Formal Reasoning 2022
Solutions Test Blocks 1, 2 and 3: Additional Test
(11/01/24)

1. Consider the statement:

$$\neg a \models a \to b$$

Does this hold?

(a) It depends on the truth values of a and b whether this holds.

(b) is correct — (b) Yes, this holds.

(c) No, this does not hold. A counter example is $v(a) = 1$ and $v(b) = 0$.

(d) No, this does not hold. A counter example is $v(a) = 1$ and $v(b) = 1$.

Answer (b) is correct.

The statement claims that $a \to b$ is a logical consequence of $\neg a$, which means that whenever $\neg a$ is true, $a \to b$ is also true. Or in terms of valuations, whenever $v(\neg a) = 1$ then $v(a \to b) = 1$. Now if $v(\neg a) = 1$ it follows that $v(a) = 0$. But in turn, that means that $v(a \to b) = 1$, which was what we had to prove.

So it doesn't depend on the truth values of a and b. And the suggested counter examples both have $v(a) = 1$ and hence $v(\neg a) = 0$, which contradicts the fact that $v(\neg a) = 1$. So these aren't counter examples at all.

2. Someone tries to formalize the sentence

There are at least three nice men.

using the dictionary

M	the domain of men
$N(x)$	x is nice

as the formula:

$$\exists x, y, z \in M[N(x) \land N(y) \land N(z)]$$

Is this correct?

(a) Yes, this is correct.

(b) No, there also needs to be a part that states that every nice man is one of x, y or z.

(c) is correct — (c) No, because this formula means that there is at least *one* nice man.

(d) No, obviously this is not a formula according to the official syntax, but it is not even a formula of predicate logic with equality.

Answer (c) is correct.

As there is no part that states that x, y, and z are not the same, this simply means that there is at least one nice man.

If we add the part that every nice man is one of x, y, or z, it would imply that we have at most three nice men.

And although the formula does not follow the official syntax, it is allowed to write the formula like this, as the rule is that the official formula only has to be used when it is explicitly stated.

3. Consider the context-free grammar G_3 over the alphabet $\Sigma = \{(,)\}$ given by the rules:

$$S \to (S) \mid SS \mid \lambda$$

The language $\mathcal{L}(G_3)$ consists of exactly all words in which the parentheses '(' and ')' are properly balanced. For example, we have `(()(()))` $\in \mathcal{L}(G_3)$.

Give the number of words in $\mathcal{L}(G_3)$ that have length six, and list all of them.

If we are looking for words of length six, it means that we have three sets of parentheses. So we get:

`((()))`   `(()())`   `(())()`   `()(())`   `()()()`

And hence there are five words of length six in $\mathcal{L}(G_3)$.

The corresponding numbers are known as the Catalan numbers. See `https://oeis.org/A000108`.

4. Someone claims that the following property

$$P(w) := [w \in V\Sigma^*]$$

is an invariant for every right linear context-free grammar. Is this correct?

(a) Yes, this is correct.

(b) No, the property should be:

$$P(w) := [w \in \Sigma^* V]$$

(c) No, the property should be:

$$P(w) := [w \in (V \cup \{\lambda\})\Sigma^*]$$

(d) is correct

(d) None of the above.

Answer (d) is correct.

Note that $w \in V\Sigma^*$ means that w consists of a nonterminal directly followed by zero or more terminals. Now take the right linear context-free grammar:

$$S \to aS \mid \lambda$$

Let us consider the three candidate invariants that are given:

- $P(w) := [w \in V\Sigma^*]$ It is clear that $P(S)$ holds and that $S \to \lambda$ is a valid production step, but $P(\lambda)$ does not hold as λ doesn't start with a nonterminal.
- $P(w) := [w \in \Sigma^* V]$ It is clear that $P(S)$ holds and that $S \to \lambda$ is a valid production step, but $P(\lambda)$ does not hold as λ doesn't end with a nonterminal.

- $P(w) := [w \in (V \cup \{\lambda\})\Sigma^*]$ It is clear that $P(S)$ holds and that $S \to aS$ is a valid production step, but $P(aS)$ does not hold as aS starts with an a and in $(V \cup \{\lambda\})\Sigma^*$ an element of Σ like a can never be followed by an element of V like S.

So none of the given options is an invariant, leaving only the 'none of the above' as the correct solution. In fact a proper invariant is

$$P(w) := [w \in \Sigma^*(V \cup \{\lambda\})].$$

5. Is there a non-deterministic finite automaton with at most two states that accepts the language $\mathcal{L}(a^*b^*)$?

   (a) Yes, there even is a deterministic finite automaton with two states that accepts this language.

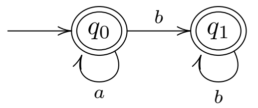
   (b) Yes, but there is not a deterministic finite automaton with at most two states that accepts this language.

   (c) No, any automaton for this language needs at least three states.

   (d) No, there is no automaton for this language at all, as it is not regular.

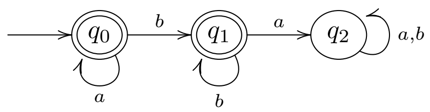
(b) is correct

First note that there is an NFA with at most two states that accepts the language $\mathcal{L}(a^*b^*)$:



However, there isn't a DFA with at most two states accepting this language. Let us construct a DFA for this language which is as small as possible.

- Obviously, we need an initial state q_0.
- And as λ is in the language it needs to be a final state.
- This state q_0 also needs an a-arrow, which can loop to itself, so we don't need a second state yet.
- However state q_0 also needs a b-arrow, which can't loop to itself, as that would accept the illegal word ba. So we need a state q_1 and a b-arrow from q_0 to q_1 and as the word b should be accepted, this q_1 must be a final state.
- However, as q_1 needs an a-arrow and any word that has a's behind b's should not be accepted, so the a-arrow from q_1 needs to go to a non-final state and because both q_0 and q_1 are final states, we certainly need a third state. So it is not possible to create such a DFA with at most two states. It is, however, possible to do it with three states, as this example shows:



6. Is there a graph with 15 vertices that all have degree 3?

   (a) Yes, the Petersen graph is an example of such a graph.
   (b) Yes, the bipartite complete graph $K_{3,15}$ is an example of such a graph.
   (c) No, because at most two vertices are allowed to have an odd degree.
   (d) No, because this would mean that the sum of the degrees would be odd.

(d) is correct

Answer (d) is correct.

Note that the Petersen graph has only 10 vertices and the bipartite complete graph $K_{3,15}$ has 18 vertices. In addition, whereas in the Petersen graph each vertex has indeed degree 3, this doesn't hold for the $K_{3,15}$ as the three distinguished points all have degree 15.

So the answer must be 'no'.

Now note that there is no limit on the number of vertices allowed to have an odd degree in general. If the graph needs to have an Euler path, then this limit is indeed a requirement.

And it is easy to see that each edge adds 1 to the degree of both the two vertices it is connected to. So the sum of the degrees in the full graph is two times the number of edges and should indeed be even, but $15 \times 3 = 45$ and 45 is odd.

7. Prove with induction that $3^n > 8$ when $n \geq 2$.

0 **Proposition:** $3^n > 8$ for all $n \geq 2$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \;:=\; 3^n > 8$$

3 **Base Case.** We show that $P(2)$ holds, i.e. we show that
$3^2 > 8$

4 This indeed holds, because $3^2 = 9 > 8$.

5 **Induction Step.** Let k be any natural number such that $k \geq 2$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that
$3^k > 8$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that
$3^{k+1} > 8$

8 This indeed holds, because

$$\begin{aligned} 3^{k+1} &= 3^k \cdot 3 \\ &\overset{\text{IH}}{>} 8 \cdot 3 \\ &= 24 \\ &> 8 \end{aligned}$$

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 2$.

8. Someone wants to formalize the English sentence

When it rains I always get wet.

using the dictionary

R	it rains
W	I get wet

as a formula of temporal logic, and is considering the two modal formulas:

$$\Box(R \to W)$$
$$R \to \Box W$$

Which of these two is the correct formalization?

(a) is correct

(a) The formula $\Box(R \to W)$ is correct because the formula $R \to \Box W$ would imply that once it has rained I will be wet forever, and that is not what the sentence says.

(b) The formula $R \to \Box W$ is correct, because it corresponds best to the sentence.

(c) Both formulas mean the same, so both are correct.

(d) Neither of these formulas is correct.

Answer (a) is correct.

The explanation why is already given in the solution itself. In addition, you could also translate the formula $\Box(R \to W)$ by *Every time that it rains I get wet* which has the same meaning as the original sentence.