Formal Reasoning 2023
Exam
(18/01/24)

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The mark for this test is the number of points divided by ten, The first 10 points are free. Good luck!

Propositional logic

1. We want to formalize the following sentence as a formula of propositional logic:

   It rains and the sun shines, when there is a rainbow.

   We use for the dictionary:

R	it rains
S	the sun shines
RB	there is a rainbow

   Which of the following formulas is a good choice for this?

   (a) $R \wedge S \to RB$
   (b) $RB \to R \wedge S$
   (c) $R \wedge S \leftrightarrow RB$
   (d) $R \wedge S \wedge RB$

2. Consider the formula of propositional logic:

   $$(a \to b \to c) \to (a \to b) \to a \to c$$

   Now answer the following two questions:

   (a) Write this formula with parentheses according to the official grammar of propositional formulas from the course notes.
   (b) Give the number of zeroes and the number of ones in the final column of this formula in its truth table. Don't give the truth table itself, but only the requested numbers.

3. Does the following hold for all formulas f of propositional logic?

   $$\not\models f \quad \text{iff} \quad \models \neg f$$

   (a) Yes, because in both cases it is stated that f is false.
   (b) Yes, because an example is $f = a \wedge \neg a$.
   (c) No, because a counter example is $f = a \vee \neg a$.
   (d) No, because a counter example is $f = a$.

Predicate logic

4. Formalize the following English sentence as a formula of predicate logic:

   I will tell the truth, the whole truth, and nothing but the truth.

   Use for the dictionary:

C	the domain of relevant claims
$I(x)$	I will tell claim x
$T(x)$	claim x is true

   Hint: Note that 'telling the truth' and 'telling the whole truth' are essentially the same thing, so focus on the sentence *I will tell the whole truth and nothing but the truth.* in your formalization.

5. Which of the following logical laws hold?

   (a) $\forall x \in D\, P(x) \equiv \neg\exists x \in D\, \neg P(x)$

   (b) $\neg\forall x \in D\, P(x) \equiv \exists x \in D\, \neg P(x)$

   (c) $\forall x \in D\, \neg P(x) \equiv \neg\exists x \in D\, P(x)$

   (d) All three of the above statements hold.

6. Someone tries to formalize:

   There are at most two intelligent men.

   They use the dictionary:

M	the domain of men
$I(x)$	x is intelligent

   However their attempted solution is:

   $$\forall x, y, z \in M[I(x) \wedge I(y) \wedge I(z) \to x = y \vee x = z]$$

   Compared to the standard solution this misses the disjunct $y = z$.

   But what does *this* formula of predicate logic with equality mean?

   (a) It means that there are at most *three* intelligent men. Because there is one disjunct less, the statement is less restrictive and there now is a possibility for this to still be true with one more intelligent man.

   (b) It still means that there are at most *two* intelligent men. The disjunct $y = z$ is not needed, because one can always take the x, y and z in such an order that this case does not occur.

   (c) It means that there is at most *one* intelligent man. If there are intelligent men m_1 and m_2, then the formula implies that they have to be the same by taking $x = m_1$ and $y = z = m_2$. In that case the conclusion of the implication becomes $m_1 = m_2 \vee m_1 = m_2$, which is logically equivalent to $m_1 = m_2$.

   (d) The formula means something different from the meanings in the three answers above.

Languages

7. Does
$$L^*\overline{L}^* = \left(L\overline{L}\right)^*$$
hold for all languages L?

Note that you should read $\overline{L}^*$ as $\left(\overline{L}\right)^*$.

   (a) Yes. We are combining L and its complement $\overline{L}$, which means that in both languages we can make any possible word, by combining parts that are in L and in $\overline{L}$.
   (b) Yes, although that language does not necessarily contain all possible words from Σ^*.
   (c) No. If we take the language $L = \{a\}$ with $\Sigma = \{a, b\}$, then $b \in L^*\overline{L}^*$ but $b \notin (L\overline{L})^*$.
   (d) No. If we take the language $L = \{a\}$ with $\Sigma = \{a, b\}$, then $b \notin L^*\overline{L}^*$ but $b \in (L\overline{L})^*$.

8. What is a regular expression for the following language?
$$\mathcal{L}(a^*b^*) \cap \mathcal{L}((ab)^*)$$
   (a) $\varnothing$
   (b) λ
   (c) $\lambda \cup ab$
   (d) $a^*b^* \cap (ab)^*$

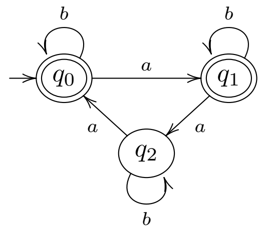
9. Give a context-free grammar for the language:
$$\{a^n b^n a^m \mid n, m \in \mathbb{N}\}$$

10. Consider the context-free grammar G_{10}:
$$S \to aA$$
$$A \to bS \mid b$$
Someone brings up the property
$$P(w) := w \text{ contains the same number of } a\text{'s and } b\text{'s}$$
and wants to show that
$$abb \notin \mathcal{L}(G_{10})$$
claiming that P is an invariant of G_{10}. Does this work?

   (a) Yes. The property P is indeed an invariant for G_{10}, because every time the rule $S \to aA$ adds an a, it has to be followed by either $A \to bS$ or $A \to b$, both of which add a b.
   (b) Yes. The only possible production for this word has to start as:
$$S \to aA \to abS$$
   but this can only continue as $abS \to abaA$, which shows that indeed the word abb is not in the language of G_{10}.
   (c) No, although the property P is an invariant for G_{10}, it cannot be used to show that $abb \notin \mathcal{L}(G_{10})$.
   (d) No, because P is not an invariant of G_{10}.

Automata

11. Give all words of length three that are accepted by the following deterministic finite automaton M_{11}:



Explain why these words are accepted.

12. Someone claims that all regular languages can be produced by a context-free grammar of which each rule has the form

$$X \to w$$

with $X \in V$ and $w \in \{\lambda\} \cup \Sigma V$.

Is this correct?

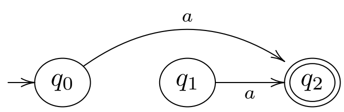
(a) Yes, this is correct. Each regular language can be accepted by a deterministic finite automaton, and this automaton can be converted to a right linear context-free grammar of this form.

(b) Yes, this is correct. In fact even all context-free languages can be produced by such a grammar.

(c) No, this is not correct. This will not work for regular languages of the form $\mathcal{L}(r_1 r_2 r_3)$, in which r_1, r_2 and r_3 are regular expressions.

(d) No, this is not correct. This only will be possible when λ is a word in the language, and there are regular languages for which this does not hold.

13. Is the following a correct non-deterministic finite automaton (an NFA) with alphabet $\Sigma = \{a\}$?



(a) Yes, this is a correct NFA, but it is not a *deterministic* finite automaton (a DFA).

(b) Yes, there is no non-determinism in this automaton, so it is a DFA, and all DFAs are NFAs.

(c) No. There is no non-determinism in this automaton, so it cannot be a *non*-deterministic finite automaton.

(d) No, because the state q_1 cannot be reached.

Discrete mathematics

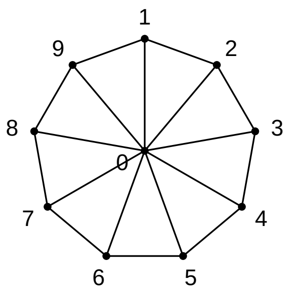
14. What is the number of different cycles in the graph K_3?

    (a) 1
    (b) 2
    (c) 6
    (d) None of the above.

15. We define the *wheel graph* W_{n+1} for $n \geq 3$ as:

    $W_{n+1} := \langle\{0, \ldots, n\}, \{(0, 1), \ldots, (0, n), (1, 2), (2, 3), \ldots, (n-1, n), (n, 1)\}\rangle$

    For example this is the wheel graph W_{10}:



    It has 10 vertices and 18 edges. Note that in this example $n = 9$, so $n + 1 = 10$.

    For which $n \geq 3$ does W_{n+1} have an Eulerian path?

    (a) For all n.
    (b) For no n.
    (c) For all even n.
    (d) For all odd n.

16. We recursively define:

$$
\begin{aligned}
f(m, 0) &= 0 && \text{for } m \geq 0 \\
f(m, n+1) &= m + f(m, n) && \text{for } m, n \geq 0
\end{aligned}
$$

    Compute $f(3, 3)$ and explain how you got your answer.

17. Someone recursively defines:

$$
\begin{aligned}
a_0 &= 1 \\
a_{n+1} &= a_n + 2n + 1 && \text{for } n \geq 0
\end{aligned}
$$

    They try to prove using induction on n that:

$$a_n = n^2$$

    Does this work?

(a) Yes, that works.

(b) No, the base case can be proven, but the induction step cannot be proven.

(c) No, the induction step can be proven, but the base case cannot be proven.

(d) No, both the base case and the induction step cannot be proven.

18. Koos has a bag holding 8 distinguishable objects, and wants to give 1 of those to Maud, 2 to Sharon and 2 to Joris (after which of course 3 objects will remain in the bag).

    In how many ways can he do this?

    (a) $\binom{8}{1} + \binom{8}{2} + \binom{8}{2} = 8 + 28 + 28 = 64$

    (b) $\binom{8}{1} + \binom{7}{2} + \binom{5}{2} = 8 + 21 + 10 = 39$

    (c) $\binom{8}{1} \cdot \binom{8}{2} \cdot \binom{8}{2} = 8 \cdot 28 \cdot 28 = 6272$

    (d) $\binom{8}{1} \cdot \binom{7}{2} \cdot \binom{5}{2} = 8 \cdot 21 \cdot 10 = 1680$

19. Someone recursively defines:

$$\begin{aligned} c(1,1) &= 1 \\ c(1,k+1) &= 0 && \text{for } k \geq 1 \\ c(n+1,1) &= n \cdot c(n,1) && \text{for } n \geq 1 \\ c(n+1,k+1) &= c(n,k) + n \cdot c(n,k+1) && \text{for } k,n \geq 1 \end{aligned}$$

    These numbers $c(n,k)$ with $1 \leq n$ and $1 \leq k \leq n$ are the…

    (a) Stirling numbers of the first kind.

    (b) Stirling numbers of the second kind.

    (c) Binomial coefficients.

    (d) None of the above.

Modal logic

20. We want to formalize the following sentence as a formula of epistemic logic:

    I don't know whether it rains.

    We use for the dictionary:

    R it rains

    Which of the following formulas is a good choice for this?

    (a) $\neg\Box R$

    (b) $\neg\Box(R \vee \neg R)$

    (c) $\neg\Box R \vee \neg\Box\neg R$

    (d) $\neg\Box R \wedge \neg\Box\neg R$

21. Give a Kripke model $\mathcal{M}_{21} = \langle W, R, V \rangle$ such that

$$\mathcal{M}_{21} \not\models a \to \Box\Diamond a$$

This is an instance of axiom scheme B (symmetry).

Define the model by giving the set W and the functions R and V. Explain why this model has the required property.

22. There is an LTL formula such that if it holds in world x_0 of an LTL model $\mathcal{M}_{22} = \langle W, R, V \rangle$, with:

$$\begin{aligned} W &= \{x_0, x_1, x_2, \dots\} \\ R(x_i) &= \{x_i, x_{i+1}, x_{i+2}, \dots\} \qquad\qquad \text{for } i \geq 0 \end{aligned}$$

then that model has to satisfy:

$$V(x_i) = \begin{cases} \{a\} & \text{if } i \text{ is even} \\ \{b\} & \text{if } i \text{ is odd} \end{cases}$$

This formula is (note that $\mathcal{G}$ binds more strongly than the conjunction):

(a) $a \land \mathcal{G}(a \lor b) \land (a \leftrightarrow \neg b) \land (a \leftrightarrow \mathcal{X}b)$
(b) $a \land \mathcal{G}(a \leftrightarrow \neg b) \land \mathcal{G}(a \leftrightarrow \mathcal{X}b)$
(c) $\mathcal{G}(a \land (a \leftrightarrow \neg b) \land (a \leftrightarrow \mathcal{X}b))$
(d) $\mathcal{G}a \land \mathcal{G}(a \leftrightarrow \neg b) \land \mathcal{G}(a \leftrightarrow \mathcal{X}b)$