Formal Reasoning 2024
Solutions Test Block 3: Discrete Mathematics and Modal Logic
(19/12/24)

Discrete Mathematics

1. What is the chromatic number of the Petersen graph?

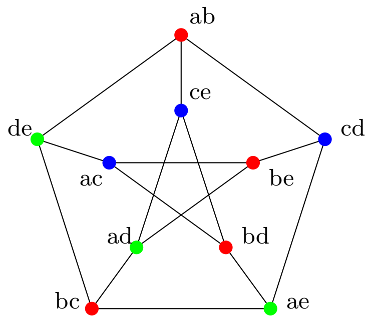
(a) is correct

   (a) 3 or less
   (b) 4
   (c) 5 or more
   (d) The Petersen graph does not have a chromatic number.

   Answer (a) is correct.

   As the Petersen graph clearly has a cycle of length five on the outside, the chromatic number must be at least 3.

   And the picture below shows that it is indeed possible to color the Petersen graph with only three colors.



2. A recursive definition of the *Ackermann function* $A(n,k)$ is:

$$A(0,k) = k+1$$
$$A(n+1,0) = A(n,1)$$
$$A(n+1,k+1) = A(n, A(n+1,k))$$

   In all these equations we have $n, k \geq 0$.

   Compute $A(2,2)$ and explain your answer by giving other relevant values of $A(n,k)$.

   Hint: First compute $A(0,k)$ and $A(1,k)$ for several $k \geq 0$.

   Let us follow the hint:

   - $A(0,0) = 0+1 = 1$
   - $A(0,1) = 1+1 = 2$
   - $A(0,2) = 2+1 = 3$
   - $A(0,3) = 3+1 = 4$

- $A(0,4) = 4 + 1 = 5$
- $A(0,5) = 5 + 1 = 6$
- $A(0,6) = 6 + 1 = 7$
- $A(1,0) = A(0,1) = 2$
- $A(1,1) = A(0, A(1,0)) = A(0,2) = 3$
- $A(1,2) = A(0, A(1,1)) = A(0,3) = 4$
- $A(1,3) = A(0, A(1,2)) = A(0,4) = 5$
- $A(1,4) = A(0, A(1,3)) = A(0,5) = 6$
- $A(1,5) = A(0, A(1,4)) = A(0,6) = 7$

Now with these values, we can compute $A(2,k)$ for $k \in \{0,1,2\}$:

- $A(2,0) = A(1,1) = 3$
- $A(2,1) = A(1, A(2,0)) = A(1,3) = 5$
- $A(2,2) = A(1, A(2,1)) = A(1,5) = 7$

3. A definition of the binomial coefficients consists of the following recursion equations:

$$\binom{0}{0} = 1 \qquad \binom{0}{k+1} = 0$$
$$\binom{n+1}{0} = 1 \qquad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

In all these equations we have $n, k \geq 0$.

From this definition, someone proves a version of the binomial theorem using induction on n. What is the induction hypothesis (IH) in this proof, when the induction step starts with 'Let k be any natural number such that $k \geq 0$'?

(a)
$$(1+x)^n = \binom{n}{0}x^0 + \cdots + \binom{n}{n}x^n$$

(b)
$$(1+x)^{n-1} = \binom{n-1}{0}x^0 + \cdots + \binom{n-1}{n-1}x^{n-1}$$

(c)
$$(1+x)^k = \binom{k}{0}x^0 + \cdots + \binom{k}{k}x^k$$

(c) is correct

(d)
$$(1+x)^{k+1} = \binom{k+1}{0}x^0 + \cdots + \binom{k+1}{k+1}x^{k+1}$$

Answer (c) is correct.

The binomial theorem states that

$$(1+x)^n = \binom{n}{0}x^0 + \cdots + \binom{n}{n}x^n$$

for all $n \geq 0$.

So the induction predicate will be

$$P(n) \quad := \quad \left[(1+x)^n = \binom{n}{0}x^0 + \cdots + \binom{n}{n}x^n\right]$$

And by definition, the Induction Hypothesis states that $P(k)$ holds, so that implies that $P(k)$ holds, which means that

$$(1+x)^k = \binom{k}{0}x^0 + \cdots + \binom{k}{k}x^k$$

4. In a D&D game, the Dungeon Master states:

> *I grab 2 of the 10 arrows without looking and fire them, hoping I didn't grab one of the 5 cursed ones.*

What is the probability that no cursed arrows are fired?

(a) Half of the arrows are cursed, therefore the probability is $\frac{1}{2} \cdot \frac{1}{2}$.

(b) After grabbing a non-cursed arrow, there are only four non-cursed arrows left, therefore the probability is $\frac{5}{10} \cdot \frac{4}{10}$.

(c) There are $\binom{10}{2}$ ways to grab two arrows, but only $\binom{5}{2}$ ways to grab two non-cursed arrows, therefore the probability is $\binom{5}{2}/\binom{10}{2}$.

(c) is correct

(d) None of the above answers gives the correct probability.

Answer (c) is correct.

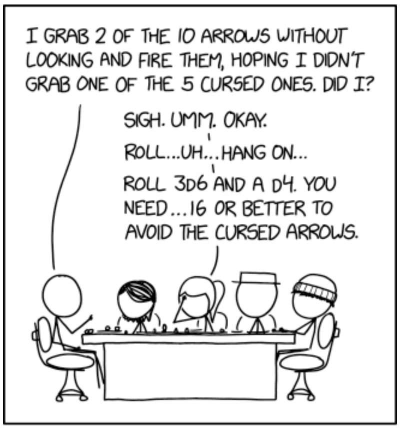
The description given in the answer clearly describes the situation.

We can also prove it as follows. As we have two categories 'cursed' and 'non-cursed' and we know exactly how many we have to pick from each category, namely two non-cursed arrows and zero cursed arrows, this is a case of the *hyper geometrical distribution.* So the formula is

$$\frac{\binom{5}{2} \cdot \binom{5}{0}}{\binom{10}{2}} = \frac{\binom{5}{2} \cdot 1}{\binom{10}{2}} = \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45} = \frac{2}{9}$$

We can also see it like this. For picking the first non-cursed arrow, there are five out of ten options, so that has a probability of $\frac{5}{10}$. Then for picking the second non-cursed arrow, there are four out of nine options, so that has a probability of $\frac{4}{9}$. Multiplying these values gives that the total probability is (again) $\frac{5}{10} \cdot \frac{4}{9} = \frac{20}{90} = \frac{2}{9}$. So the options with $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and $\frac{5}{10} \cdot \frac{4}{10} = \frac{20}{100} = \frac{2}{10} = \frac{1}{5}$ are wrong.

Note that this question is based on this XKCD comic:



I GOT *WAY* MORE ANNOYING TO PLAY D&D WITH ONCE I LEARNED THAT OUR DM HAS A COMBINATORICS DEGREE AND CAN'T RESIST PUZZLES.

The original can be found at `https://xkcd.com/3015`. Can you check whether the answer by the dungeon master makes sense?

Modal Logic

5. Give an example of a formula of doxastic logic that contains both modalities, and give its meaning, using for the dictionary:

In	I am inside
Out	I am outside

There are a lot of correct options. Let us provide a very simple one:

$$\Box \mathsf{In} \to \Diamond \mathsf{In}$$

The meaning of this formula in doxastic logic is: *If I believe that I am inside, then it is not against my belief that I am inside.* Note that the exercise didn't state that all atomic propositions need to be used nor that the formula should hold.

Note also that the given formula is an instance of the axiom D, for seriality.

6. Do the formulas of axiom scheme D, $\Box f \to \Diamond f$, all follow from those of axiom scheme T, $\Box f \to f$?

(a) is correct

(a) Yes. If we have $\Box f \to f$ and $\Box\neg f \to \neg f$, then because $\Box\neg f \to \neg f \equiv \neg\neg f \to \neg\Box\neg f \equiv f \to \Diamond f$, it follows that $\Box f \to \Diamond f$.

(b) Yes. All formulas of axiom scheme D are logically equivalent to a formula of axiom scheme T.

(c) No. But this would be the case if one interchanges T and D.

(d) No. And this is not true with T and D interchanged either.

Answer (a) is correct.

Note that D is about seriality and T about reflexivity. And in the theory of Kripke semantics, we have seen that if a model is reflexive, it is always serial as well, but not the other way round.

Here, we have the same situation. The explanation given in the answer proves one direction, namely that if axiom T holds, then automatically axiom D also holds. The only detail in the explanation that is missing is that if $\Box f \to f$ and $f \to \Diamond f$ hold, then $\Box f \to \Diamond f$ also holds.

However, there is no equivalence. Let f be the statement *God exists.* Then the formula $\Box f \to \Diamond f$ has the following meaning in doxastic logic: *If I believe that God exists, then it is not against my belief that God doesn't exist.* Or in a different formulation: *If I believe that God exists, then as far as I believe God might exist.* Note that these statements indeed hold in doxastic logic. However, if we look at the meaning of the reflexive $\Box f \to f$ in doxastic logic we get that: *If I believe that God exists, then God exists.* And this is typically not accepted as a valid statement.

7. Give the minimum number of worlds in a Kripke model $\mathcal{M}$ such that:

$$\mathcal{M} \models \Diamond a \wedge \neg\Box a$$

(a) 1

(b) is correct (b) 2

(c) 3 or more

(d) There does not exist a Kripke model with this property.

Answer (b) is correct.

The statement

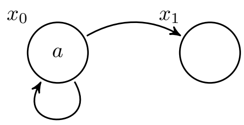
$$\mathcal{M} \models \Diamond a \wedge \neg\Box a$$

means that in all worlds x_i in the model $\mathcal{M}$ the statement

$$x_i \Vdash \Diamond a \wedge \neg\Box a$$

should hold.

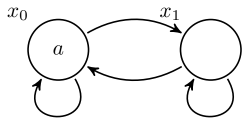
By definition, a model should have at least one world, say x_0. Now to make $x_0 \Vdash \Diamond a$ hold, it is necessary that there is a world that is accessible from x_0 in which a holds. This can be accomplished by putting x_0 in $R(x_0)$ and $a \in V(x_0)$, or in other words, by drawing a reflexive arrow from x_0 to x_0 and state that a holds in x_0. However, if we also want to have that $x_0 \Vdash \neg\Box a$ it means that there has to be a world that is accessible from x_0 for which a does not hold. This follows directly from the fact that $\neg\Box a \equiv \Diamond\neg a$. However, as x_0 is the only world accessible from x_0 and $x_0 \Vdash a$, it is clear that we need a second world x_1 such that $x_1 \in R(x_0)$ and $a \notin V(x_1)$. Or in other words, we need a second world x_1, which is accessible from x_0 and where a does not hold. Now both $x_0 \Vdash \Diamond a$ and $x_0 \Vdash \neg\Box a$ hold, so also $x_0 \Vdash \Diamond a \wedge \neg\Box a$ holds.



However, in this model, $x_1 \not\Vdash \Diamond a \wedge \neg\Box a$ so certainly

$$\mathcal{M} \not\models \Diamond a \wedge \neg\Box a$$

Fortunately, we can easily fix this by making both x_0 and x_1 accessible from x_1 and keeping the valuation the same.



So if $\mathcal{M} = \langle\{x_0, x_1\}, R, V\rangle$, where $R(x_0) = R(x_1) = \{x_0, x_1\}$ and $V(x_0) = \{a\}$ and $V(x_1) = \varnothing$ then

$$\mathcal{M} \models \Diamond a \wedge \neg\Box a$$

holds.

Hence in particular, it is possible to make the statement true with a model that has only two worlds, but it is not possible with a model having only one world.

8. A *Kripke frame* $\langle W, R\rangle$ consists of the first two components of a Kripke model $\langle W, R, V\rangle$. What can be the Kripke frames of an LTL Kripke model?

   (a) Any frame $\langle W, R\rangle$ with R a function such that $R(x) \subseteq W$ for all $x \in W$.

   (b) The frame $\langle W, R\rangle$ with

$$W = \{x_i \mid i \geq 0\}$$
$$R(x_i) = \{x_j \mid j \geq i\} \qquad \text{for } i \geq 0$$

(b) is correct

   (c) The frame $\langle W, R\rangle$ with

$$W = \{x_i \mid i \geq 0\}$$
$$R(x_i) = \{x_{i+1}\} \qquad \text{for } i \geq 0$$

   (d) None of the above answers is correct.

   Answer (b) is correct.

   By definition, LTL Kripke models only differ in terms of the valuation. The worlds of an LTL Kripke model are the set $W = \{x_i \mid i \in \mathbb{N}\})$ and this set is equal to the proposed set $W = \{x_i \mid i \geq 0\}$. And the worlds accessible to a given world x_i are exactly all subsequent worlds x_j, for which $i \leq j$. Now this is the same set at the proposed set $R(x_i) = \{x_j \mid j \geq i\}$ for $i \geq 0$.

   Note that *Any* $\langle W, R\rangle$, *with* R *a function such that* $R(x) \subseteq W$ *for all* $x \in W$ can't be right as this would allow for a frame with only one world x_0 and where $R(x_0) = \varnothing$, but that does not lead to an LTL Kripke model.

   Note also that the option where $R(x_i) = \{x_{i+1}\}$ would not lead to an LTL Kripke model as the transitive closure is missing, i.e. if $R(x_0) = \{x_1\}$ and $R(x_1) = \{x_2\}$ (which is the case in the proposed solution) then it should be that $x_2 \in R(x_0)$ as well, but it isn't.