Formal Reasoning 2024
Solutions Exam
(16/01/25)

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The mark for this test is the number of points divided by ten, The first 10 points are free. Good luck!

Propositional logic

1. Consider the following English sentence:

   I do not work on Wednesday and Friday.

   We use the following dictionary:

W	I work on Wednesday
F	I work on Friday

   Which formula of propositional logic encodes the meaning of this sentence using this dictionary?

   (a) $\neg W \land F$

   (b) $\neg W \lor F$

   (c) $\neg(W \land F)$

   (d) is correct (d) $\neg(W \lor F)$

   Answer (d) is correct.

   The direct translation of the sentence is $\neg W \land \neg F$, but that is not one of the options. However, by applying De Morgan we get that $\neg W \land \neg F \equiv \neg(W \lor F)$. Note that none of the other options is also logically equivalent.

2. A truth table for formulas with n atoms has 2^n rows. This means there are 2^{2^n} possible different columns in such a table. For example, if the atoms are a and b, then $n = 2$, which means that a truth table with these two atoms has $2^2 = 4$ rows, and that there are $2^4 = 16$ possible columns.

   Is there for each of those possible columns a formula of propositional logic that has exactly that column?

   (a) is correct (a) Yes, one can get any column with exactly one 1 using a formula that is a conjunction of atoms and negations of atoms, and then get arbitrary columns as disjunctions of such formulas.

   (b) Yes, one can get any column with exactly one 1 using a formula that is a disjunction of atoms and negations of atoms, and then get arbitrary columns as conjunctions of such formulas.

   (c) No, there are not enough formulas for that, as the function 2^{2^n} grows very very fast (for $n = 9$ this already is larger than the number of atoms in the universe).

(d) No, the columns with only zeroes or with only ones are not the column of a formula of propositional logic.

Answer (a) is correct.

We provide an example for $n = 2$ with the atoms a and b. Let us assume we want to find a formula f for the following column:

a	b	f
0	0	1
0	1	0
1	0	0
1	1	1

The positions where we have a 1 are in the rows where $a = 0$ and $b = 0$ and where $a = 1$ and $b = 1$. This means that we can make columns for conjunctions of atoms and negations of atoms to get each one. And then we can combine these formulas with a disjunction. In this example that would lead to this truth table:

a	b	$\neg a$	$\neg b$	$\neg a \wedge \neg b$	$a \wedge b$	$f = (\neg a \wedge \neg b) \vee (a \wedge b)$
0	0	1	1	1	0	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	0	1	1

Note that there is nothing special about this example. It works the same for larger examples.

This type of formulas is called 'full disjunctive normal form'.

3. The principles of *Modus Ponens* and *Modus Tollens* are:

$$(f \to g) \wedge f \vDash g \qquad \text{Modus Ponens}$$
$$(f \to g) \wedge \neg g \vDash \neg f \qquad \text{Modus Tollens}$$

Here f and g are arbitrary formulas of propositional logic. Do these principles hold for all f and g?

(a) is correct

(a) Both of these principles hold for all f and g.

(b) Modus Ponens holds for all f and g, but Modus Tollens does not.

(c) Modus Tollens holds for all f and g, but Modus Ponens does not.

(d) Neither of these principles hold for all f and g.

Answer (a) is correct.

Let us first have a look at the Modus Ponens. Let v be a model (valuation) such that $v((f \to g) \wedge f) = 1$. Then automatically $v(f \to g) = 1$ and $v(f) = 1$. Note that there are three ways to get $v(f \to g) = 1$, namely $v(f) = 0$ and $v(g) = 0$, $v(f) = 0$ and $v(g) = 1$, and $v(f) = 1$ and $v(g) = 1$. However, as we already know that $v(f) = 1$, it must be that we are in the third situation, so in particular $v(g) = 1$. So for any model v with $v((f \to g) \wedge f) = 1$, it follows that $v(g) = 1$. And $(f \to g) \wedge f \vDash g$ holds.

Now let us look at the Modus Tollens. Let v be a model such that $v((f \to g) \wedge \neg g) = 1$. Then automatically $v(f \to g) = 1$ and $v(\neg g) = 1$. Note that there are three ways to get $v(f \to g) = 1$, namely $v(f) = 0$ and $v(g) = 0$, $v(f) = 0$ and $v(g) = 1$, and $v(f) = 1$ and $v(g) = 1$. However, as $v(\neg g) = 1$, we know that $v(g) = 0$. And it must be that we are in the first situation, so in particular $v(f) = 0$. So for any model v with $v((f \to g) \wedge \neg g) = 1$, it follows that $v(f) = 0$ and hence $v(\neg f) = 1$. And $(f \to g) \wedge \neg g \vDash \neg f$ holds.

So both Modus Ponens and Modus Tollens hold.

Predicate logic

4. Consider the following English sentence:

   Not all men own a smartphone.

   We use the following dictionary:

H	humans
T	things
$M(x)$	x is a man
$S(x)$	x is a smartphone
$O(x,y)$	x owns y

   Give a formula of predicate logic *which does not contain a universal quantifier,* that encodes the meaning of this sentence using this dictionary.

   The sentence *Not all men own a smartphone.* is equivalent to *There exists a man who doesn't own a smartphone.* which is equivalent to *There exists a man such that there is no thing that he owns and is a smartphone.* and this sentence can be translated to the formula:

$$\exists h \in H\, [M(h) \wedge \neg \exists t \in T\, [S(t) \wedge O(h,t)]]$$

5. Consider the structure $M_5 := (\mathbb{R}, \cdot)$, where $\cdot$ denotes multiplication, as well as the interpretation I_5:

R	$\mathbb{R}$
$M(x,y,z)$	$x \cdot y = z$

   Does the following hold?

$$(M_5, I_5) \vDash \forall x, z \in R\, \exists y \in R\, M(x,y,z)$$

   (a) Yes, one can always take $y = 1$, because multiplying by one does not change the value.

   (b) Yes, one can always take $y = z = 0$, because zero times anything is zero.

   (c) No, this does not hold in the case that $x \cdot y \neq 1$ and $z = 1$.

(d) is correct (d) No, this does not hold in the case that $x = 0$ and $z \neq 0$.

   Answer (d) is correct.

   The statement claims that in model (M_5, I_5) for each pair of real numbers x and z, there exists a real number y such that $x \cdot y = z$. This certainly doesn't hold if $x = 0$ and $z \neq 0$, as for any real number y, the product $0 \cdot y = 0$, so it can't be equal to z.

Note that if $x \neq 0$ the statement does hold. In case also $z = 0$, then we can take $y = 0$. And in case $z \neq 0$, we can take $y = \frac{1}{x}$ which exists as a real number because $x \neq 0$.

In particular, this also indicate why the other 'No' option is incorrect. The fact that $x \cdot y \neq 1$ and $z = 1$ for some y doesn't mean that such a y doesn't exist. Because we have seen that such a y always exists as long as $x \neq 0$.

6. Consider the following English sentence:

   There exist exactly two men.

   We use the following dictionary:

H	humans
$M(x)$	x is a man

   Someone proposes the following formula of predicate logic with equality that encodes the meaning of this sentence using this dictionary:

   $$\exists x, y \in H\,[M(x) \wedge M(y)]$$

   Is this correct?

   (a) Yes, this is correct.

   (b) No, this formula states that there are at most two men, but it is also true if there is only one man.

   (c) No, this formula states that there are at least two men, but it is also true if there are more than two men.

   (d) is correct

   (d) No, this formula is also true if there is only one man or if there are more than two men.

   Answer (d) is correct.

   As there is no additional clause like $x \neq y$, it is allowed to have only one man as we can take x and y to be the same man. In addition, if there are more than two men in H, the formula is also true. As long as we take real men for x and y the formula will hold, even if we take the same men.

Languages

7. Does there exist a finite language (i.e., a language with only finitely many words in it) with the two properties $L \not\subseteq \{\lambda\}$ and $LL \subseteq L$?

   (a) Yes, as $L = \varnothing$ is such a language.

   (b) Yes, as $L = \{\lambda\}$ is such a language.

   (c) is correct

   (c) No, languages with these two properties exist, but they are necessarily infinite.

   (d) No, a language with these two properties does not exist at all, not even as an infinite language.

Answer (c) is correct.

Languages with the properties $L \not\subseteq \{\lambda\}$ and $LL \subseteq L$ do exist. In the course notes we have already seen that if L is the language over $\Sigma = \{a, b\}$ where each word has an even number of a's, then even $L^* \subseteq L$. And as $LL \subseteq L^*$, we get that $LL \subseteq L$. Obviously, this language L is infinite and hence $L \not\subseteq \{\lambda\}$.

Now assume that we have a finite language L such that $L \not\subseteq \{\lambda\}$, which means that there is a word $w \in L$ such that $|w| > 0$. As L is finite we can have a look at the longest words within L:

$$L' := \{w \in L \mid \forall v \in L\, [\, |w| \geq |v|\,]\} \subseteq L$$

This set is not empty so we can take $w \in L'$ and assume that $|w| = n > 0$. Then $ww \in LL$ and $|ww| = 2n$. However, as $n > 0$ we know that $2n > n$, which means that ww is longer than the longest word in L. So it certainly isn't in L. Hence there cannot be a finite language L such that $L \not\subseteq \{\lambda\}$ for which $LL \subseteq L$.

Another way to see that a language with these two properties has to be infinite, goes like this. As $L \not\subseteq \{\lambda\}$, there has to be a word $w \in L$ with length n where $n \geq 1$. And as $LL \subseteq L$ it follows that $ww \in L$ with length $2n$. And as $LL \subseteq L$ it follows that $wwww \in L$ with length $4n$. And as $LL \subseteq L$ it follows that $wwwwwwww \in L$ with length $8n$. And so on, and on, and on. This implies that for each natural number $m \in \mathbb{N}$ we can find a word in L with a length that is larger than m. So L cannot be finite.

Note that if $L \subseteq \{\lambda\}$ does hold, we only have two options: $L = \varnothing$ and $L = \{\lambda\}$ and for both situations it does hold that $LL \subseteq L$.

8. Give a regular expression for the language over the alphabet $\Sigma = \{a, b\}$:

$$L_8 := \overline{\mathcal{L}\big((a \cup b)^* aa (a \cup b)^*\big)}$$

Note that the language $\mathcal{L}\big((a \cup b)^* aa (a \cup b)^*\big)$ is the language where every word contains two consecutive a's, so its complement L_8 is the language that does not contain two consecutive a's. So if $w \in L_8$ then, there are three options: w doesn't contain any a's, each a is immediately followed by a series of at least one b, or each a is immediately followed by a series of at least one b, except for the last a which is also the last symbol in the word. These observations lead to the regular expression

$$b^*((abb^*)^*)(\lambda \cup a)$$

Also correct:

$$(ab \cup b)^*(\lambda \cup a)$$

9. Consider as G_9:

$$S \to SS \mid aS$$

Is this a correct context-free grammar?

(a) Yes, and the language it defines is:

$$\mathcal{L}(G_9) = \varnothing$$

(a) is correct

(b) Yes, and the language it defines is:

$$\mathcal{L}(G_9) = \{a^n \mid n \geq 1\}$$

(c) No, a context-free grammar needs to have a rule with λ on the right-hand side.

(d) No, a context-free grammar needs to have a non-terminal besides the start symbol S.

Answer (a) is correct.

Each context-free grammar is a triple $\langle \Sigma, V, R \rangle$ where Σ is an alphabet, V a set of nonterminals, and R a set of production rules of the form $X \to w$ where $X \in V$ and $w \in (\Sigma \cup V)^*$.

In case of G_9 we can take $\Sigma = \{a\}$, $V = \{S\}$, and $R = \{S \to SS, S \to aS\}$. And it is clear that all requirements for a grammar are met.

So it is a correct context-free grammar. However, as there are no rules $X \to w$ where $w \in \Sigma^*$, it is clear that there are no finite productions, so $\mathcal{L}(G_9) = \varnothing$.

10. Consider the context-free grammar G_{10} with alphabet $\Sigma = \{a\}$:

$$S \to aA \mid \lambda$$
$$A \to AS$$

Someone wants to show that $\mathcal{L}(G_{10}) = \{\lambda\}$ by giving a production for λ and showing that the following property is an invariant of this grammar:

$$P(w) := \big[w \in \{S, \lambda\} \text{ or } w \text{ contains } A\big]$$

Does this work?

(a) Yes, this works, as the only word in $\{a\}^*$ that satisfies this invariant is λ.

(a) is correct

(b) Yes, this works. Although this is not an invariant, indeed $\mathcal{L}(G_{10}) = \{\lambda\}$.

(c) No, this does not work, as there is no production for λ.

(d) No, this does not work, there are words $u, v \in \{S, A, a\}^*$ such that $P(u)$ holds, $P(v)$ does not hold, and $u \to v$ is a replacement according to one of the rules of the grammar.

Answer (a) is correct.

First note that $\mathcal{L}(G_{10}) \subseteq \{a\}^*$. In addition, note that there is a production for λ: $S \to \lambda$.

It is also clear that $P(a^n)$ does not hold for any $n > 0$. So if P is an invariant, then it can be used to show that $\mathcal{L}(G_{10}) = \{\lambda\}$.

And P is indeed an invariant. Obviously, $P(S)$ holds as $S \in \{S, \lambda\}$. Now if $u, v \in \{S, A, a\}^*$ such that $P(u)$ holds and $u \to v$ there are two options:

- $u = S$: In this case $v = aA$ or $v = \lambda$ and in both situations $P(v)$ holds.
- u contains A and hence v contains A as none of the three rules remove an A.

Note that the situation $u = \lambda$ is not an option as then there doesn't exist a v with $\lambda \to v$.

So P is indeed an invariant and hence it can be used to show that there are no words beside λ in $\mathcal{L}(G_{10})$ and because of the production for λ we get that $\mathcal{L}(G_{10}) = \{\lambda\}$.

Automata

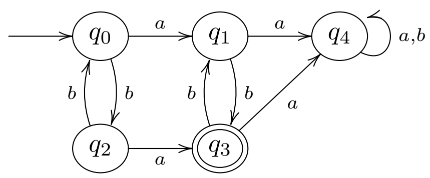
11. There is a unique deterministic finite automaton with five states for the language:

$$L_{11} = \{w \in \{a,b\}^* \mid w \text{ contains exactly one } a \text{ and an odd number of } b\text{'s}\}$$

Give the right linear context-free grammar that one gets by converting this automaton to a grammar using the method which is explained in the course. Use for the non-terminals the set $V = \{S, A, B, C, D\}$.

You do not need to give/describe the automaton, and you should not optimize the grammar.

Such an automaton will look like this, but obviously, the names of the states can be different.



Now if we identify the state q_0 with S, q_1 with A, q_2 with B, q_3 with C, and q_4 with D, and apply the default algorithm, we get this right linear context-free grammar:

$$\begin{array}{lcl} S & \to & aA \mid bB \\ A & \to & aD \mid bC \\ B & \to & aC \mid bS \\ C & \to & aD \mid bA \mid \lambda \\ D & \to & aD \mid bD \end{array}$$

12. What is the minimum number of states in a deterministic finite automaton for the language:

$$L_{12} = \{w \in \{a,b\}^* \mid w \text{ contains } aa\}$$

(a) 1

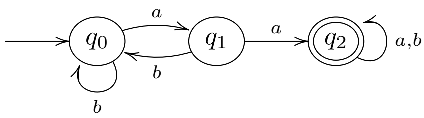
(b) 2

(c) is correct — (c) 3

(d) 4 or more

Answer (c) is correct.

Such a DFA needs to keep track of the current number of consecutive a's being read. So it needs to distinguish the situations where currently zero, one, and two consecutive a's have been read. So three states are needed. Note that there indeed exists an automaton with three states that accepts this language:



13. What is the minimum number of states in a *non*-deterministic finite automaton for the language:

$$L_{13} = \{w \in \{a,b\}^* \mid w \text{ does not contain } aa\}$$

(a) 1

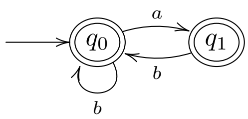
(b) is correct (b) 2

(c) 3

(d) 4 or more

Answer (b) is correct.

This is an NFA with two states that accepts L_{13}:



Now if we assume that there is also an NFA with only a single state q_0 that accepts L_{13}, we get a contradiction. As we need to accept the word a, this single state needs to be a final state and we need to have at least some a-transition from q_0. However, as we only have one state, it must go from q_0 to q_0. But then we cannot prevent that the word aa is also accepted. So it can't be done with a single state and hence two is the minimum.

Discrete mathematics

14. How many non-isomorphic trees are there with exactly four vertices?

(a) 1 or less

(b) is correct (b) 2

(c) 3

(d) 4 or more

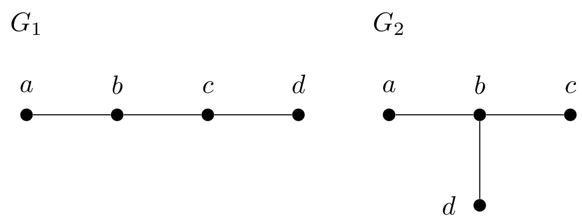
Answer (b) is correct.

Note that a tree is a connected graph that has no cycles.

We try to create as many trees, based on the vertices with the highest degree.

- If there was a tree with four vertices and the highest degree of the vertices was 1, then this graph would not be connected and hence it cannot be a tree.
- A tree where the highest degree of the vertices is 2 does exist. As it should be connected, it follows that all vertices should be in a single line, as junctions would imply a degree 3 or higher. There is only one way to do this. See G_1 below.
- A tree where the highest degree of the vertices is 3 also exists. In order to create such a tree, we start by taking a single vertex and then connect three other vertices to it. Note that it is not possible to connect some of these three vertices directly with each other, as that would create loops and loops are not allowed in trees. So there is only one way to create such a tree with four vertices where the highest degree is 2. See G_2 below.
- It is not possible to create a tree with four vertices where the highest degree is 4 (or more), as this would require that one vertex is connected to at least four other vertices, which would require at least five vertices, but there are only four vertices.

And this is what these trees look like:



Note that these two trees are indeed non-isomorphic as isomorphic graphs should have the same degrees.

15. How many non-isomorphic graphs are there that have exactly three vertices and contain an Eulerian path?

    (a) 1 or less

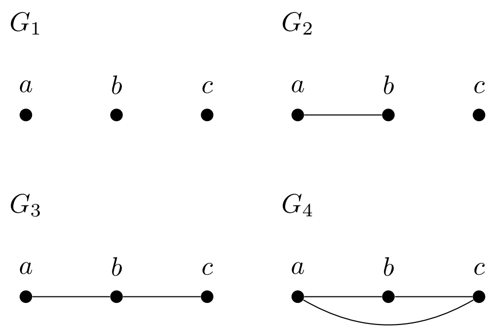
    (b) 2

    (c) is correct (c) 3

    (d) 4 or more

Answer (c) is correct.

There are four non-isomorphic graphs that have exactly three vertices:



Note that according to the definition in the course notes, an Eulerian path, in a graph $\langle V, E\rangle$, is a path in which every edge from E is included exactly once. An Eulerian circuit, or Eulerian cycle, is an Eulerian path that is a cycle as well.

As G_1 does not have any path at all, it certainly has no path in which every edge from E is included exactly once, even though there are no edges! The reason is that a path needs to consist of at least one edge. So G_1 does not have an Eulerian path.

Note that G_2 does have an Eulerian path, namely $a \to b$. The fact that vertex c is not visited doesn't matter.

As G_3 and G_4 are connected graphs, we can apply Euler's theorem and get that G_3 has an Eulerian path (for instance $a \to b \to c$) and that G_4 even has an Eulerian cycle, which by definition is also an Eulerian path (for instance $a \to b \to c \to a$).

So there are three non-isomorphic graphs with three vertices that have an Eulerian path. In addition, for G_3 we have that $a \to b \to c$ is an Eulerian path.

16. We define a sequence a_n for $n \geq 0$ using the recursion equations:

$$\begin{aligned} a_0 &= 1 \\ a_{n+1} &= a_n + 2n + 3 \qquad\qquad \text{for all } n \geq 0 \end{aligned}$$

Give the value of a_{44}.

Hint: Compute the first elements of this sequence, and note the pattern. In your answer you may give those first elements and the pattern if you worry about computation errors in your final answer.

Do *not* use induction to prove that this pattern holds. Just giving the value (if correct) is sufficient for full points.

Let us follow up on the hint and compute some values of a_n:

$$\begin{array}{lllllllll} a_0 & = & 1 \\ a_1 & = & a_{0+1} & = & a_0 + 2 \cdot 0 + 3 & = & 1 + 0 + 3 & = & 4 \\ a_2 & = & a_{1+1} & = & a_1 + 2 \cdot 1 + 3 & = & 4 + 2 + 3 & = & 9 \\ a_3 & = & a_{2+1} & = & a_2 + 2 \cdot 2 + 3 & = & 9 + 4 + 3 & = & 16 \\ a_4 & = & a_{3+1} & = & a_3 + 2 \cdot 3 + 3 & = & 16 + 6 + 3 & = & 25 \end{array}$$

So it seems that the pattern is

$$a_n = (n+1)^2$$

And hence, $a_{44} = (44+1)^2 = 45^2 = 2025$. Knowing Freek, the answer was predictable, but it is also easy to compute without a calculator.

$$\begin{array}{r} 4\,5 \\ \times\;\; 4\,5 \\ \hline 2\,2\,5 \\ 1\,8\,0\;\;\, \\ \hline 2\,0\,2\,5 \end{array}$$

17. When proving a statement using induction, we need to show that $P(n)$ holds for all n starting at the start value. But in the induction step we already assume that $P(k)$ is true (the induction hypothesis), starting at this same value. Why do we need to prove anything then?

    (a) The variables n and k are different, which means that knowing $P(k)$ for all k does not give us $P(n)$ for all n.

    (b) The induction step starts one position later than the start value of the proposition, which means that we still need to prove the base case.

    (c) We only may use $P(k)$ to prove $P(k+1)$, which means that we may not use the induction hypothesis outside of the context of the induction step.

    (c) is correct

    (d) It is not the case that we may use that the induction hypothesis is already true when proving the induction step.

    Answer (c) is correct.

    The IH states that we assume that $P(k)$ holds for some k such that k is larger or equal than the start value. In particular, we do not assume that $P(k)$ holds for all k larger or equal than the start value, as in that case we would be done with the proof. So within the context of the induction step, we are actually only looking at a single k, whereas in the end we need to prove it for all n larger or equal to the start value in general.

    The first option is not correct. If we would know that $P(k)$ holds for all k larger or equal than the start value, then we would also know that for all n larger or equal than the start value.

    The second option is not correct. The induction step does not start one position later than the start value. It starts at the same start value! The conclusion of the induction step is one position later.

    The fourth option is not correct. We certainly may assume that the IH holds within the induction step.

18. How can we see the following equality?

$$\binom{n}{k} = \binom{n}{n-k}$$

    (a) We can note that choosing k objects corresponds to selecting the $n-k$ objects that are not chosen.

    (b) We can compare the coefficients of $(x+y)^n$ and $(y+x)^n$, and use that $x+y=y+x$.

(c) We may use the formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

twice, and note that $n - (n-k) = k$.

(d) is correct

(d) All of the above.

Answer (d) is correct.

The combinatorial interpretation of $\binom{n}{k}$ is indeed the number of ways we can choose k elements out of a set of n different elements and write them on the left of the blackboard, and write all the remaining $n-k$ elements on the right of the blackboard. Obviously, if we would take $n-k$ out of the same set but now write the elements on the right of the blackboard and write the remaining k elements on the left of the blackboard, we end up with exactly the same possibilities. However, the last method would compute $\binom{n}{n-k}$. So these are equal.

The general version of the binomial theorem states that

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Now if we use that $(x+y)^n = (y+x)^n$ (which indeed holds!), then we get

$$(y+x)^n = \binom{n}{0}y^n + \binom{n}{1}y^{n-1}x + \binom{n}{2}y^{n-2}x^2 + \cdots + \binom{n}{n-1}yx^{n-1} + \binom{n}{n}x^n$$

By comparing the coefficients of the monomials $x^{n-k}y^n$ and y^nx^{n-k}, then for the first one we get $\binom{n}{k}$ and for the second one $\binom{n}{n-k}$. However, as multiplication is commutative, the monomials $x^{n-k}y^k$ and y^kx^{n-k} are the same, so their coefficients should also be the same. So $\binom{n}{k} = \binom{n}{n-k}$.

We can also use the formula to show that these binomial coefficients are the same.

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n!}{(n-k)! \cdot k!} = \frac{n!}{(n-k)! \cdot (n-(n-k))!} = \binom{n}{n-k}$$

So all methods work.

19. In how many ways can one divide a collection of six distinguishable objects into three non-distinguishable non-empty groups?

Give the name of the numbers that are used for counting this, give the kind of brackets that are used for writing these numbers, and give a relevant part of the corresponding triangle which you used to compute your answer.

This is the classical interpretation of Stirling numbers of the second kind. Hence the number of ways to divide six distinguishable objects into three non-distinguishable non-empty groups is $\left\{ {6 \atop 3} \right\}$.

The top of the triangle looks like this:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 7 & & 6 & & 1 & & \\
 & 1 & & 15 & & 25 & & 10 & & 1 & \\
1 & & 31 & & \boxed{90} & & 65 & & 15 & & 1
\end{array}$$

The value $\left\{ {6 \atop 3} \right\}$ is marked and it turns out to be 90.

Modal logic

20. Does the axiom scheme T, which is $\Box f \to f$, hold in doxastic logic?

    (a) Yes, as you can believe true statements.

    (b) Yes, as you can believe false statements.

    (c) No, as you can believe true statements.

    (d) is correct (d) No, as you can believe false statements.

    Answer (d) is correct.

    Doxastic logic is the classical example where the axiom scheme T, $\Box f \to f$ does not hold. It means that *if I believe that f holds, then f holds.* However, this is not true, exactly for the reason that is explained in the last option.

21. Give a serial Kripke model $\mathcal{M}_{21}$ such that:

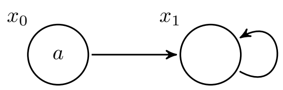
    $$\mathcal{M}_{21} \not\models a \to \Box\Diamond a$$

    Write your answer as a tuple $\langle W, R, V \rangle$.

    (If you do not know how to write a Kripke model as a tuple, you may describe your model in some other way for partial points.)

    As this formula is an instance of the symmetry scheme B, it means that our counter example should have a non-symmetric accessibility relation. In addition, it does need to be serial, so each world must have at least one outgoing arrow.

    So take for instance the model $\mathcal{M}_{21}$ given by



    Note that it is not symmetric (as there is no arrow from x_1 to x_0), but it is serial.

    This model is mathematically described by the tuple $\langle W, R, V \rangle$ where

    $$\begin{aligned} W &= \{x_0, x_1\} \\ R(x_0) &= \{x_1\} \\ R(x_1) &= \{x_1\} \\ V(x_0) &= \{a\} \\ V(x_1) &= \varnothing \end{aligned}$$

    This is the corresponding $\Vdash$-table:

	a	$\Diamond a$	$\Box\Diamond a$	$a \to \Box\Diamond a$
x_0	1	0	0	0
x_1	0	0	0	0

    Hence $x_0 \nVdash a \to \Box\Diamond a$ and hence $\mathcal{M}_{21} \not\models a \to \Box\Diamond a$. In fact, also $x_1 \nVdash a \to \Box\Diamond a$, but that is not important anymore.

22. Is there an LTL Kripke model $\mathcal{M}_{22}$ such that the following holds?

$$\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\neg a \wedge \mathcal{X} a)$$

(a) Yes. It is very well possible that $\neg a$ is true in x_0, with a being true in x_1.

(b) Yes. The truth of a in x_i is unrelated to the truth of a in x_{i+1}, this holds for any $i \geq 0$.

(c) No there is no such model: if $\neg a$ is true at all moments, it is not possible that a is true in any world of the model either, which means that $\mathcal{X} a$ then cannot be true.

(c) is correct

(d) No, one cannot use the symbol $\Vdash$ in this way.

Answer (c) is correct.

As the $\mathcal{G}$ operator distributes over conjunctions, it follows that

$$\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\neg a) \wedge \mathcal{G}(\mathcal{X} a)$$

should hold. However, this means that both $\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\neg a)$ and $\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\mathcal{X} a)$ should hold. The first claim implies that $a \notin V(x_i)$ for all $i \geq 0$. The second claim implies that $a \in V(x_i)$ for all $i \geq 1$. So there is a contradiction in x_1 where both $\neg a$ and a should hold, which cannot be the case. Hence such a model does not exist.

Note that the other 'No' answer is incorrect: it is allowed to write

$$\mathcal{M}_{22}, x_0 \Vdash \mathcal{G}(\neg a \wedge \mathcal{X} a)$$

and

$$x_0 \Vdash \mathcal{G}(\neg a \wedge \mathcal{X} a)$$

and these mean the same thing under the assumption that somehow it is known that we are talking about model $\mathcal{M}_{22}$.