Formal Reasoning 2023
Test Block 3: Discrete Mathematics and Modal Logic
(21/12/23)

There are six multiple choice questions and two open questions (questions 4 and 7). Each multiple choice question is worth 10 points, and the open questions are worth 15 points each. The mark for this test is the number of points divided by ten, and the first ten points are free. Good luck!

Discrete Mathematics

1. How many non-isomorphic trees are there with exactly five vertices?
   (a) 2
   (b) 3
   (c) 4
   (d) more than 4

2. Is there a graph that has an Eulerian cycle, but no Hamiltonian cycle?
   (a) Yes, the Petersen graph is an example.
   (b) Yes, the cube graph is an example.
   (c) Yes, $K_{2,4}$ is an example.
   (d) No.

3. The sums of the rows of Pascal's triangle are always a power of two. For example, we have $1 + 4 + 6 + 4 + 1 = 16 = 2^4$.

   What are the sums of the rows in the triangle of the Stirling numbers of the first kind?
   (a) Also powers of two.
   (b) Powers of two minus one.
   (c) The Bell numbers.
   (d) Factorials.

4. We define a sequence a_n recursively by the recursion equations:

$$\begin{cases} a_0 & = 0 \\ a_{n+1} & = 3a_n + 1 \quad \text{for all } n \geq 0 \end{cases}$$

   Now consider the following proof in which two parts have been omitted:

   Proposition $a_n = (3^n - 1)/2$ for all $n \geq 0$.

   Proof by induction on n. We first define our predicate P as:

$$P(n) := \big[a_n = (3^n - 1)/2\big]$$

Base case. We show that $P(0)$ holds, i.e. we show that $a_0 = (3^0 - 1)/2$.
This indeed holds because

$$a_0 = 0 \qquad \text{by definition}$$

and

$$(3^0 - 1)/2 = (1 - 1)/2 = 0$$

Induction step. Let k be any natural number such that $k \geq 0$.

We now show that $P(k + 1)$ also holds, i.e. we show that

$$a_{k+1} = (3^{k+1} - 1)/2$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$ $\square$

Give the two missing parts of this proof.

In ASCII you can use _ and ^ for subscript and superscript, so you may write

$$a_{k+1} = \frac{3^{k+1} - 1}{2}$$

as

```
a_(k+1) = (3^(k+1) - 1)/2
```

Modal Logic

5. Consider the following English sentence:

   I do not know that Albany is the capital of New York State, but I know that I don't know this.

   We use as dictionary:

   A Albany is the capital of New York State

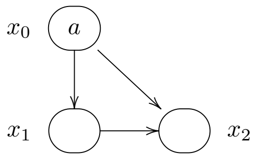
   Now which formula of epistemic logic corresponds to the meaning of this sentence?

(a) $(\Box\neg A) \wedge (\Box\Box\neg A)$
(b) $(\Diamond\neg A) \wedge (\Diamond\Diamond\neg A)$
(c) $(\Diamond\neg A) \wedge (\Box\Diamond\neg A)$
(d) none of the above

6. In which of the following logics does axiom scheme T (reflexivity) hold?

   (a) deontic logic
   (b) doxastic logic
   (c) temporal logic
   (d) all of the above

7. Consider the following Kripke model:



   Does

   $$x_0 \Vdash \Diamond\Box a$$

   hold? Explain your answer.

   You can write this statement using ASCII in the style

```
x_0 ||- <>[]a
```

   Make sure to explicitly state which formulas hold in which worlds.

8. Which LTL formula states that a will be true infinitely often but also that a will be false infinitely often?

   (a) $(\mathcal{G}\mathcal{F}a) \wedge (\mathcal{G}\mathcal{F}\neg a)$
   (b) $(\mathcal{F}\mathcal{G}a \wedge (\mathcal{F}\mathcal{G}\neg a)$
   (c) $(\mathcal{G}\mathcal{G}a) \wedge (\mathcal{G}\mathcal{G}\neg a)$
   (d) $(\mathcal{F}\mathcal{F}a) \wedge (\mathcal{F}\mathcal{F}\neg a)$