Formal Reasoning 2023
Solutions Test Block 3: Discrete Mathematics and Modal Logic

(21/12/23)

Discrete Mathematics

1. How many non-isomorphic trees are there with exactly five vertices?

   (a) 2

   (b) 3 — (b) is correct

   (c) 4

   (d) more than 4

   Answer (b) is correct.

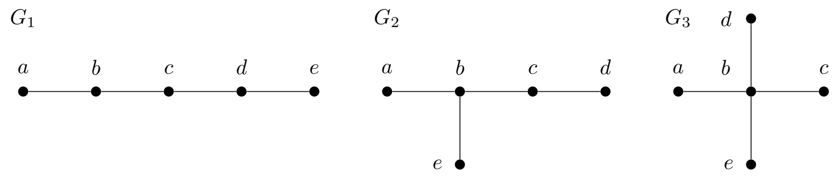
   Note that a tree is a connected graph that has no cycles.

   We try to create as many trees, based on the vertices with the highest degree.

   - If there was a tree with five vertices and the highest degree of the vertices was 1, then this graph would not be connected and hence it cannot be a tree.
   - A tree where the highest degree of the vertices is 2 does exist. As it should be connected, it follows that all vertices should be in a single line, as junctions would imply a degree 3 or higher. There is only one way to do this. See G_1 below.
   - A tree where the highest degree of the vertices is 3 also exists. In order to create such a tree, we start by taking a single vertex and then connect three other vertices to it. Note that it is not possible to connect some of these three vertices directly with each other, as that would create loops and loops are not allowed in trees. So there is only one way to create such a tree with four vertices where the highest degree is 3.

     We still need to add the fifth vertex. This one cannot be connected to the vertex with degree 3, as this would lead to a vertex with degree 4, and the assumption was that the highest degree was 3. So it must be added to one of the three vertices with degree one. However, no matter which one of the three vertices we pick, we will get three isomorphic trees. So there is only one non-isomorphic tree with five vertices where the highest degree of the vertices is 3. See G_2 below.
   - In the previous case, we have already described that we could create a tree with a vertex of degree 4, by taking a single vertex and connecting all other four vertices to it, without any other connections, in order to prevent loops. By construction, it follows that this is the only non-isomorphic tree with this property. See G_3 below.
   - As there are only five vertices, it is not possible to have a tree with a vertex that has a degree higher than 4.

And this is what these trees look like:



2. Is there a graph that has an Eulerian cycle, but no Hamiltonian cycle?

   (a) Yes, the Petersen graph is an example.

   (b) Yes, the cube graph is an example.

(c) is correct

   (c) Yes, $K_{2,4}$ is an example.

   (d) No.

   Answer (c) is correct.

   The graph $K_{2,4}$ is a complete bipartite graph with six vertices, which can be divided in two red vertices and four blue vertices such that each edge connects a red and a blue vertex, and, in addition, all red vertices are connected to all blue vertices. So, in particular, the degree of the red vertices is 4 and the degree of the blue vertices is 2. And as it is a connected graph with at least two vertices, Euler's theorem states that there indeed exists an Eulerian path.

   However, it has no Hamiltonian cycle. If it had a Hamiltonian cycle, we may assume that the cycle started at a red vertex. The second vertex will be one of the four blue vertices. The third vertex will be the other red vertex. The fourth vertex will be one of the three blue vertices that are not yet used. The fifth vertex should be the first red vertex. (Obviously, it needs to be red, but if it was the second red vertex, then there are two edges between the same two vertices which is not allowed in our graphs.) However, we can only reuse the first red vertex if we finish our cycle. But we still have two blue vertices that are not part of the cycle, which is in contradiction with the fact that the cycle is a Hamiltonian cycle.

   So the graph doesn't have a Hamiltonian cycle.

   Note that the Petersen graph doesn't have an Eulerian cycle nor a Hamiltonian cycle. And the cube graph does have a Hamiltonian cycle, but not an Eulerian cycle.

3. The sums of the rows of Pascal's triangle are always a power of two. For example, we have $1 + 4 + 6 + 4 + 1 = 16 = 2^4$.

   What are the sums of the rows in the triangle of the Stirling numbers of the first kind?

   (a) Also powers of two.

   (b) Powers of two minus one.

   (c) The Bell numbers.

(d) is correct

   (d) Factorials.

Answer (d) is correct.

The triangle of the Stirling numbers of the first kind has as its top:

					1					
				1		1				
			2		3		1			
		6		11		6		1		
	24		50		35		10		1	
120		274		225		85		15		1

If we add the rows, we get $1 = 1!$, $2 = 2!$, $6 = 3!$, $24 = 4!$, $120 = 5!$, and $720 = 6!$. So we get the factorials.

Knowing that the Stirling numbers of the first kind count permutations, it also makes sense that these numbers are related to factorials.

4. We define a sequence a_n recursively by the recursion equations:

$$\begin{cases} a_0 & = 0 \\ a_{n+1} & = 3a_n + 1 \quad \text{for all } n \geq 0 \end{cases}$$

Now consider the following proof in which two parts have been omitted:

Proposition $a_n = (3^n - 1)/2$ for all $n \geq 0$.

Proof by induction on n. We first define our predicate P as:

$$P(n) := \left[a_n = (3^n - 1)/2\right]$$

Base case. We show that $P(0)$ holds, i.e. we show that $a_0 = (3^0 - 1)/2$.

This indeed holds because

$$a_0 = 0 \qquad \text{by definition}$$

and

$$(3^0 - 1)/2 = (1 - 1)/2 = 0$$

Induction step. Let k be any natural number such that $k \geq 0$.

We now show that $P(k + 1)$ also holds, i.e. we show that

$$a_{k+1} = (3^{k+1} - 1)/2$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$ $\square$

Give the two missing parts of this proof.

In ASCII you can use _ and ^ for subscript and superscript, so you may write

$$a_{k+1} = \frac{3^{k+1} - 1}{2}$$

as

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a_(k+1) = (3^(k+1) - 1)/2
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The text in the first box should be similar to:

Assume that we already know that $P(k)$ holds, i.e., we assume that

$$a_k = (3^k - 1)/2 \qquad \text{(Induction Hypothesis)}$$

And the text in the second box should be similar to:

This indeed holds because

$$\begin{aligned} a_{k+1} &= 3a_k + 1 && \text{by definition} \\ &= 3\frac{3^k - 1}{2} + 1 && \text{by the induction hypothesis} \\ &= \frac{3 \cdot 3^k - 3 + 2}{2} && \text{by high school algebra} \\ &= \frac{3^{k+1} - 1}{2} && \text{by high school algebra} \end{aligned}$$

Modal Logic

5. Consider the following English sentence:

   I do not know that Albany is the capital of New York State, but I know that I don't know this.

   We use as dictionary:

   A Albany is the capital of New York State

   Now which formula of epistemic logic corresponds to the meaning of this sentence?

   (a) $(\Box\neg A) \wedge (\Box\Box\neg A)$
   (b) $(\Diamond\neg A) \wedge (\Diamond\Diamond\neg A)$
   (c) is correct (c) $(\Diamond\neg A) \wedge (\Box\Diamond\neg A)$
   (d) none of the above

   Answer (c) is correct.

   The sentence is a conjunction and, as the second part is *I know that I don't know this*, the translation basically has the form $f \wedge (\Box f)$ where f is the formula that matches the first part of the sentence. Now the default translation of the first part, *I do not know that Albany is the capital of New York State*, would be $\neg\Box A$. So the default translation of the whole sentence would be $(\neg\Box A) \wedge (\Box\neg\Box A)$, but if we look at the given options, we see that this formula is not one of the options.

However, as we know that $\neg\Box A \equiv \Diamond\neg A$, we see that the default translation is equivalent to $(\Diamond\neg A) \wedge (\Box\Diamond\neg A)$, which is indeed one of the options.

If we look at the other two options, it is clear that they can't be correct.

- The formula $(\Box\neg A) \wedge (\Box\Box\neg A)$ states on its left hand *I know that Albany is not the capital of New York State*, which is clearly not correct.
- And the formula $(\Diamond\neg A) \wedge (\Diamond\Diamond\neg A)$ has a correct part on the left, but the right part is equivalent to *It is not the case that I know that I know that Albany is the capital of New York State*, which is also wrong.

6. In which of the following logics does axiom scheme T (reflexivity) hold?

   (a) deontic logic

   (b) doxastic logic

   (c) temporal logic — (c) is correct

   (d) all of the above

   Answer (c) is correct.

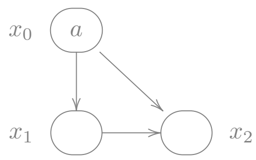
   The reflexivity axiom is $\Box f \to f$.

   If we interpret this in deontic logic, we get: *If I must do f, then I do f.* This is clearly not true for all formulas f, take for instance f to be *waiting for the red light.*

   If we interpret it in doxastic logic, it means: *If I believe f, then f is true.* This is also not true. Take for instance f to be *I will be the next prime-minister of The Netherlands*, then the fact that I believe it doesn't make it true.

   If we interpret it in temporal logic (both in general temporal logic as defined in this course as in the specific variant LTL), it means: *If it is always true, then it is true now.* This clearly holds as *now* is an instance of *always.*

7. Consider the following Kripke model:



   Does

$$x_0 \Vdash \Diamond\Box a$$

   hold? Explain your answer.

   You can write this statement using ASCII in the style

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x_0 ||- <>[]a
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Make sure to explicitly state which formulas hold in which worlds.

The formula $\Diamond\Box a$ does hold in world x_0. It means that there is a world which is accessible from world x_0 for which it holds that in all of its accessible worlds the proposition a holds. This holds if we take x_2 to be the accessible world from x_0. As world x_2 has no accessible worlds, the formula $\Box a$ holds vacuously in x_2. So we get $x_2 \Vdash \Box a$. And as x_2 is indeed accessible from x_0 we get that $x_0 \Vdash \Diamond\Box a$ holds.

We can also demonstrate this with a $\Vdash$-table:

$\Vdash$	a	$\Box a$	$\Diamond\Box a$
x_0	1	0	1
x_1	0	0	1
x_2	0	1	0

In the table we can see that both $x_0 \Vdash \Diamond\Box a$ and $x_1 \Vdash \Diamond\Box a$ hold, but that $x_2 \Vdash \Diamond\Box a$ does not hold. So in particular $\models \Diamond\Box$ doesn't hold, but that was not the question.

8. Which LTL formula states that a will be true infinitely often but also that a will be false infinitely often?

(a) is correct

   (a) $(\mathcal{G}\mathcal{F}a) \wedge (\mathcal{G}\mathcal{F}\neg a)$
   (b) $(\mathcal{F}\mathcal{G}a \wedge (\mathcal{F}\mathcal{G}\neg a)$
   (c) $(\mathcal{G}\mathcal{G}a) \wedge (\mathcal{G}\mathcal{G}\neg a)$
   (d) $(\mathcal{F}\mathcal{F}a) \wedge (\mathcal{F}\mathcal{F}\neg a)$

   Answer (a) is correct.

   This is the meaning of the formulas:

   (a) $(\mathcal{G}\mathcal{F}a) \wedge (\mathcal{G}\mathcal{F}\neg a)$: *It is always true that there will be a moment in the future such that proposition a is true and it is always true that there will be a moment in the future such that proposition a is false.* This indeed matches the sentence that a will be infinitely often true and also infinitely often false.

   (b) $(\mathcal{F}\mathcal{G}a \wedge (\mathcal{F}\mathcal{G}\neg a)$: *There is a moment in the future such that from that moment on a is always true and there is a moment in the future such that from that moment on a is always false.* There is no model that makes this formula true as it implies that from a certain moment on a will be both true and false. As there are models that make the sentence true, for instance by alternating a and $\neg a$ continuously, this formula doesn't fit with the sentence. So this formula certainly isn't a good translation of the original sentence.

   (c) $(\mathcal{G}\mathcal{G}a) \wedge (\mathcal{G}\mathcal{G}\neg a)$: Note that a double $\mathcal{G}\mathcal{G}$ doesn't add anything to a single $\mathcal{G}$ as they both mean that a proposition is always true. So the formula can be reduced to $(\mathcal{G}a) \wedge (\mathcal{G}\neg a)$ and this formula means: *Proposition a is always true and proposition a is always false.* So again a contradiction and there are no models that mke this formula true.. So this formula certainly isn't a good translation of the original sentence.

(d) $(\mathcal{F}\mathcal{F}a) \wedge (\mathcal{F}\mathcal{F}\neg a)$: Note that a double $\mathcal{F}\mathcal{F}$ doesn't add anything to a single $\mathcal{F}$ as they both mean that a proposition is true sometime in the future. So the formula can be reduced to $(\mathcal{F}a) \wedge (\mathcal{F}\neg a)$ and this formula means: *There is a moment in the future such that a is true and there is a moment in the future such that a is false.* So a model for which a is true exactly once (and hence automatically false infinitely often) makes this formula true, but it doesn't make the original sentence true. So this formula certainly isn't a good translation of the original sentence.