Formal Reasoning 2023
Solutions Exam
(18/01/24)

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The mark for this test is the number of points divided by ten, The first 10 points are free. Good luck!

Propositional logic

1. We want to formalize the following sentence as a formula of propositional logic:

   It rains and the sun shines, when there is a rainbow.

   We use for the dictionary:

R	it rains
S	the sun shines
RB	there is a rainbow

   Which of the following formulas is a good choice for this?

   (a) $R \wedge S \to RB$
   (b) $RB \to R \wedge S$ — (b) is correct
   (c) $R \wedge S \leftrightarrow RB$
   (d) $R \wedge S \wedge RB$

   Answer (b) is correct.

   The sentence *It rains and the sun shines, when there is a rainbow.* is equivalent to *If there is a rainbow, then it rains and the sun shines.* And this is clearly the formula $RB \to R \wedge S$.

   Technically, as there are no parentheses in natural language, one could also interpret the sentence as *It rains and if there is a rainbow, the sun shines.* However, as this is an interpretation that doesn't match any of the solutions, it should be clear that this can't be the proper interpretation.

2. Consider the formula of propositional logic:

   $$(a \to b \to c) \to (a \to b) \to a \to c$$

   Now answer the following two questions:

   (a) Write this formula with parentheses according to the official grammar of propositional formulas from the course notes.
   (b) Give the number of zeroes and the number of ones in the final column of this formula in its truth table. Don't give the truth table itself, but only the requested numbers.

(a) The formula should be written as follows:

$$\underbrace{\left(\underbrace{\left(a \to \underbrace{(b \to c)}_{\to}\right)}_{\to} \to \underbrace{\left(\underbrace{(a \to b)}_{\to} \to \underbrace{(a \to c)}_{\to}\right)}_{\to}\right)}_{\to}$$

(b) And the corresponding truth table is:

a	b	c	$b \to c$	$a \to b \to c$	$a \to b$	$a \to c$	$(a \to b) \to a \to c$	$(a \to b \to c) \to (a \to b) \to a \to c$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

So the formula is a tautology and hence it has zero zeroes and eight ones in its final column.

3. Does the following hold for all formulas f of propositional logic?

$$\not\models f \ \text{ iff } \ \models \neg f$$

(a) Yes, because in both cases it is stated that f is false.

(b) Yes, because an example is $f = a \land \neg a$.

(c) No, because a counter example is $f = a \lor \neg a$.

(d) is correct (d) No, because a counter example is $f = a$.

Answer (d) is correct.

If we take $f = a$ then there are two valuations v_1 and v_2 possible: $v_1(a) = 0$ and $v_2(a) = 1$. Because of v_1 it follows that $\not\models f$ holds. However, it also follows that $v_2(\neg f) = v_2(\neg a) = 0$, so also $\not\models \neg f$ holds. And obviously $\models \neg f$ does not hold. So $\not\models f$ iff $\models \neg f$ does not hold.

Note that $f = a \lor \neg a$ is not a counter example as $a \lor \neg a$ is a tautology and hence $\not\models a \lor \neg a$ does not hold and $\models \neg(a \lor \neg a)$ also does not hold, so $\not\models f$ iff $\models \neg f$ does hold.

Predicate logic

4. Formalize the following English sentence as a formula of predicate logic:

I will tell the truth, the whole truth, and nothing but the truth.

Use for the dictionary:

C	the domain of relevant claims
$I(x)$	I will tell claim x
$T(x)$	claim x is true

Hint: Note that 'telling the truth' and 'telling the whole truth' are essentially the same thing, so focus on the sentence *I will tell the whole truth and nothing but the truth.* in your formalization.

After applying the hint, the translation becomes quite straightforward. These are several options, but there may be more:

$$(\forall x \in C \; (I(x) \leftrightarrow T(x)))$$
$$(\forall x \in C \; ((T(x) \rightarrow I(x)) \wedge (\neg T(x) \rightarrow \neg I(x))))$$
$$((\forall x \in C \; (T(x) \rightarrow I(x))) \wedge (\forall x \in C \; (\neg T(x) \rightarrow \neg I(x))))$$

5. Which of the following logical laws hold?

   (a) $\forall x \in D\, P(x) \equiv \neg\exists x \in D\, \neg P(x)$

   (b) $\neg\forall x \in D\, P(x) \equiv \exists x \in D\, \neg P(x)$

   (c) $\forall x \in D\, \neg P(x) \equiv \neg\exists x \in D\, P(x)$

   (d) All three of the above statements hold.

(d) is correct

   Answer (d) is correct.

   The exercise uses the De Morgan laws for quantifiers, which express how negations can be 'moved' over quantifiers. In particular we have:

$$\neg\exists x \in D\, \neg P(x) \equiv \forall x \in D\, \neg\neg P(x) \equiv \forall x \in D\, P(x)$$
$$\neg\forall x \in D\, P(x) \equiv \exists x \in D\, \neg P(x)$$
$$\neg\exists x \in D\, P(x) \equiv \forall x \in D\, \neg P(x)$$

   So all of the given laws are correct.

6. Someone tries to formalize:

   There are at most two intelligent men.

   They use the dictionary:

M	the domain of men
$I(x)$	x is intelligent

   However their attempted solution is:

$$\forall x, y, z \in M[I(x) \wedge I(y) \wedge I(z) \rightarrow x = y \vee x = z]$$

   Compared to the standard solution this misses the disjunct $y = z$.

   But what does *this* formula of predicate logic with equality mean?

   (a) It means that there are at most *three* intelligent men. Because there is one disjunct less, the statement is less restrictive and there now is a possibility for this to still be true with one more intelligent man.

   (b) It still means that there are at most *two* intelligent men. The disjunct $y = z$ is not needed, because one can always take the x, y and z in such an order that this case does not occur.

(c) It means that there is at most *one* intelligent man. If there are intelligent men m_1 and m_2, then the formula implies that they have to be the same by taking $x = m_1$ and $y = z = m_2$. In that case the conclusion of the implication becomes $m_1 = m_2 \lor m_1 = m_2$, which is logically equivalent to $m_1 = m_2$.

(c) is correct

(d) The formula means something different from the meanings in the three answers above.

Answer (c) is correct.

It means that there is at most *one* intelligent man.

The explanation given in the option already explains why you can't have two or more intelligent men according to this formula. In addition, it is clear that the formula holds if you have a domain with at most one intelligent man.

Languages

7. Does

$$L^*\overline{L}^* = \left(L\overline{L}\right)^*$$

hold for all languages L?

Note that you should read $\overline{L}^*$ as $\left(\overline{L}\right)^*$.

(a) Yes. We are combining L and its complement $\overline{L}$, which means that in both languages we can make any possible word, by combining parts that are in L and in $\overline{L}$.

(b) Yes, although that language does not necessarily contain all possible words from Σ^*.

(c) No. If we take the language $L = \{a\}$ with $\Sigma = \{a, b\}$, then $b \in L^*\overline{L}^*$ but $b \notin (L\overline{L})^*$.

(c) is correct

(d) No. If we take the language $L = \{a\}$ with $\Sigma = \{a, b\}$, then $b \notin L^*\overline{L}^*$ but $b \in (L\overline{L})^*$.

Answer (c) is correct.

If we take the language $L = \{a\}$ with $\Sigma = \{a, b\}$, then $b \in L^*\overline{L}^*$ as $b = \lambda b$ where $\lambda \in L^*$ and $b \in \overline{L}^*$ as $b \in \overline{L}$. In addition, $b \notin (L\overline{L})^*$ as words in $(L\overline{L})^*$ are either λ or start with an a.

Note that this answer automatically excludes all other answers.

8. What is a regular expression for the following language?

$$\mathcal{L}(a^*b^*) \cap \mathcal{L}((ab)^*)$$

(a) $\varnothing$

(b) λ

(c) $\lambda \cup ab$

(c) is correct

(d) $a^*b^* \cap (ab)^*$

Answer (c) is correct.

Note that

$$\mathcal{L}(a^*b^*) = \{\lambda, \quad a, b, \quad aa, ab, bb, \quad aaa, aab, abb, bbb, \quad \dots\}$$

and

$$\mathcal{L}((ab)^*) = \{\lambda, ab, abab, ababab, \dots\}$$

So it is clear that the intersection of these two languages is $\{\lambda, ab\}$ and $\{\lambda, ab\} = \mathcal{L}(\lambda \cup ab)$.

In addition, note that

- $ab \notin \mathcal{L}(\varnothing)$,
- $ab \notin \mathcal{L}(\lambda)$,
- and $a^*b^* \cap (ab)^*$ is not a regular expression at all.

9. Give a context-free grammar for the language:

$$\{a^n b^n a^m \mid n, m \in \mathbb{N}\}$$

This language is generated by this grammar:

$$S \to AB$$
$$A \to aAb \mid \lambda$$
$$B \to aB \mid \lambda$$

The non-terminal A takes care of creating $a^n b^n$ and the non-terminal B takes care of adding a^m.

10. Consider the context-free grammar G_{10}:

$$S \to aA$$
$$A \to bS \mid b$$

Someone brings up the property

$$P(w) := w \text{ contains the same number of } a\text{'s and } b\text{'s}$$

and wants to show that

$$abb \notin \mathcal{L}(G_{10})$$

claiming that P is an invariant of G_{10}. Does this work?

(a) Yes. The property P is indeed an invariant for G_{10}, because every time the rule $S \to aA$ adds an a, it has to be followed by either $A \to bS$ or $A \to b$, both of which add a b.

(b) Yes. The only possible production for this word has to start as:

$$S \to aA \to abS$$

but this can only continue as $abS \to abaA$, which shows that indeed the word abb is not in the language of G_{10}.

(c) No, although the property P is an invariant for G_{10}, it cannot be used to show that $abb \notin \mathcal{L}(G_{10})$.

(d) is correct

(d) No, because P is not an invariant of G_{10}.

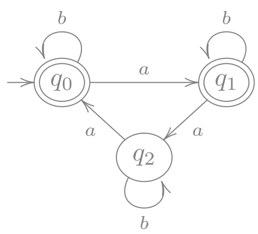
Answer (d) is correct.

The property P is not an invariant. Take $v = S$ and $v' = aA$. Then $P(S)$ holds as there are zero a's and zero b's in S and $S \to aA$ is a valid production step, but $P(aA)$ does not hold as it contains one a and zero b's, and one is not the same as zero.

Note that the explanation in the second option why $abb \notin G_{10}$ is correct, but the 'Yes' is not correct as this claims that P is an invariant, which it is not.

Automata

11. Give all words of length three that are accepted by the following deterministic finite automaton M_{11}:



Explain why these words are accepted.

We systematically list all words of length three and indicate in which state they end. Exactly those words that end in q_0 or q_1 are accepted.

aaa: q_0 accepted	baa: q_2 not accepted
aab: q_2 not accepted	bab: q_1 accepted
aba: q_2 not accepted	bba: q_1 accepted
abb: q_1 accepted	bbb: q_0 accepted

The pattern is that the words ending in the non-final state q_2 are exactly those words that contain precisely two a's.

Using the modulo operator it can be expressed as: word w ends in state q_i if and only if the number of a's in w modulo 3 equals i.

12. Someone claims that all regular languages can be produced by a context-free grammar of which each rule has the form

$$X \to w$$

with $X \in V$ and $w \in \{\lambda\} \cup \Sigma V$.

Is this correct?

(a) is correct

(a) Yes, this is correct. Each regular language can be accepted by a deterministic finite automaton, and this automaton can be converted to a right linear context-free grammar of this form.

(b) Yes, this is correct. In fact even all context-free languages can be produced by such a grammar.

(c) No, this is not correct. This will not work for regular languages of the form $\mathcal{L}(r_1r_2r_3)$, in which r_1, r_2 and r_3 are regular expressions.

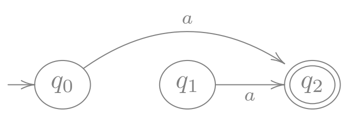
(d) No, this is not correct. This only will be possible when λ is a word in the language, and there are regular languages for which this does not hold.

Answer (a) is correct.

The algorithm to convert a DFA to a context-free grammar is indeed following this principle. Each state gets mapped to a non-terminal in V. And each transition corresponds with a single letter from Σ, followed by a non-terminal. In addition, final states are mapped to λ options in the grammar.

Note in addition that it is not the case that all context-free languages can be produced by such a specific context-free grammar. As this specific type of grammars is a subset of the right-linear grammars, this transformation would imply that all context-free languages can be produced by a right-linear context-free grammar. But that would imply that all context-free grammars are in fact regular, which is not true.

13. Is the following a correct non-deterministic finite automaton (an NFA) with alphabet $\Sigma = \{a\}$?



(a) is correct

(a) Yes, this is a correct NFA, but it is not a *deterministic* finite automaton (a DFA).

(b) Yes, there is no non-determinism in this automaton, so it is a DFA, and all DFAs are NFAs.

(c) No. There is no non-determinism in this automaton, so it cannot be a *non*-deterministic finite automaton.

(d) No, because the state q_1 cannot be reached.

Answer (a) is correct.

For an NFA there are no restrictions on the number of incoming or outgoing transitions. So this is indeed an NFA. However, as state q_2 does not have an outgoing a transition, it is not a DFA.

Discrete mathematics

14. What is the number of different cycles in the graph K_3?

(a) 1

(b) 2

(c) is correct

(c) 6

(d) None of the above.

Answer (c) is correct.

For each cycle we first have to choose a starting point, and then a direction. For the starting point we have three options, and given the starting point, we have two directions. So there are six different cycles. Note that a cycle is defined as a path where the starting point and the end point are the same and that paths are defined as ordered edges. So although the cycles $(1,2),(2,3),(3,1)$ and $(2,3),(3,1),(1,2)$ look the same in the graph, mathematically, they are different.

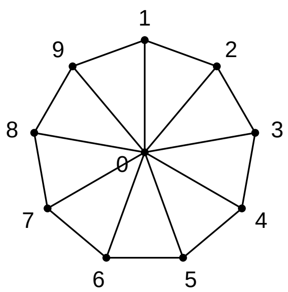
So these are all the different cycles:

$(1,2),(2,3),(3,1)$	$(2,3),(3,1),(1,2)$	$(3,1),(1,2),(2,3)$
$(1,3),(3,2),(2,1)$	$(2,1),(1,3),(3,2)$	$(3,2),(2,1),(1,3)$

15. We define the *wheel graph* W_{n+1} for $n \geq 3$ as:

$$W_{n+1} := \langle \{0,\dots,n\}, \{(0,1),\dots,(0,n),(1,2),(2,3),\dots,(n-1,n),(n,1)\} \rangle$$

For example this is the wheel graph W_{10}:



It has 10 vertices and 18 edges. Note that in this example $n = 9$, so $n + 1 = 10$.

For which $n \geq 3$ does W_{n+1} have an Eulerian path?

(b) is correct

(a) For all n.

(b) For no n.

(c) For all even n.

(d) For all odd n.

Answer (b) is correct.

The wheel graph is a connected graph, so it has an Eulerian path if and only if the number of vertices with an odd degree is at most two. However, W_{n+1} has at least n vertices of odd degree, namely, the n vertices on the outside, which all have degree 3. And as $n \geq 3$ this means that each W_{n+1} has at least n vertices of odd degree, which is more than two. So none of these wheel graphs have an Eulerian path.

16. We recursively define:

$$f(m,0) = 0 \qquad \text{for } m \geq 0$$
$$f(m,n+1) = m + f(m,n) \qquad \text{for } m,n \geq 0$$

Compute $f(3,3)$ and explain how you got your answer.

Note that

$$\begin{aligned} f(3,3) &= 3 + f(3,2) \\ &= 3 + (3 + f(3,1)) \\ &= 3 + (3 + (3 + f(3,0))) \\ &= 3 + (3 + (3 + 0)) \\ &= 9 \end{aligned}$$

17. Someone recursively defines:

$$a_0 = 1$$
$$a_{n+1} = a_n + 2n + 1 \qquad \text{for } n \geq 0$$

They try to prove using induction on n that:

$$a_n = n^2$$

Does this work?

(a) Yes, that works.

(b) No, the base case can be proven, but the induction step cannot be proven.

(c) No, the induction step can be proven, but the base case cannot be proven.

(c) is correct

(d) No, both the base case and the induction step cannot be proven.

Answer (c) is correct.

The base case cannot be proven. We would have to prove that $a_0 = 0^2$. However, we know that $a_0 = 1$ and $0^2 \neq 1$.

The induction step can be proven. If we assume the Induction Hypothesis that $P(k)$ holds, or more precisely that $a_k = k^2$, then we get

$$\begin{aligned} a_{k+1} &= a_k + 2k + 1 \\ &\overset{\text{IH}}{=} k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

which exactly means that $P(k+1)$ holds as well.

18. Koos has a bag holding 8 distinguishable objects, and wants to give 1 of those to Maud, 2 to Sharon and 2 to Joris (after which of course 3 objects will remain in the bag).

In how many ways can he do this?

(a) $\binom{8}{1}+\binom{8}{2}+\binom{8}{2}=8+28+28=64$

(b) $\binom{8}{1}+\binom{7}{2}+\binom{5}{2}=8+21+10=39$

(c) $\binom{8}{1}\cdot\binom{8}{2}\cdot\binom{8}{2}=8\cdot 28\cdot 28=6272$

(d) is correct

(d) $\binom{8}{1}\cdot\binom{7}{2}\cdot\binom{5}{2}=8\cdot 21\cdot 10=1680$

Answer (d) is correct.

Note that the order in which presents are picked is not important.

This is an algorithm for Koos.

- Koos starts by picking one of the objects and giving it to Maud. This can be done in $\binom{8}{1}=8$ ways.
- Then he picks two objects out of the remaining seven and gives these to Sharon. This can be done in $\binom{7}{2}=\frac{7!}{2!\cdot 5!}=\frac{7\cdot 6}{2}=21$ ways.
- Then he picks two objects out of the remaining five and gives these to Joris. This can be done in $\binom{5}{2}=\frac{5!}{2!\cdot 3!}=\frac{5\cdot 4}{2}=10$ ways.

So in total there are $8\cdot 21\cdot 10=1680$ ways to do this.

19. Someone recursively defines:

$$\begin{aligned} c(1,1) &= 1 \\ c(1,k+1) &= 0 && \text{for } k\geq 1 \\ c(n+1,1) &= n\cdot c(n,1) && \text{for } n\geq 1 \\ c(n+1,k+1) &= c(n,k)+n\cdot c(n,k+1) && \text{for } k,n\geq 1 \end{aligned}$$

These numbers $c(n,k)$ with $1\leq n$ and $1\leq k\leq n$ are the...

(a) is correct

(a) Stirling numbers of the first kind.

(b) Stirling numbers of the second kind.

(c) Binomial coefficients.

(d) None of the above.

Answer (a) is correct.

Let us compute some $c(n,k)$:

$n\downarrow$ $k\rightarrow$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	2	3	1	0	0	0
4	6	11	6	1	0	0
5	24	50	35	10	1	0

And if we restrict this table to the cells where $1\leq k\leq n$ we get exactly the Stirling numbers of the first kind.

Modal logic

20. We want to formalize the following sentence as a formula of epistemic logic:

    I don't know whether it rains.

    We use for the dictionary:

    $$R \quad \text{it rains}$$

    Which of the following formulas is a good choice for this?

    (a) $\neg\Box R$
    (b) $\neg\Box(R \vee \neg R)$
    (c) $\neg\Box R \vee \neg\Box\neg R$
    (d) is correct (d) $\neg\Box R \wedge \neg\Box\neg R$

    Answer (d) is correct.

    The sentence *I don't know whether it rains.* is basically equivalent to the sentence *I don't know that it rains but I also don't know that it doesn't rain.* And this sentence translates to the formula: $\neg\Box R \wedge \neg\Box\neg R$.

    The other formulas can be translated as:

    - $\neg\Box R$ *I don't know that it rains.*
    - $\neg\Box(R \vee \neg R)$ *I don't know that it rains or not.* And as $R \vee \neg R$ is a tautology, this could even be seen as *I don't know that true holds.*
    - $\neg\Box R \vee \neg\Box\neg R$ *I don't know that it rains or I don't know that it doesn't rain.*

    All of these are clearly different from the original sentence.

21. Give a Kripke model $\mathcal{M}_{21} = \langle W, R, V \rangle$ such that

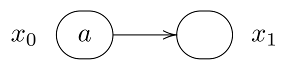
    $$\mathcal{M}_{21} \not\models a \to \Box\Diamond a$$

    This is an instance of axiom scheme B (symmetry).

    Define the model by giving the set W and the functions R and V. Explain why this model has the required property.

    As this is an instance of the symmetry scheme B, it means that our counter example should have a non-symmetric accessibility relation.

    So take for instance the model $\mathcal{M}_{21}$ given by



    This model is mathematically described by

    $$\begin{aligned} W &= \{x_0, x_1\} \\ R(x_0) &= \{x_1\} \\ R(x_1) &= \varnothing \\ V(x_0) &= \{a\} \\ V(x_1) &= \varnothing \end{aligned}$$

This is the corresponding $\Vdash$-table:

	a	$\Diamond a$	$\Box\Diamond a$	$a \to \Box\Diamond a$
x_0	1	0	0	0
x_1	0	0	1	1

Hence $x_0 \nVdash a \to \Box\Diamond a$ and hence $\mathcal{M}_{21} \nvDash a \to \Box\Diamond a$.

22. There is an LTL formula such that if it holds in world x_0 of an LTL model $\mathcal{M}_{22} = \langle W, R, V\rangle$, with:

$$W = \{x_0, x_1, x_2, \dots\}$$
$$R(x_i) = \{x_i, x_{i+1}, x_{i+2}, \dots\} \qquad \text{for } i \geq 0$$

then that model has to satisfy:

$$V(x_i) = \begin{cases} \{a\} & \text{if } i \text{ is even} \\ \{b\} & \text{if } i \text{ is odd} \end{cases}$$

This formula is (note that $\mathcal{G}$ binds more strongly than the conjunction):

(a) $a \wedge \mathcal{G}(a \vee b) \wedge (a \leftrightarrow \neg b) \wedge (a \leftrightarrow \mathcal{X}b)$

(b) is correct

(b) $a \wedge \mathcal{G}(a \leftrightarrow \neg b) \wedge \mathcal{G}(a \leftrightarrow \mathcal{X}b)$

(c) $\mathcal{G}(a \wedge (a \leftrightarrow \neg b) \wedge (a \leftrightarrow \mathcal{X}b))$

(d) $\mathcal{G}a \wedge \mathcal{G}(a \leftrightarrow \neg b) \wedge \mathcal{G}(a \leftrightarrow \mathcal{X}b)$

Answer (b) is correct.

So we are looking for a formula that only allows the model that alternates a and b every step and starts with a.

The formula $f := a \wedge \mathcal{G}(a \leftrightarrow \neg b) \wedge \mathcal{G}(a \leftrightarrow \mathcal{X}b)$ enforces this. If $x_0 \Vdash f$ then by the non-modal a we get that $a \in V(x_0)$. And by the $\mathcal{G}(a \leftrightarrow \neg b)$ we get that $b \notin V(x_0)$. So $V(x_0) = \{a\}$ as requested.

In addition, the $\mathcal{G}(a \leftrightarrow \mathcal{X}b)$ enforces that $b \in V(x_1)$. And the $\mathcal{G}(a \leftrightarrow \neg b)$ in turn enforces that $a \notin V(x_1)$.

And from this together with (again) $\mathcal{G}(a \leftrightarrow \mathcal{X}b)$ it follows that $b \notin V(x_2)$. So $\mathcal{G}(a \leftrightarrow \neg b)$ enforces that $a \in V(x_2)$.

So the world x_2 has the same properties as world x_0. Hence we can continue like this and get indeed that in a model that matches this formula in the worlds with an even index only a holds, and in the worlds with an odd index only b holds. Exactly as was requested.

Note that the first formula also allows for a model where $V(x_0) = \{a\}$, $V(x_1) = \{b\}$, and $V(x_i) = \{a\}$ for $i \geq 2$. This is essentially caused by the lacking $\mathcal{G}$lobals around $(a \leftrightarrow \neg b)$ and $(a \leftrightarrow \mathcal{X}(b))$.

Note also that the third formula has no matching models as a has to hold in all worlds due to the $\mathcal{G}(a)$ part, but at the same time, should not hold in world x_1 as $x_0 \Vdash a$ implies $x_1 \Vdash b$ and $x_1 \Vdash b$ implies $x_1 \Vdash \neg a$. So there is a contradiction in x_1.

And the fourth formula is basically the same as the third one, but now the $\mathcal{G}$lobals are distributed over the conjunctions, which is something that doesn't change the meaning. And hence also the fourth solution has no models.