Formal Reasoning 2022
Solutions Test Blocks 1, 2 and 3: Additional Test
(11/01/23)

1. Someone formalizes the sentence

   I am inside if and only if I am not outside.

   as
   $$\mathsf{In} \leftrightarrow \neg\mathsf{Out}$$
   using the dictionary

In	I am inside
Out	I am outside

   and notes the following logical equivalences:
   $$\mathsf{In} \leftrightarrow \neg\mathsf{Out} \equiv (\mathsf{In} \to \neg\mathsf{Out}) \land (\neg\mathsf{Out} \to \mathsf{In})$$
   $$\mathsf{In} \leftrightarrow \neg\mathsf{Out} \equiv (\mathsf{In} \to \neg\mathsf{Out}) \land (\neg\mathsf{In} \to \mathsf{Out})$$
   Now which part of these statements corresponds to the 'if' part of the sentence, and which to the 'only if' part?

   (a) The 'if' part corresponds to $(\mathsf{In} \to \neg\mathsf{Out})$ and the 'only if' part corresponds to $(\neg\mathsf{Out} \to \mathsf{In})$.

   (b) The 'if' part corresponds to $(\neg\mathsf{Out} \to \mathsf{In})$ and the 'only if' part corresponds to $(\mathsf{In} \to \neg\mathsf{Out})$.

   (b) is correct

   (c) The 'if' part corresponds to $(\mathsf{In} \to \neg\mathsf{Out})$ and the 'only if' part corresponds to $(\neg\mathsf{In} \to \mathsf{Out})$.

   (d) The 'if' part corresponds to $(\neg\mathsf{In} \to \mathsf{Out})$ and the 'only if' part corresponds to $(\mathsf{In} \to \neg\mathsf{Out})$.

   Answer (b) is correct.

   The 'if' part is

   I am inside if I am not outside.

   So 'if I am not outside then I am inside', which corresponds to $(\neg\mathsf{Out} \to \mathsf{In})$. Likewise, the 'only if' part is

   I am inside only if I am not outside.

   So 'if I am inside, it must be that I am not outside', which corresponds to $(\mathsf{In} \to \neg\mathsf{Out})$.

2. Consider the following formula of predicate logic:
   $$\forall x \in D\ R(x, a) \to Q(b)$$
   Which of the following is this formula written according to the official grammar from the course notes?

(a)

$$((\forall x \in D\, R(x,a)) \rightarrow Q(b))$$

(a) is correct

(b)

$$(\forall x \in D\, (R(x,a) \rightarrow Q(b)))$$

(c)

$$(\forall x \in D\, [R(x,a)] \rightarrow Q(b))$$

(d)

$$(\forall x \in D\, [R(x,a) \rightarrow Q(b)])$$

Answer (a) is correct.

Because the quantifier binds stronger than the implication, the correct answer must be $((\forall x \in D\, R(x,a)) \rightarrow Q(b))$ or $(\forall x \in D\, [R(x,a)] \rightarrow Q(b))$. However, the official notation does not allow square brackets, so it must be $((\forall x \in D\, R(x,a)) \rightarrow Q(b))$.

3. The six kinds of regular expressions are

$$\begin{gathered} a, b, c \ldots \\ r^* \\ r_1 r_2 \\ r_1 \cup r_2 \\ \lambda \\ \varnothing \end{gathered}$$

but couldn't some have been expressed in terms of the others?

(a) λ^* can be used in the place of $\varnothing$.

(b) is correct

(b) $\varnothing^*$ can be used in the place of λ.

(c) Neither is the case.

(d) Both are the case.

Answer (b) is correct.

First note that λ^* cannot be used to replace $\varnothing$ as

$$\mathcal{L}(\lambda^*) = (\mathcal{L}(\lambda))^* = \{\lambda\}^* = \{\lambda\} \neq \varnothing = \mathcal{L}(\varnothing)$$

However, $\varnothing^*$ can be used to replace λ as

$$\mathcal{L}(\varnothing^*) = (\mathcal{L}(\varnothing))^* = \varnothing^* = \{\lambda\} = \mathcal{L}(\lambda)$$

We omit the formal proof by induction that this substitution also works when the λ that we replace is a subexpression of a more complex regular expression.

4. Consider the context-free grammar G_4 given by:

$$\begin{aligned} S &\rightarrow aAa \mid Sbb \mid a \\ A &\rightarrow abaS \end{aligned}$$

Give an invariant that shows that $aabaa \notin \mathcal{L}(G_4)$. Don't explain your answer: just giving the invariant is sufficient.

In order to simplify the argument we introduce some new notation

$$|w|_s \quad := \quad \text{the number of occurrences of symbol } s \text{ in word } w$$

Now we take as invariant:

$$P(w) \quad := \quad |w|_S + |w|_A + |w|_a \text{ is odd}$$

Or in natural language the number of non-terminals together with the number of a's is odd.

We prove that this is indeed an invariant:

- $P(S)$ holds as $|S|_S + |S|_A + |S|_a = 1 + 0 + 0 = 1$ and 1 is indeed odd.
- Now assume that v and v' are such that $P(v)$ holds and $v \to v'$. We show that $P(v')$ also holds. From $P(v)$ it follows that $|v|_S + |v|_A + |v|_a$ is odd. We check that $P(v')$ holds for each possible step $v \to v'$.
  - $S \to aAa$: in this case we get that

    $$\begin{aligned} |v'|_S + |v'|_A + |v'|_a &= (|v|_S - 1) + (|v|_A + 1) + (|v|_a + 2) \\ &= |v|_S + |v|_A + |v|_a + 2 \end{aligned}$$

    which is an odd number, as we add 2 to a number that is known to be odd.
  - $S \to Sbb$: in this case we get that

    $$|v'|_S + |v'|_A + |v'|_a \quad = \quad |v|_S + |v|_A + |v|_a$$

    which is known to be odd.
  - $S \to a$: in this case we get that

    $$\begin{aligned} |v'|_S + |v'|_A + |v'|_a &= (|v|_S - 1) + |v|_A + (|v|_a + 1) \\ &= |v|_S + |v|_A + |v|_a \end{aligned}$$

    which is known to be odd.
  - $A \to abaS$: in this case we get that

    $$\begin{aligned} |v'|_S + |v'|_A + |v'|_a &= (|v|_S + 1) + (|v|_A - 1) + (|v|_a + 2) \\ &= |v|_S + |v|_A + |v|_a + 2 \end{aligned}$$

    which is an odd number, as we add 2 to a number that is known to be odd.

  So in all cases $|v'|_S + |v'|_A + |v'|_a$ is odd and hence $P(v')$ indeed holds.

So $P(w)$ is indeed an invariant of the grammar G_4. And as $P(aabaa)$ does not hold, it is clear that $aabaa \notin \mathcal{L}(G_4)$.

We don't provide a proof for it, but this is also an invariant that can be used to show that $aabaa \notin \mathcal{L}(G_4)$:

$$\begin{aligned} P(w) \quad := \quad & w \in \{S, a, aAa, Sbb, abb, aAabb, Sbbbb, abbbb\} \\ & \text{or the length of } w \text{ is at least 6} \end{aligned}$$

5. Let M_5 be some deterministic finite automaton with

$$\mathcal{L}(M_5) = \{w \in \{a,b\}^* \mid w \text{ does not contain } aaa\}$$

Does the machine M_5 necessarily have a sink state?

(a) Yes, after you have processed three a's in a row, you will be in a sink state.

(b) Yes, every finite automaton can have a sink state.

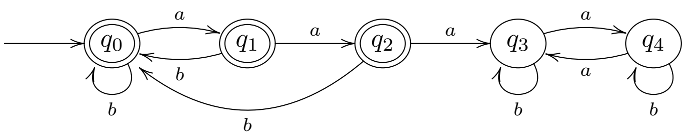
(c) is correct

(c) No, there can be multiple states that are not final, with transitions between them.

(d) No, this language is infinite, so from every state, it has to be possible to get to a final state that accepts the input.

Answer (c) is correct.

The following automaton does accept $\mathcal{L}(M_5)$, but after processing aaa, you are not in a sink, as q_3 has a transition to q_4.



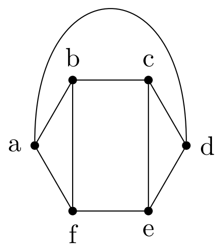
Note that the only requirement is that once in q_3 there are no transitions possible that will lead to a final state. This is typically achieved by a sink, but it can also be done by a more complex sink-like construction.

Note that the last answer is incorrect. If it were possible to reach a final state after processing aaa, then words that are not in the language will be accepted by the automaton.

6. - Give a planar graph G_6 that has six vertices that all have degree three, and which is not bipartite. Write your answer in the form of a pair $\langle V, E\rangle$.

   You do not need to explain why your graph has these properties.
   - Does this graph have a Hamiltonian cycle? *That* you have to explain.

Take for instance the graph G_6:



As a tuple this is the graph $G_6 := \langle V, E\rangle$ where

$$V := \{a, b, c, d, e, f\}$$
$$E := \{(a,b), (b,c), (c,d), (d,e), (e,f), (f,a), (a,d), (b,f), (c,e)\}$$

Now let us discuss the required properties:

- G_6 clearly has six vertices.
- All vertices have degree three.
- G_6 is clearly planar.
- G_6 is not bipartite as it contains the subgraph $\langle\{a,b,f\},\{(a,b),(b,f),(f,a)\}\rangle$ which is isomorphic to K_3 which means that its chromatic number is three, which means that it is not bipartite.

Note that this graph G_6 does have a Hamiltonian cycle: the cycle $a \to b \to c \to d \to e \to f \to a$ visits all vertices exactly once, except for the fact that the starting vertex is the same as the ending vertex.

7. Someone tries to define the factorial sequence $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 6$, ..., but does not get it completely right, because in the definition the multiplication is with n instead of $n+1$:

$$a_0 = 1$$
$$a_{n+1} = n \cdot a_n$$

What is the value of a_4 according to this incorrect definition?

(a) 6
(b) 24
(c) 120
(d) None of the above.

(d) is correct

Answer (d) is correct.

We can compute the following values:

$$\begin{array}{lllllllll} a_0 & = & & & & & & & 1 \\ a_1 & = & a_{0+1} & = & 0 \cdot a_0 & = & 0 \cdot 1 & = & 0 \\ a_2 & = & a_{1+1} & = & 1 \cdot a_1 & = & 1 \cdot 0 & = & 0 \\ a_3 & = & a_{2+1} & = & 2 \cdot a_2 & = & 2 \cdot 0 & = & 0 \\ a_4 & = & a_{3+1} & = & 3 \cdot a_3 & = & 3 \cdot 0 & = & 0 \end{array}$$

So none of the recursively defined numbers is correct.

8. Consider all Kripke models $\mathcal{M}$ that have the property $\mathcal{M} \vDash a$. Is it the case that for all these models also $\mathcal{M} \vDash \Diamond a$?

(a) Yes, if something is *actually* the case (in this instance: 'a'), then it is obviously *possible* that it is the case.

(b) Yes, a will be true in all worlds of $\mathcal{M}$, so whatever world we access, a will hold there.

(c) No, this holds if and only if the model $\mathcal{M}$ is *serial* (the property that corresponds to the axiom scheme D, which is $\Box f \to \Diamond f$.)

(c) is correct

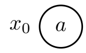
(d) No, this holds if and only if the model $\mathcal{M}$ is *reflexive* (the property that corresponds to the axiom scheme T, one of which forms is $f \to \Diamond f$.)

Answer (c) is correct.

We start by proving the 'if' part of the statement. Let $\mathcal{M}$ be a serial model such that $\mathcal{M} \vDash a$ holds. This means that for each world x in this model $\mathcal{M}$ $x \Vdash a$ holds. Now let x_i be an arbitrary world in model $\mathcal{M}$. Then, as $\mathcal{M}$ is serial, by definition there has to be a world x_j that is accessible from x_i. However, by assumption we know that $x_j \Vdash a$ holds. And this means automatically that $x_i \Vdash \Diamond a$ holds as well.

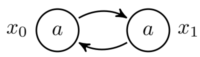
We continue by proving the 'only if' part of the statement. Let $\mathcal{M}$ be a model that has the properties $\mathcal{M} \Vdash a$ and $\mathcal{M} \vDash \Diamond a$. Then in particular $\mathcal{M} \vDash \Diamond a$ holds. So for each world x in $\mathcal{M}$ we have that $x \Vdash \Diamond a$ holds. Now this means that each world x has at least one accessible world where a holds. So in particular it means that each world x has at least one accessible world, regardless of the valuations in that accessible world. Hence $\mathcal{M}$ is by definition a serial model.

We provide some examples that indicate why the other options are indeed wrong. First we give the model $\mathcal{M}_8$:

$\mathcal{M}_8 \quad :=$ 

As $x_0 \Vdash a$ and x_0 is the only world in $\mathcal{M}_8$ it follows that $\mathcal{M}_8 \vDash a$ holds. However, as x_0 has no accessible worlds $x_0 \Vdash \Diamond a$ clearly does not hold and hence $\mathcal{M}_8 \vDash \Diamond a$ also does not hold. So the 'Yes, ...' answers are clearly wrong.

Next we give the model $\mathcal{M}'_8$:

$\mathcal{M}'_8 \quad :=$ 

Now both $\mathcal{M}'_8 \vDash a$ and $\mathcal{M}'_8 \vDash \Diamond a$ hold. Note that this model is not reflexive. So the answer about reflexive models is also clearly wrong.