# Formal Reasoning 2024
# Exam
**(16/01/25)**

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The mark for this test is the number of points divided by ten, The first 10 points are free. Good luck!

## Propositional logic

1. Consider the following English sentence:

   *I do not work on Wednesday and Friday.*

   We use the following dictionary:

   | | |
   |---|---|
   | $W$ | I work on Wednesday |
   | $F$ | I work on Friday |

   Which formula of propositional logic encodes the meaning of this sentence using this dictionary?

   (a) $\neg W \wedge F$
   (b) $\neg W \vee F$
   (c) $\neg(W \wedge F)$
   (d) $\neg(W \vee F)$

2. A truth table for formulas with $n$ atoms has $2^n$ rows. This means there are $2^{2^n}$ possible different columns in such a table. For example, if the atoms are $a$ and $b$, then $n = 2$, which means that a truth table with these two atoms has $2^2 = 4$ rows, and that there are $2^4 = 16$ possible columns.

   Is there for each of those possible columns a formula of propositional logic that has exactly that column?

   (a) Yes, one can get any column with exactly one 1 using a formula that is a conjunction of atoms and negations of atoms, and then get arbitrary columns as disjunctions of such formulas.
   (b) Yes, one can get any column with exactly one 1 using a formula that is a disjunction of atoms and negations of atoms, and then get arbitrary columns as conjunctions of such formulas.
   (c) No, there are not enough formulas for that, as the function $2^{2^n}$ grows very very fast (for $n = 9$ this already is larger than the number of atoms in the universe).
   (d) No, the columns with only zeroes or with only ones are not the column of a formula of propositional logic.

3. The principles of *Modus Ponens* and *Modus Tollens* are:

| | |
|---|---|
| $(f \to g) \wedge f \vDash g$ | Modus Ponens |
| $(f \to g) \wedge \neg g \vDash \neg f$ | Modus Tollens |

Here $f$ and $g$ are arbitrary formulas of propositional logic. Do these principles hold for all $f$ and $g$?

(a) Both of these principles hold for all $f$ and $g$.

(b) Modus Ponens holds for all $f$ and $g$, but Modus Tollens does not.

(c) Modus Tollens holds for all $f$ and $g$, but Modus Ponens does not.

(d) Neither of these principles hold for all $f$ and $g$.

## Predicate logic

4. Consider the following English sentence:

*Not all men own a smartphone.*

We use the following dictionary:

| | |
|---|---|
| $H$ | humans |
| $T$ | things |
| $M(x)$ | $x$ is a man |
| $S(x)$ | $x$ is a smartphone |
| $O(x, y)$ | $x$ owns $y$ |

Give a formula of predicate logic *which does not contain a universal quantifier,* that encodes the meaning of this sentence using this dictionary.

5. Consider the structure $M_5 := (\mathbb{R}, \cdot)$, where $\cdot$ denotes multiplication, as well as the interpretation $I_5$:

| | |
|---|---|
| $R$ | $\mathbb{R}$ |
| $M(x, y, z)$ | $x \cdot y = z$ |

Does the following hold?

$$(M_5, I_5) \vDash \forall x, z \in R\, \exists y \in R\, M(x, y, z)$$

(a) Yes, one can always take $y = 1$, because multiplying by one does not change the value.

(b) Yes, one can always take $y = z = 0$, because zero times anything is zero.

(c) No, this does not hold in the case that $x \cdot y \neq 1$ and $z = 1$.

(d) No, this does not hold in the case that $x = 0$ and $z \neq 0$.

6. Consider the following English sentence:

*There exist exactly two men.*

We use the following dictionary:

| | |
|---|---|
| $H$ | humans |
| $M(x)$ | $x$ is a man |

Someone proposes the following formula of predicate logic with equality that encodes the meaning of this sentence using this dictionary:

$$\exists x, y \in H\,[M(x) \wedge M(y)]$$

Is this correct?

(a) Yes, this is correct.

(b) No, this formula states that there are at most two men, but it is also true if there is only one man.

(c) No, this formula states that there are at least two men, but it is also true if there are more than two men.

(d) No, this formula is also true if there is only one man or if there are more than two men.

## Languages

7. Does there exist a finite language (i.e., a language with only finitely many words in it) with the two properties $L \not\subseteq \{\lambda\}$ and $LL \subseteq L$?

   (a) Yes, as $L = \varnothing$ is such a language.

   (b) Yes, as $L = \{\lambda\}$ is such a language.

   (c) No, languages with these two properties exist, but they are necessarily infinite.

   (d) No, a language with these two properties does not exist at all, not even as an infinite language.

8. Give a regular expression for the language over the alphabet $\Sigma = \{a, b\}$:

$$L_8 := \overline{\mathcal{L}\big((a \cup b)^* aa (a \cup b)^*\big)}$$

9. Consider as $G_9$:

$$S \to SS \mid aS$$

   Is this a correct context-free grammar?

   (a) Yes, and the language it defines is:

$$\mathcal{L}(G_9) = \varnothing$$

   (b) Yes, and the language it defines is:

$$\mathcal{L}(G_9) = \{a^n \mid n \geq 1\}$$

   (c) No, a context-free grammar needs to have a rule with $\lambda$ on the right-hand side.

   (d) No, a context-free grammar needs to have a non-terminal besides the start symbol $S$.

10. Consider the context-free grammar $G_{10}$ with alphabet $\Sigma = \{a\}$:

$$S \to aA \mid \lambda$$
$$A \to AS$$

Someone wants to show that $\mathcal{L}(G_{10}) = \{\lambda\}$ by giving a production for $\lambda$ and showing that the following property is an invariant of this grammar:

$$P(w) := \big[w \in \{S, \lambda\} \text{ or } w \text{ contains } A\big]$$

Does this work?

(a) Yes, this works, as the only word in $\{a\}^*$ that satisfies this invariant is $\lambda$.
(b) Yes, this works. Although this is not an invariant, indeed $\mathcal{L}(G_{10}) = \{\lambda\}$.
(c) No, this does not work, as there is no production for $\lambda$.
(d) No, this does not work, there are words $u, v \in \{S, A, a\}^*$ such that $P(u)$ holds, $P(v)$ does not hold, and $u \to v$ is a replacement according to one of the rules of the grammar.

## Automata

11. There is a unique deterministic finite automaton with five states for the language:

$L_{11} = \{w \in \{a, b\}^* \mid w \text{ contains exactly one } a \text{ and an odd number of } b\text{'s}\}$

Give the right linear context-free grammar that one gets by converting this automaton to a grammar using the method which is explained in the course. Use for the non-terminals the set $V = \{S, A, B, C, D\}$.

You do not need to give/describe the automaton, and you should not optimize the grammar.

12. What is the minimum number of states in a deterministic finite automaton for the language:

$$L_{12} = \{w \in \{a, b\}^* \mid w \text{ contains } aa\}$$

(a) 1
(b) 2
(c) 3
(d) 4 or more

13. What is the minimum number of states in a *non*-deterministic finite automaton for the language:

$$L_{13} = \{w \in \{a, b\}^* \mid w \text{ does not contain } aa\}$$

(a) 1
(b) 2
(c) 3
(d) 4 or more

## Discrete mathematics

14. How many non-isomorphic trees are there with exactly four vertices?

    (a) 1 or less
    (b) 2
    (c) 3
    (d) 4 or more

15. How many non-isomorphic graphs are there that have exactly three vertices and contain an Eulerian path?

    (a) 1 or less
    (b) 2
    (c) 3
    (d) 4 or more

16. We define a sequence $a_n$ for $n \geq 0$ using the recursion equations:

    $$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= a_n + 2n + 3 \qquad\qquad \text{for all } n \geq 0 \end{aligned}$$

    Give the value of $a_{44}$.

    *Hint:* Compute the first elements of this sequence, and note the pattern. In your answer you may give those first elements and the pattern if you worry about computation errors in your final answer.

    Do *not* use induction to prove that this pattern holds. Just giving the value (if correct) is sufficient for full points.

17. When proving a statement using induction, we need to show that $P(n)$ holds for all $n$ starting at the start value. But in the induction step we already assume that $P(k)$ is true (the induction hypothesis), starting at this same value. Why do we need to prove anything then?

    (a) The variables $n$ and $k$ are different, which means that knowing $P(k)$ for all $k$ does not give us $P(n)$ for all $n$.
    (b) The induction step starts one position later than the start value of the proposition, which means that we still need to prove the base case.
    (c) We only may use $P(k)$ to prove $P(k+1)$, which means that we may not use the induction hypothesis outside of the context of the induction step.
    (d) It is not the case that we may use that the induction hypothesis is already true when proving the induction step.

18. How can we see the following equality?

    $$\binom{n}{k} = \binom{n}{n-k}$$

    (a) We can note that choosing $k$ objects corresponds to selecting the $n-k$ objects that are not chosen.

(b) We can compare the coefficients of $(x + y)^n$ and $(y + x)^n$, and use that $x + y = y + x$.

(c) We may use the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

twice, and note that $n - (n - k) = k$.

(d) All of the above.

19. In how many ways can one divide a collection of six distinguishable objects into three non-distinguishable non-empty groups?

    Give the name of the numbers that are used for counting this, give the kind of brackets that are used for writing these numbers, and give a relevant part of the corresponding triangle which you used to compute your answer.

## Modal logic

20. Does the axiom scheme $T$, which is $\Box f \to f$, hold in doxastic logic?

    (a) Yes, as you can believe true statements.

    (b) Yes, as you can believe false statements.

    (c) No, as you can believe true statements.

    (d) No, as you can believe false statements.

21. Give a serial Kripke model $\mathcal{M}_{21}$ such that:

$$\mathcal{M}_{21} \not\models a \to \Box\Diamond a$$

    Write your answer as a tuple $\langle W, R, V \rangle$.

    (If you do not know how to write a Kripke model as a tuple, you may describe your model in some other way for partial points.)

22. Is there an LTL Kripke model $\mathcal{M}_{22}$ such that the following holds?

$$\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\neg a \land \mathcal{X} a)$$

    (a) Yes. It is very well possible that $\neg a$ is true in $x_0$, with $a$ being true in $x_1$.

    (b) Yes. The truth of $a$ in $x_i$ is unrelated to the truth of $a$ in $x_{i+1}$, this holds for any $i \geq 0$.

    (c) No there is no such model: if $\neg a$ is true at all moments, it is not possible that $a$ is true in any world of the model either, which means that $\mathcal{X} a$ then cannot be true.

    (d) No, one cannot use the symbol $\Vdash$ in this way.