Formal Reasoning 2024
Solutions Test Block 1: Propositional and Predicate Logic
(23/09/24)

Propositional logic

1. We want to formalize the English sentence

   It rains, but I don't get wet.

   For this we use the dictionary:

R	it rains
W	I get wet

   What is a correct formula of propositional logic, that corresponds to the meaning of this sentence?

   (a) is correct

   (a) $R \wedge \neg W$
   (b) $R \rightarrow \neg W$
   (c) $R \rightarrow W \vee \neg W$
   (d) None of the above.

   Answer (a) is correct.

   The sentence

   It rains, but I don't get wet.

   states that two things are true: the fact that it rains and the fact that I don't get wet. So the conjunction seems to be the best way to translate this sentence: $R \wedge \neg W$.

   Note that both other formulas don't state that it rains. They only state something if it rains.

2. What is the official form of the formula

   $$\neg a \wedge a$$

   according to the official grammar for formulas of propositional logic from the course notes?

   (a) $\neg a \wedge a$
   (b) $\neg(a \wedge a)$
   (c) $(\neg a) \wedge a$
   (d) $(\neg a \wedge a)$

   (d) is correct

   Answer (d) is correct.

   First we look at the binding strength. As the unary negation ($\neg$) binds stronger than the binary conjunction ($\wedge$), the structure of the formula is $((\neg a) \wedge a)$. However, as binary operators take a single pair of parentheses and unary operators don't take parentheses, the formula according to the official syntax will be $(\neg a \wedge a)$.

3. Consider the statement

$$(\neg f \leftrightarrow \neg g) \equiv \neg(f \leftrightarrow g)$$

where f and g are formulas of propositional logic. What is the case?

(a) For all formulas f and g, this holds.

(b) There are formulas f and g for which this holds and there are formulas f and g for which this does not hold.

(c) is correct

(c) For all formulas f and g, this does not hold.

(d) This is not a correct statement, because the symbol $\equiv$ should not occur inside a formula of propositional logic.

Answer (c) is correct.

The $\equiv$-statement claims that the truth tables for the formulas $\neg f \leftrightarrow \neg g$ and $\neg(f \leftrightarrow g)$ are the same. As f and g are arbitrary formulas, we can't create a truth table, but we can check what happens on individual rows. Obviously, each row in the truth table for the formula $\neg f \leftrightarrow \neg g$ has a 1 or a 0. So we make a case distinction on that:

- Assume that $\neg f \leftrightarrow \neg g$ has a 1 in the truth table. Because of the $\leftrightarrow$ this only happens when the truth values of $\neg f$ and $\neg g$ are the same. Now because of the $\neg$ that also means that the truth values of f and g are the same. Which in turn means that the truth table of $f \leftrightarrow g$ has a 1. However, that implies that the truth table of $\neg(f \leftrightarrow g)$ has a 0. So if the formula on the left has a 1 then the formula on the right has a 0. Hence the statement will certainly not be true for all formulas. But it could still be true for formulas that only have 0's on the left.
- Assume that $\neg f \leftrightarrow \neg g$ has a 0 in the truth table. Because of the $\leftrightarrow$ this only happens when the truth values of $\neg f$ and $\neg g$ are not the same. Now because of the $\neg$ that also means that the truth values of f and g are not the same. Which in turn means that the truth table of $f \leftrightarrow g$ has a 0. However, that implies that the truth table of $\neg(f \leftrightarrow g)$ has a 1. So if the formula on the left has a 0 then the formula on the right has a 1.

Hence we have seen that no matter which truth value the formula on the left has, the formula on the right has the opposite truth value. So there are no formulas f and g such that the formulas on the left and on the right have the same truth table. And the statement is wrong for all combinations of f and g.

4. Let f be an arbitrary proposition. Does the following hold?

If not $\models f$, then $\models \neg f$.

Explain your answer.

It doesn't hold for all formulas f, as we can create a counterexample for which *not* $\models f$ holds, but $\models \neg f$ does not. Take $f := a$, where a is an atomic proposition. By definition there are two rows in the truth table for a: one with a 0 and one with a 1. So f is not a tautology, so *not* $\Vdash f$

indeed holds. However, $\neg f$, which is $\neg a$, also has a 1 and a 0 in its truth table. So $\neg f$ is also not a tautology, so $\vDash \neg f$ indeed does not hold.

Note that there are formulas f for which it does hold. Take for instance $f := a \wedge \neg a$ which has only 0's in its truth table and hence it is not a tautology. However $\neg f$ has only 1's in its truth table, so $\vDash \neg f$ does hold for this specific example.

Predicate logic

5. Which formula of predicate logic gives the meaning of the English sentence:

   All intelligent men are nice.

   For this we use the dictionary:

M	the domain of men
$I(x)$	x is intelligent
$N(x)$	x is nice

   (a) is correct

   (a) $\neg\exists x \in M\, [I(x) \wedge \neg N(x)]$

   (b) $\neg\exists x \in M\, [I(x) \to \neg N(x)]$

   (c) $\neg\exists x \in M\, [I(x) \wedge N(x)]$

   (d) $\neg\exists x \in M\, [I(x) \to N(x)]$

   Answer (a) is correct.

   The default translation of this sentence would lead to the formula:

   $$\forall x \in M\, [I(x) \to N(x)]$$

   However, that is not one of the options. So we will use logical laws to transform this formula into one of the given options:

$\forall x \in M\, [I(x) \to N(x)]$	
$\equiv\ \neg\neg(\forall x \in M\, [I(x) \to N(x)])$	double negation
$\equiv\ \neg(\exists x \in M\, [\neg(I(x) \to N(x))])$	De Morgan
$\equiv\ \neg(\exists x \in M\, [\neg(\neg I(x) \vee N(x))])$	material implication
$\equiv\ \neg(\exists x \in M\, [\neg\neg I(x) \wedge \neg N(x))])$	De Morgan
$\equiv\ \neg(\exists x \in M\, [I(x) \wedge \neg N(x))])$	double negation
$\equiv\ \neg\exists x \in M\, [I(x) \wedge \neg N(x))]$	removing superfluous parentheses

   And this is indeed one of the options.

   Note that two of the other options combine an $\exists$ with an implication ($\to$), which makes them very unlikely. And the other option with an $\exists$ and a conjunction ($\wedge$) actually states that there are no men that are both intelligent and nice.

6. Does the following hold?

   $$\exists y \in W\, [\forall x \in M\, L(x,y)] \vDash \forall x \in M\, [\exists y \in W\, L(x,y)]$$

   (a) Yes, because these two formulas mean the same, and therefore are logically equivalent.

(b) is correct

(b) Yes, but it does not hold in the other direction.

(c) No, but it does hold in the other direction.

(d) No, because the symbol $\models$ should have been the symbol $\equiv$.

Answer (b) is correct.

The statement claims that the formula $\forall x \in M\,[\exists y \in W\, L(x, y)]$ is a logical consequence of the formula $\exists y \in W\,[\forall x \in M\, L(x, y)]$. So this should hold independent of the models at hand! The formula on the left claims that there exists a special element y in domain W for which $L(x, y)$ holds for all elements x in domain M. The formula on the right claims that for each element x in domain M, there exists a y in domain W, such that $L(x, y)$ holds. Hence the difference between the two is that for the left one we first pick this specific element y that will have $L(x, y)$ for all x, whereas on the right for each element x we are allowed to choose a different y with $L(x, y)$, which is a weaker statement. In particular, it follows that the formula on the right holds as for each x in M we can pick always that same specific element y in W that we get from the assumption that the formula on the left holds.

If we would swap the left and the right formula, our assumption would be that for each x in M we have a possibly different y in W such that $L(x, y)$ holds, which is not strong enough to claim that there is also a special y in W that will make $L(x, y)$ hold for all x at the same time.

So $\forall x \in M\,[\exists y \in W\, L(x, y)]$ is indeed a logical consequence of the formula $\exists y \in W\,[\forall x \in M\, L(x, y)]$ and the statement holds, but not in the other direction.

7. Give an interpretation I_7 in a structure M_7 such that:

$$(M_7, I_7) \models \forall x, y, z \in D\,[R(x, z) \wedge R(y, z) \rightarrow \neg R(x, y)]$$

Use an interpretation where the domain D is interpreted as the set of natural numbers $\mathbb{N}$. Explain your answer.

Take for instance the structure M_7:

Domain(s)	Natural numbers $\mathbb{N}$
Relation(s)	differs by one

and the interpretation I_7:

D	$\mathbb{N}$
$R(x, y)$	$x + 1 = y$

In the model (M_7, I_7) the statement means: for all combinations of natural numbers x, y and z, if $x + 1 = z$ and $y + 1 = z$, then $x + 1 \neq y$. This holds as from $x + 1 = z$ and $y + 1 = z$ it follows that $x = y$. And hence $x + 1 = y$ does not hold.

Alternative interpretations (without specifying a formal structure) of $R(x, y)$ that also work:

- x and y are both negative;

- $x + y = -1$;
- $x + y$ is an odd number;
- $x < 1$ and $y > 1$.

Can you figure out why?

8. Consider the following formula of predicate logic:

$$\exists x \in W \, \forall y \in W \, [x = y \to L(k, y)]$$

We use the dictionary:

W	the domain of women
k	Koos
$L(x, y)$	x loves y

What is the meaning of this formula?

(a) *Koos loves exactly one woman.*

(b) is correct (b) *Koos loves at least one woman.*

(c) *Koos loves at most one woman.*

(d) None of the above.

Answer (b) is correct.

The formula states that *there exists a woman x for whom it holds that for all women y, if x is the same woman as y, then Koos loves y.* So first, a special woman x is indicated. Then, all women y are compared to this special woman x. Obviously, as x is a woman, one of these y's will actually be the same as x. So for that y, the part $x = y$ holds, and hence the part $L(k, y)$ also holds. Which means that Koos loves y, and as y is the same as x, we can also state that Koos loves this special woman x. So in particular, Koos loves at least one woman.

As there is an implication ($\to$) in the formula and not an equivalence ($\leftrightarrow$), it is allowed that Koos loves more than one woman, as there are no restrictions given for the case that $x \neq y$.