Formal Reasoning 2023
Solutions Test Block 1: Propositional and Predicate Logic
(25/09/23)

Propositional logic

1. The English sentence:

   I'm inside, but only if it rains.

   is ambiguous. Consider the following three formulas of propositional logic:

   (i) $\mathsf{In} \leftrightarrow \mathsf{R}$
   (ii) $\mathsf{In} \rightarrow \mathsf{R}$
   (iii) $\mathsf{R} \rightarrow \mathsf{In}$

   We use as dictionary:

In	I'm inside
R	it rains

   Which of these formulas describe possible meanings for this sentence?

   (a) is correct

   (a) (i) and (ii)
   (b) (i) and (iii)
   (c) (ii) and (iii)
   (d) all three

   Answer (a) is correct.

   If the sentence *I'm inside, but only if it rains.* holds, it certainly implies that if I am inside, it must rain. So $\mathsf{In} \rightarrow \mathsf{R}$ certainly holds. In particular, this formula represents the typical mathematical translation of the sentence.

   However, it is argued that the mathematical translation does not match nicely with the concept of natural language. And therefore, many people also consider the formula $\mathsf{In} \leftrightarrow \mathsf{R}$ as a matching formula.

   So these two are considered to describe possible meanings.

   Note that the third option $\mathsf{R} \rightarrow \mathsf{In}$ is not considered to describe a possible meaning as it omits the 'certain' part of *when I am inside, it must rain.*

2. Give a formula of propositional logic that best matches the meaning of:

   If it doesn't rain, or if I'm inside, then I stay dry.

   Use as dictionary:

In	I'm inside
R	it rains
D	I'm dry

Hint: Consider an appropriate logical law.

If you only can think of a solution that needs connectives other than implications, then you can give that answer for partial points.

First, we give a straightforward translation:

$$((\neg \mathsf{R} \vee \mathsf{In}) \rightarrow \mathsf{D})$$

And now we apply the logical law $A \rightarrow B \equiv \neg A \vee B$ where $A = \mathsf{R}$ and $B = \mathsf{In}$ and we get

$$((\mathsf{R} \rightarrow \mathsf{In}) \rightarrow \mathsf{D})$$

3. How should we read the formula

$$a \vee b \wedge c$$

and how many ones are there in its column in its truth table?

(a) $(a \vee b) \wedge c$ and the number of ones in its truth table is five

(b) $(a \vee b) \wedge c$ and the number of ones in its truth table is not five

(c) $a \vee (b \wedge c)$ and the number of ones in its truth table is three

(d) is correct (d) $a \vee (b \wedge c)$ and the number of ones in its truth table is not three

Answer (d) is correct.

As the conjunction binds stronger than the disjunction the formula should be read as

$$(a \vee (b \wedge c))$$

and its truth table is:

a	b	c	$(b \wedge c)$	$(a \vee (b \wedge c))$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

So the formula has five ones, which is not three.

4. Which of the following answers is correct?

(a) $a \rightarrow (b \rightarrow c) \models (a \rightarrow b) \rightarrow c$ and $(a \rightarrow b) \rightarrow c \models a \rightarrow (b \rightarrow c)$

(b) $a \rightarrow (b \rightarrow c) \models (a \rightarrow b) \rightarrow c$ and $(a \rightarrow b) \rightarrow c \not\models a \rightarrow (b \rightarrow c)$

(c) is correct (c) $a \rightarrow (b \rightarrow c) \not\models (a \rightarrow b) \rightarrow c$ and $(a \rightarrow b) \rightarrow c \models a \rightarrow (b \rightarrow c)$

(d) $a \rightarrow (b \rightarrow c) \not\models (a \rightarrow b) \rightarrow c$ and $(a \rightarrow b) \rightarrow c \not\models a \rightarrow (b \rightarrow c)$

Answer (c) is correct.

Let us have a look at this truth table:

a	b	c	$(b \to c)$	$(a \to (b \to c))$	$(a \to b)$	$((a \to b) \to c)$
0	0	0	1	1	1	0
0	0	1	1	1	1	1
0	1	0	0	1	1	0
0	1	1	1	1	1	1
1	0	0	1	1	0	1
1	0	1	1	1	0	1
1	1	0	0	0	1	0
1	1	1	1	1	1	1

From the table it follows that

$$((a \to b) \to c) \vDash (a \to (b \to c))$$

as on all rows where $((a \to b) \to c)$ has a 1, $(a \to (b \to c))$ also has a 1.

However, it also follows that

$$(a \to (b \to c)) \nvDash ((a \to b) \to c)$$

as there are rows where $(a \to (b \to c))$ has a 1, but $((a \to b) \to c)$ has a 0, for instance in the first row.

A different way of solving this is by looking for models in which the formulas don't hold. If

$$(a \to (b \to c))$$

is false in model v, it means that $v(a) = 1$ and $v(b \to c) = 0$, which in its turn can only be if $v(b) = 1$ and $v(c) = 0$.

It is easy to see that in this same model $v(a \to b) = 1$ and hence $v((a \to b) \to c) = 0$.

So in the only model where

$$(a \to (b \to c))$$

is false,

$$((a \to b) \to c)$$

is also false. This automatically implies that

$$((a \to b) \to c) \vDash (a \to (b \to c))$$

However, if we define the model v' by stating that $v'(a) = v'(b) = v'(c) = 0$, then it follows that $v'(a \to (b \to c)) = 1$ and $v'((a \to b) \to c) = 0$. So in particular, the statement

$$(a \to (b \to c)) \nvDash ((a \to b) \to c)$$

does not hold.

Predicate logic

5. What is the form according to the official grammar from the course notes of the following formula?

$$\exists x, y \in D\, [x \neq y]$$

   (a) $\exists x \in D\, [\exists y \in D\, [\neg(x = y)]]$

(b) is correct (b) $(\exists x \in D\, (\exists y \in D\, \neg(x = y)))$

   (c) $(\exists x \in D\, (\exists y \in D\, (\neg(x = y))))$

   (d) none of the above

   Answer (b) is correct.

   Note that the comma, square brackets, and the $\neq$ are not allowed. In addition, the $\neg(x = y)$ does not require additional parentheses around it.

6. Translate as a formula of predicate logic with equality:

   Every human has exactly two parents.

   Use as dictionary

H	domain of humans
$P(x, y)$	x is a parent of y

   So for each human there are at least and at most two different humans who are their parents. The default pattern now states:

$$\begin{array}{llll} \forall x \in H & [& \exists p_1, p_2 \in H & [& p_1 \neq p_2 \wedge P(p_1, x) \wedge P(p_2, x) \\ & & & & \wedge \\ & & & & \forall y \in H[P(y, x) \to (y = p_1) \vee (y = p_2)] \\ & & &] & \\ &] & & & \end{array}$$

7. Let f be the formula $\forall x \in M\, \exists y \in W\, L(x, y)$ and let g be the formula $\exists y \in W\, \forall x \in M\, L(x, y)$. Which of the following answers is correct?

   (a) $f \vDash g$ and $g \vDash f$

   (b) $f \vDash g$ and $g \nvDash f$

(c) is correct (c) $f \nvDash g$ and $g \vDash f$

   (d) $f \nvDash g$ and $g \nvDash f$

   Answer (c) is correct.

   So we basically have to check two statements:

$$\forall x \in M\, \exists y \in W\, L(x, y) \quad \vDash \quad \exists y \in W\, \forall x \in M\, L(x, y) \qquad (1)$$

$$\exists y \in W\, \forall x \in M\, L(x, y) \quad \vDash \quad \forall x \in M\, \exists y \in W\, L(x, y) \qquad (2)$$

   Note that statement (1) is not correct. Take as structure

$M_1 :=$

Domain(s)	$\mathbb{N}$
Relation(s)	less than

and as interpretation

$$I_1 := \boxed{\begin{array}{ll} M & \mathbb{N} \\ W & \mathbb{N} \\ L(x,y) & x < y \end{array}}$$

Then $\forall x \in M\, \exists y \in W\, L(x,y)$ does hold in model (M_1, I_1), because once x is chosen, we can choose $y := x + 1$ and then certainly $x < y$. However, the formula $\exists y \in W\, \forall x \in M\, L(x,y)$ does not hold as this would imply that there exists a natural number that is larger than all natural numbers, which is certainly not true. So statement (1) is not correct.

However, statement (2) does hold. Note that $\exists y \in W\, \forall x \in M\, L(x,y)$ in (M_1, I_1) means that we can choose a single element in W, let us call it w, such that $\forall x \in M\, L(x,w)$ holds. Now the formula $\forall x \in M\, \exists y \in W\, L(x,y)$ states that given any $x \in M$ we can always find an element $y \in W$ such that $L(x,y)$. And this is true as we can always pick $y := w$ and then indeed $L(x,w)$ holds for each x. So statement (2) is correct.

8. Let f be the formula $\forall x, y, z \in N\, [L(x,y) \to L(x,z) \lor L(z,y)]$. Now consider the following structure M with interpretations I_1 and I_2:

$$M := (\mathbb{N}, <, \leq)$$

$$I_1 := \boxed{\begin{array}{ll} N & \mathbb{N} \\ L(x,y) & x < y \end{array}}$$

$$I_2 := \boxed{\begin{array}{ll} N & \mathbb{N} \\ L(x,y) & x \leq y \end{array}}$$

Which of the following is correct?

(a) is correct

(a) $(M, I_1) \vDash f$ and $(M, I_2) \vDash f$
(b) $(M, I_1) \vDash f$ and $(M, I_2) \nvDash f$
(c) $(M, I_1) \nvDash f$ and $(M, I_2) \vDash f$
(d) $(M, I_1) \nvDash f$ and $(M, I_2) \nvDash f$

Answer (a) is correct.

So first we have to determine the meaning of

$$\forall x, y, z \in N\, [L(x,y) \to L(x,z) \lor L(z,y)]$$

in the two models.

In model (M, I_1) it means: for each combination of natural numbers x, y, and z, if $x < y$ then $x < z$ or $z < y$, or both. This is true. Let us assume that $x < y$. Now we make a case distinction on the relation between x and z.

- $x < z$ holds. Then automatically '$x < z$ or $z < y$' holds, so the statement holds.
- $x < z$ does not hold and hence $z \leq x$ holds. But, as $x < y$ it follows that also $z < y$. And hence automatically '$x < z$ or $z < y$' holds, so the statement holds.

So the statement holds in all cases in this model.

And in model (M, I_2) it means: for each combination of natural numbers x, y, and z, if $x \leq y$ then $x \leq z$ or $z \leq y$, or both. This is also true. Let us assume that $x \leq y$. Now we make a case distinction on the relation between x and z.

- $x \leq z$ holds. Then automatically '$x \leq z$ or $z \leq y$' holds, so the statement holds.
- $x \leq z$ does not hold and hence $z < x$ holds. But, as $x \leq y$ it follows that also $z \leq y$. And hence automatically '$x \leq z$ or $z \leq y$' holds, so the statement holds.

So the statement holds in all cases in this model.

And the final conclusion is that the statements hold in both models.