent to $E \leftrightarrow \neg S$, so if that one is not a proper formalization, this one is also not a proper formalization.

3. Which of the following logical equivalences does not hold?

(a) $a \to \neg b \equiv \neg a \vee \neg b$

(b) $a \to \neg b \equiv \neg(a \wedge b)$

(c) is correct

(c) $a \to \neg b \equiv \neg a \to b$

(d) $a \to \neg b \equiv b \to \neg a$

Answer (c) is correct.

Let us create a truth table with all formulas included:

a	b	$\neg a$	$\neg b$	$a \to \neg b$	$\neg a \vee \neg b$	$a \wedge b$	$\neg(a \wedge b)$	$\neg a \to b$	$b \to \neg a$
0	0	1	1	1	1	0	1	0	1
0	1	1	0	1	1	0	1	1	1
1	0	0	1	1	1	0	1	1	1
1	1	0	0	0	0	1	0	1	0

So $a \to \neg b$ is not equivalent to $\neg a \to b$ as those columns are different. And the other options indeed have the same column as $a \to b$.

It is also possible to show this using logical laws:

- $a \to \neg b \equiv \neg a \vee \neg b$ by definition.
- $\neg a \vee \neg b \equiv \neg(a \wedge b)$ by De Morgan.
- $\neg a \vee \neg b \equiv \neg b \vee \neg a$ by commutativity.
- $\neg b \vee \neg a \equiv b \to \neg a$ by definition.

4. How many models v of propositional logic are there in which the formula $a \vee b \to b \wedge a$ does not hold. Give them all. Explain your answer.

   First, we write the formula with the brackets to make the structure exactly clear:

   $$((a \vee b) \to (b \wedge a))$$

   Then, we try to find models for which this formula does not hold. As it is an implication, this means that we need models where the premise does hold, but the conclusion does not. So $a \vee b$ should hold and $a \wedge b$ should not hold. There are three models for which the first formula holds: $v_1(a) = 0$ and $v_1(b) = 1$, $v_2(a) = 1$ and $v_2(b) = 0$, and $v_3(a) = 1$ and $v_3(b) = 1$. However, the second formula does not hold in v_1 and v_2, but it does hold in v_3. So all models in which the given formula does not hold are v_1 and v_2.

   Of course, we could have created a truth table, but we wanted to explain it also in a different way.

a	b	$a \vee b$	$b \wedge a$	$((a \vee b) \to (b \wedge a))$
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

   In this truth table the rows with the 0's in the last column coincide exactly with the models v_1 and v_2 mentioned above.

Predicate logic

5. We want to formalize the following English sentence:

   There are tall men that are intelligent.

   For the dictionary we use:

M	men
$T(x)$	x is tall
$I(x)$	x is intelligent

   Which of the following formulas of predicate logic is the best formalization of this?

   (a) $\forall x \in M\,[T(x) \to I(x)]$

   (b) $\forall x \in M\,[T(x) \wedge I(x)]$

   (c) $\exists x \in M\,[T(x) \to I(x)]$

   (d) is correct (d) $\exists x \in M\,[T(x) \wedge I(x)]$

   Answer (d) is correct.

   Let us give the translations of the given formulas:

   (a) $\forall x \in M\,[T(x) \to I(x)]$
   This means *All tall men are intelligent.* This does not allow the situation that there are tall men who are not intelligent, which is possible in the original sentence. So it is not a good translation.

(b) $\forall x \in M\,[T(x) \wedge I(x)]$

This means *All men are tall and intelligent.* The combination of $\forall$ and $\wedge$ is always suspicious and usually wrong. In this case, it is also really wrong as this formula does not allow the situation that there are tall men who are not intelligent, which is possible in the original sentence. So it is not a good translation.

(c) $\exists x \in M\,[T(x) \rightarrow I(x)]$

This means *There exists a man for whom it holds that if he is tall, then he is intelligent.* The combination of $\exists$ and $\rightarrow$ is always suspicious and usually wrong. In this case, it is also really wrong as this formula is already true in situations when there are no tall men at all, which is not possible in the original sentence. So it is not a good translation.

(d) $\exists x \in M\,[T(x) \wedge I(x)]$

This means *There exists a tall and intelligent man.* This is correct. The statement with 'there are' seems to imply that there must be at least two tall and intelligent men, but it doesn't do that explicitly. It actually only states that there is at least one tall and intelligent man. So the 'there exists' which also states that there is at least one tall and intelligent man fits quite nicely. So it is a good translation.

6. Formalize the following English sentence as a formula of predicate logic with equality:

   Only Sharon loves Koos.

   Write the formula according to the official grammar from the course notes and use for the dictionary:

H	humans
k	Koos
s	Sharon
$L(x, y)$	x loves y

   If you don't recall the official grammar: you will already get points for a formula with the proper meaning.

   The sentence can be rephrased as:

   Sharon loves Koos and if some human loves Koos, then it is Sharon.

   This would lead to the formula:

$$\underbrace{\left(L(s,k) \wedge \underbrace{\left(\forall h \in H \underbrace{\left(L(h,k) \rightarrow \underbrace{(h = s)}_{=} \right)}_{\rightarrow} \right)}_{\forall} \right)}_{\wedge}$$

7. Which of the following models is *not* suitable for showing that

$$\exists x \in D\, \forall y \in D\; R(x,y) \not\equiv \forall y \in D\, \exists x \in D\; R(x,y)$$

(a) $(\mathbb{N}, =)$, where D is interpreted as $\mathbb{N}$ and $R(x,y)$ as $x = y$

(b) is correct

(b) $(\mathbb{N}, <)$, where D is interpreted as $\mathbb{N}$ and $R(x,y)$ as $x < y$

(c) $(\mathbb{Z}, =)$, where D is interpreted as $\mathbb{Z}$ and $R(x,y)$ as $x = y$

(d) $(\mathbb{Z}, <)$, where D is interpreted as $\mathbb{Z}$ and $R(x,y)$ as $x < y$

Answer (b) is correct.

As usual we look at the meaning of all formulas in the given models. A model is suitable to show that the formulas are not equivalent if in the model the left formula evaluates to a different truth value than the right formula. Hence a model is not suitable if both formulas evaluate to the same truth value.

(a) $(\mathbb{N}, =)$, where D is interpreted as $\mathbb{N}$ and $R(x,y)$ as $x = y$

The left formula means *There exists a natural number x such that for all natural numbers y, it holds that $x = y$.* As this would imply that all natural numbers are the same, it evaluates to false.

The right formula means *For each natural number y, there exists a natural number x such that $x = y$.* This evaluates to true, as we can always take $x = y$.

So this model is suitable for showing that the formulas are *not* logically equivalent.

(b) $(\mathbb{N}, <)$, where D is interpreted as $\mathbb{N}$ and $R(x,y)$ as $x < y$

The left formula means *There exists a natural number x such that for all natural numbers y, it holds that $x < y$.* This evaluates to false as there is no natural number x such that $x < 0$.

The right formula means *For each natural number y, there exists a natural number x such that $x < y$.* This also doesn't hold, as for each $y = 0$ we can't find an x such that $x < 0$.

So this model is *not* suitable for showing that the formulas are *not* logically equivalent.

(c) $(\mathbb{Z}, =)$, where D is interpreted as $\mathbb{Z}$ and $R(x,y)$ as $x = y$

The left formula means *There exists an integer x such that for all integers y, it holds that $x = y$.* As this would imply that all integers are the same, it evaluates to false.

The right formula means *For each integer y, there exists an integer x such that $x < y$.* This holds, as for each y we can take $x = y - 1$.

So this model is suitable for showing that the formulas are *not* logically equivalent.

(d) $(\mathbb{Z}, <)$, where D is interpreted as $\mathbb{Z}$ and $R(x,y)$ as $x < y$

The left formula means *There exists an integer x such that for all integers y, it holds that $x < y$.* This evaluates to false. No matter which smart x we choose, we can always take $y = x$ and then $x < y$ doesn't hold.

The right formula means *For each integer y, there exists an integer x such that $x < y$.* This holds, as for each y we can take $x = y - 1$.

So this model is suitable for showing that the formulas are *not* logically equivalent.

8. Does the following logical equivalence hold?

$$\forall x \in M\, [\forall y \in M\ T(y) \to T(x)] \equiv \forall x \in M\, [\forall y \in M [T(y) \to T(x)]]$$

The difference between the two formulas being compared here, is that the right formula has additional brackets.

(a) Yes, those two formulas are the same.

(b) Yes, these formulas are logically equivalent. The formulas differ, but they both mean that if a man y is tall, then a man x is also tall.

(c) is correct

(c) No. In the model (men, tall) where M is interpreted as 'men' and $T(x)$ as 'x is tall', the left formula holds but the right formula does not hold, because some men are tall and some men aren't.

(d) No. In the model (men, tall) where M is interpreted as 'men' and $T(x)$ as 'x is tall', the left formula does not hold but the right formula does hold, because some men are tall and some men aren't.

Answer (c) is correct.

Due to the fact that the quantifiers bind strongly, the missing brackets on the left imply that we should parse the formula as

$$\forall x \in M\ \big[\big(\forall y \in M\ T(y)\big) \to T(x)\big]$$

In particular this means that if $\forall y \in M\ T(y)$ doesn't hold, then automatically the complete formula on the left does hold.

And the formula on the right says that for each pair of x and y in M either both have property T or both don't have property T. In particular, this implies that either all elements have the property T or no elements have this property.

So if there is a model such that some elements of the domain do have the property T and some do not have the property T, then the formula on the left holds, but the formula on the right does not.

In the given model (men, tall) where M is interpreted as 'men' and $T(x)$ as 'x is tall', this is indeed the case because some men are tall and some men aren't.

Note that if the model is chosen in such a way that the right formula holds, then the left formula also holds. As we have seen, if the right formula holds, then either all elements have the property T, which implies that the conclusion of the implication on the left holds, so the implication itself holds, or, no elements have the property T, which implies that the premise of the implication on the left does not hold, so the implication itself holds.

The first and the second answer make no sense because the answer is 'no'.