Formal Reasoning 2022
Solutions Test Block 3: Discrete Mathematics and Modal Logic
(22/12/22)

Discrete Mathematics

1. We are looking for a graph with the following properties:

   - Each vertex in the graph has degree three.
   - The graph has a Hamiltonian cycle.
   - The graph is not bipartite.

   Which of the following graphs satisfies this?

   (a) The cube graph.
   (b) The Petersen graph.
   (c) The graph $K_{3,3}$.
   (d) is correct (d) None of the above.

   Answer (d) is correct.

   Let's look at the three graphs:

   (a) The cube graph.
   - Each vertex in the graph has degree three. ✔
   - The graph has a Hamiltonian cycle. ✔
   - The graph is bipartite. ✘

   So it does not comply with all three requirements.

   (b) The Petersen graph.
   - Each vertex in the graph has degree three. ✔
   - The graph does not have a Hamiltonian cycle. ✘
   - The graph is not bipartite. ✔

   So it does not comply with all three requirements.

   (c) The graph $K_{3,3}$.
   - Each vertex in the graph has degree three. ✔
   - The graph has a Hamiltonian cycle. ✔
   - The graph is bipartite. ✘

   So it does not comply with all three requirements.

   Hence the conclusion is that 'none of the above' is the correct answer.

2. We define a sequence recursively by:

$$a_0 = 0$$
$$a_{n+1} = a_n + 2n$$

   Prove by induction that $a_n = n(n-1)$ for all $n \geq 0$. Follow the template.

   0 **Proposition:**

$a_n = n(n-1)$ for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \quad := \quad a_n = n(n-1)$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that

$a_0 = 0 \cdot (0-1)$

4 This indeed holds, because

$$a_0 = 0 = 0 \cdot -1 = 0 \cdot (0-1)$$

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that

$a_k = k(k-1)$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that

$a_{k+1} = (k+1)((k+1)-1)$

8 This indeed holds, because

$$\begin{aligned} a_{k+1} &= a_k + 2k \\ &\overset{\text{IH}}{=} k(k-1) + 2k \\ &= k^2 - k + 2k \\ &= k^2 + k \\ &= (k+1)k \\ &= (k+1)((k+1)-1) \end{aligned}$$

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

3. The equality

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots \binom{n}{n} = 0$$

holds for every $n > 0$. For example for $n = 3$ this becomes:

$$\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0$$

How can one see that this equality is correct for all $n > 0$?

(a) This follows from the fact that each row of Pascal's triangle is symmetric, i.e., we have $\binom{n}{k} = \binom{n}{n-k}$ for each $0 \leq k \leq n$.

(b) This follows from the binomial theorem, by considering the expansion of $(1-1)^n$.

(b) is correct

(c) Both arguments work for all $n > 0$.

(d) Neither argument works for all $n > 0$.

Answer (b) is correct.

Newton's Binomial Theorem states that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

So if $x = -1$, we get

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n}$$

And $(1-1)^n = 0$ for $n > 0$.

Note that the argument about Pascal's triangle doesn't hold for even n. For instance for $n = 4$ we get

$$\binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 1 - 4 + 6 - 4 + 1 = 0$$

and we see that the symmetric values $\binom{n}{k}$ and $\binom{n}{n-k}$ have the same sign and do not cancel out.

4. In how many ways can we separate 5 distinguishable objects in 3 non-distinguishable non-empty piles?

   (a) $\left[{5 \atop 3}\right]$

   (b) is correct — (b) $\left\{{5 \atop 3}\right\}$

   And what is the outcome as a natural number?

   (a) is correct — (a) 25

   (b) 35

   (c) 65

   (d) 85

Answer (a) is correct.

As we are talking about distinguishable objects and indistinguishable non-empty piles, we need to use the Stirling numbers of the second kind. So the answer is $\left\{{5 \atop 3}\right\}$ which is 25 as can be seen in the triangle:

						1						
					1		1					
				1		3		1				
			1		7		6		1			
		1		15		[25]		10		1		
	1		31		90		65		15		1	
1		63		301		350		140		21		1

And these are all options:

$$\begin{gathered}\{1,2,3\}+\{4\}+\{5\}, \quad \{1,2,4\}+\{3\}+\{5\}, \quad \{1,3,4\}+\{2\}+\{5\}, \quad \{2,3,4\}+\{1\}+\{5\}, \\ \{1,2,5\}+\{3\}+\{4\}, \quad \{1,3,5\}+\{2\}+\{4\}, \quad \{2,3,5\}+\{1\}+\{4\}, \\ \{1,4,5\}+\{2\}+\{3\}, \quad \{2,4,5\}+\{1\}+\{3\}, \\ \{3,4,5\}+\{1\}+\{2\}, \\ \{1,2\}+\{3,4\}+\{5\}, \quad \{1,3\}+\{2,4\}+\{5\}, \quad \{1,4\}+\{2,3\}+\{5\}, \\ \{1,2\}+\{3,5\}+\{4\}, \quad \{1,3\}+\{2,5\}+\{4\}, \quad \{1,5\}+\{2,3\}+\{4\}, \\ \{1,2\}+\{4,5\}+\{3\}, \quad \{1,4\}+\{2,5\}+\{3\}, \quad \{1,5\}+\{2,4\}+\{3\}, \\ \{1,3\}+\{4,5\}+\{2\}, \quad \{1,4\}+\{3,5\}+\{2\}, \quad \{1,5\}+\{3,4\}+\{2\}, \\ \{2,3\}+\{4,5\}+\{1\}, \quad \{2,4\}+\{3,5\}+\{1\}, \quad \{2,5\}+\{3,4\}+\{1\}\end{gathered}$$

Modal Logic

5. Consider the dictionary:

R	it rains
U	I use an umbrella

Using this dictionary, what does the strict implication

$$\Box(R \to U)$$

mean in deontic logic?

(a) When it rains, I know I use an umbrella.

(b) When it rains, I believe I use an umbrella.

(c) is correct — (c) When it rains, I must use an umbrella.

(d) When it rains, I always use an umbrella.

Answer (c) is correct.

As deontic logic is about obligation, it must be the option with 'must' in it.

6. Give a Kripke model $\mathcal{M}$ that shows that axiom scheme B does not generally hold, i.e., that there is a modal formula f such that

$$\not\models (f \to \Box\Diamond f)$$

Write your model $\mathcal{M}$ as a triple $\langle W, R, V\rangle$ and explicitly give W, R, V and a formula f. You don't need to explain your answer.

Let $\mathcal{M} = \langle W, R, V\rangle$ where

$$W = \{x_0, x_1\}$$

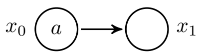
and the accessibility function R is defined by

$$\begin{aligned} R(x_0) &= \{x_1\} \\ R(x_1) &= \emptyset \end{aligned}$$

and the valuation function V is defined by

$$\begin{aligned} V(x_0) &= \{a\} \\ V(x_1) &= \emptyset \end{aligned}$$

As a diagram it looks like this:

$\mathcal{M} \quad :=$ 

Now if we take $f = a$ then we get that $a \to \Box\Diamond a$ does not hold in all worlds of this model. This follows from the following $\Vdash$-table:

world $\Vdash$	a	$\Diamond a$	$\Box\Diamond a$	$a \to \Box\Diamond a$
x_0	1	0	0	0
x_1	0	0	1	1

From this $\Vdash$-table it follows that $x_0 \nVdash a \to \Box\Diamond a$, and hence $\mathcal{M} \nvDash a \to \Box\Diamond a$, and hence also $\nvDash a \to \Box\Diamond a$. Hence axiom scheme B does not generally hold.

7. Does $\vDash_T f$ imply that $\vDash_D f$ for all modal formulas f?

   (a) Yes, because all serial models are reflexive.

   (b) Yes, because all reflexive models are serial.

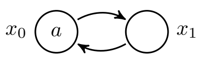
   (c) is correct — (c) No, because not all serial models are reflexive.

   (d) No, because not all reflexive models are serial.

   Answer (c) is correct.

   Note that $\vDash_T f$ means that f holds in all reflexive models and that $\vDash_D f$ means that f holds in all serial models. In addition, note that every reflexive model is also a serial model, but not all serial models are reflexive. Hence there are more serial models than reflexive models and hence the claim that f holds in all serial models is stronger than the claim that f holds in all reflexive models.

   For instance, if we take formula $f = \Box a \to a$, then this formula indeed holds in all reflexive models as it is an instance of axiom scheme T which by definition holds in all reflexive models. However, this formula does not hold in all serial models. Take for instance this serial model

   $\mathcal{M} \quad :=$ 

   For this model we get the following $\Vdash$-table:

world $\Vdash$	a	$\Box a$	$\Box a \to a$
x_0	0	1	0
x_1	1	0	1

   So $x_0 \nVdash \Box a \to a$, so $\mathcal{M} \nvDash \Box a \to a$, and so $\vDash_T \Box a \to a$ but not $\vDash_D \Box a \to a$.

8. We want to formalize the sentence

   I will work until I leave home.

   as a formula of LTL. We take this sentence to mean that I will stop working (at least for some time) when I leave home, and we will allow for the possibility that I leave home immediately, in which case I actually won't work at all before that.

   We use the dictionary:

W	I work
H	I am at home

Now consider the following two LTL formulas:

$$(W \wedge H)\, \mathcal{U}\, (\neg W \wedge \neg H)$$

$$(W\, \mathcal{U}\, \neg H) \wedge (H\, \mathcal{U}\, \neg W)$$

Which of these is (are) a good formalization of the sentence?

(a) The first.

(b) The second.

(c) Both. (c) is correct

(d) Neither.

Answer (c) is correct.

Note that the most direct solution seems

$$W\, \mathcal{U}\, \neg H$$

However, this solution also allows models where W and $\neg H$ are true at the same moment, as $W\, \mathcal{U}\, \neg H$ does not enforce that W no longer holds when $\neg H$ holds, which is what is intended in the sentence above. The first solution fixes this by explicitly adding the H before the $\mathcal{U}$ and the $\neg W$ after the $\mathcal{U}$. So models in which this formula is true, match with the intended meaning of the sentence.

The second formula seems completely different, but it can be proven that these two formulas are logically equivalent. And hence they satisfy the same models.

At first glance it may seem that for the second formula we can choose two possibly different moments for the $\mathcal{U}$ operator. However, if we assume that $W\, \mathcal{U}\, \neg H$ is true because $\neg H$ is true in moment j_0 and if we assume that $H\, \mathcal{U}\, \neg W$ is true because $\neg W$ is true in moment j_1 and $j_0 < j_1$, then it follows that at moment j_0 both $\neg H$ and H are true, which cannot be the case. Hence j_0 cannot be before j_1. Likewise if $j_1 < j_0$ we get that at moment j_1 both $\neg W$ and W are true, which cannot be the case either. So $j_0 = j_1$, and then it easily follows that before j_0 the formulas W and H are true and at j_0 the formulas $\neg H$ and $\neg W$. But that is clearly equivalent to the first formula

$$(W \wedge H)\, \mathcal{U}\, (\neg W \wedge \neg H)$$

So both formulas are good formalizations of the sentence.