Formal Reasoning 2024
Solutions Test Blocks 1, 2 and 3: Additional Test
(09/01/25)

1. Consider the English sentence:

   If it doesn't rain, then I don't get wet.

   Which of the following formulas of propositional logic corresponds to the meaning of this sentence? We use the dictionary:

R	it rains
W	I get wet

   Hint: Use logical laws, or use a truth table to check the correctness of your answer.

   (a)
   $$\neg(R \wedge W)$$

   (b)
   $$\neg(R \wedge \neg W)$$

   (c)
   $$\neg(\neg R \wedge W)$$

(c) is correct

   (d)
   $$\neg(\neg R \wedge \neg W)$$

   Answer (c) is correct.

   The default translation using an implication would be

   $$\neg R \to \neg W$$

   Using logical laws we can rewrite it to one of the options:

   $$\begin{aligned} \neg R \to \neg W \quad &\equiv \quad \neg\neg R \vee \neg W \quad && \text{applying } a \to b \equiv \neg a \vee b \\ &\equiv \quad \neg(\neg R \wedge W) && \text{applying De Morgan} \end{aligned}$$

   Hopefully, it is clear that the different options are not logically equivalent. If not, then have a look at the truth table for all of these formulas:

R	W	$\neg R$	$\neg W$	$\neg R \to \neg W$	$\neg(R \wedge W)$	$\neg(R \wedge \neg W)$	$\neg(\neg R \wedge W)$	$\neg(\neg R \wedge \neg W)$
0	0	1	1	1	1	1	1	0
0	1	1	0	0	1	1	0	1
1	0	0	1	1	1	0	1	1
1	1	0	0	1	0	1	1	1

   Note that for lack of space not all intermediate columns are included. only the columns of $\neg R \to \neg W$ and $\neg(\neg R \wedge W)$ are equal, so only these two are equivalent.

2. Consider the interpretation I_2, that maps N to the set of natural numbers $\mathbb{N}$, and $L(x, y)$ to $x \leq y$. Which of the following statements does *not* hold?

(a)

$$((\mathbb{N}, \leq), I_2) \vDash \forall x \in N \, \exists y \in N \, L(x, y)$$

(b)

$$((\mathbb{N}, \leq), I_2) \vDash \forall x \in N \, \exists y \in N \, L(y, x)$$

(c)

$$((\mathbb{N}, \leq), I_2) \vDash \exists x \in N \, \forall y \in N \, L(x, y)$$

(d)

$$((\mathbb{N}, \leq), I_2) \vDash \exists x \in N \, \forall y \in N \, L(y, x)$$

(d) is correct

Answer (d) is correct.

Let us have a look at the meaning of the four statements:

- $((\mathbb{N}, \leq), I_2) \vDash \forall x \in N \, \exists y \in N \, L(x, y)$: For each natural number x there is a natural number y such that $x \leq y$. This is true, as we can take $y = x$ because $x \leq x$, once x is chosen.
- $((\mathbb{N}, \leq), I_2) \vDash \forall x \in N \, \exists y \in N \, L(y, x)$: For each natural number x there is a natural number y such that $y \leq x$. This is true, as we can take $y = x$ because $x \leq x$, once x is chosen.
- $((\mathbb{N}, \leq), I_2) \vDash \exists x \in N \, \forall y \in N \, L(x, y)$: There exists a natural number x such that for all natural numbers y it holds that $x \leq y$. This is true as we can take $x = 0$ and $0 \leq y$ for all natural numbers y.
- $((\mathbb{N}, \leq), I_2) \vDash \exists x \in N \, \forall y \in N \, L(y, x)$: There exists a natural number x such that for all natural numbers y it holds that $y \leq x$. This is not true as it would imply that the natural numbers are bounded. However, if x is chosen, then we can always take $y = x + 1$ and then $x + 1 \leq x$ doesn't hold.

3. Consider the language

$$L_3 := \overline{\mathcal{L}\big(a^*b^*\big)} \cap \mathcal{L}\big((ab)^*\big)$$

with alphabet $\Sigma = \{a, b\}$. Give one word that is in L_3, give one word that is in $\overline{\mathcal{L}\big(a^*b^*\big)}$ but not in $\mathcal{L}\big((ab)^*\big)$, and give one word that is in $\mathcal{L}\big((ab)^*\big)$ but not in $\overline{\mathcal{L}\big(a^*b^*\big)}$.

Make sure that the length of each of these three words is at least two and at most four.

As L_3 is the intersection of the languages $\overline{\mathcal{L}\big(a^*b^*\big)}$ and $\mathcal{L}\big((ab)^*\big)$ it means that words in L_3 should be in both of these languages. And words that are not in L_3 should not be in at least one of these languages.

- The language $\mathcal{L}\big(a^*b^*\big)$ consists of words that start with some number of a's (possibly zero) by some number of b's (possibly zero).
- The language $\overline{\mathcal{L}\big(a^*b^*\big)}$ consists of words that contain the substring ba.
- The language $\mathcal{L}\big((ab)^*\big)$ consists of words that have zero or more consecutive blocks of ab (and nothing else).

Now, as $abab$ contains ba and it consists of two consecutive blocks ab, it is in both parts of the intersection and hence it is in L_3.

And ba obviously contains ba, so it is in $\overline{\mathcal{L}(a^*b^*)}$, but it does not consist of zero or more consecutive blocks of ab, so it is not in $\mathcal{L}((ab)^*)$.

And ab consists of one consecutive block of ab, so it is in $\mathcal{L}((ab)^*)$, but it doesn't contain ba so it is not in $\overline{\mathcal{L}(a^*b^*)}$.

Note that all these three words have a length that is at least two and at most four.

4. Consider the following context-free grammar G_4:

$$S \to aSb \mid \lambda$$

The language $L_4 := \mathcal{L}(G_4)$ is context-free but not regular.

Which of the following properties is *not* an invariant that can be used to show that $abba \notin L_4$?

(a)
$$P_4(w) := \big[w = S \text{ or } w = \lambda \text{ or } w \text{ ends with } b\big]$$

(b)
$$P_4(w) := \big[w \text{ is of the form } a^n u b^n \text{ with } n \in \mathbb{N} \text{ and } u \in \{S, \lambda\}\big]$$

(c)
$$P_4(w) := \big[w \text{ does not contain } ba\big]$$

(c) is correct

(d)
$$P_4(w) := \big[w \in \mathcal{L}\big(a^*(S \cup \lambda)b^*\big)\big]$$

Answer (c) is correct.

It is clear that
$$P_4(w) := \big[w \text{ does not contain } ba\big]$$
is not an invariant, as $P(Sa)$ holds, and $Sa \to aSba$, but $P(aSba)$ clearly does not hold.

The other three are indeed invariants.

- $P_4(w) := \big[w = S \text{ or } w = \lambda \text{ or } w \text{ ends with } b\big]$:
  (a) $P_4(S)$ clearly holds.
  (b) If v and v' are such that $P(v)$ holds and $v \to v'$, then $v = S$ or $v = ub$ for some u.
    - If $v = S$ then $v' = aSb$ or $v' = \lambda$ and in both situations $P(v')$ clearly holds.
    - If $v = ub$ then $v' = u'b$ for some u' and hence $P(v')$ clearly holds.
- $P_4(w) := \big[w \text{ is of the form } a^n u b^n \text{ with } n \in \mathbb{N} \text{ and } u \in \{S, \lambda\}\big]$:
  (a) $P_4(S)$ clearly holds as $S = a^0 S b^0$ and $0 \in \mathbb{N}$.
  (b) If v and v' are such that $P(v)$ holds and $v \to v'$, then $v = a^n S b^n$ for some $n \in \mathbb{N}$. And hence $v' = a^{n+1} S b^{n+1}$ or $v' = a^n b^n$ and in both situations $P(v')$ clearly holds.

- $P_4(w) := \big[w \in \mathcal{L}\big(a^*(S \cup \lambda)b^*\big)\big]$:
  (a) $P_4(S)$ clearly holds as $S \in \mathcal{L}(a^*(S \cup \lambda)b^*)$.
  (b) If v and v' are such that $P(v)$ holds and $v \to v'$, then $v = a^n S b^m$ for some $n, m \in \mathbb{N}$. And hence $v' = a^{n+1} S b^{m+1}$ or $v' = a^n b^m$ and in both situations $P(v')$ clearly holds.

5. Consider all deterministic finite automata with alphabet $\Sigma = \{a\}$ and exactly two states. How many different languages do these automata accept?

   (a) 2 or less

   (b) 4 or less, but more than 2

   (c) is correct (c) 8 or less, but more than 4

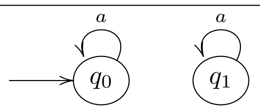
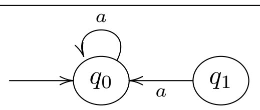
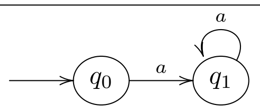
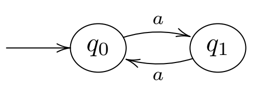
   (d) more than 8

   Answer (c) is correct.

   There are three orthogonal choices:

   - $\delta(q_0, a) = q_0$ or $\delta(q_0, a) = q_1$
   - $\delta(q_1, a) = q_1$ or $\delta(q_1, a) = q_0$
   - the finite states are $\varnothing$, $\{q_0\}$, $\{q_1\}$, or $\{q_0, q_1\}$

   So there are $2 \cdot 2 \cdot 4 = 16$ different automata. However, not all these automata actually accept different languages. The table below provides the four possible structures without indicating the final states and relates the choice for the final states to the accepted language.

<table>
<tr><th>Finite states</th><th>$\varnothing$</th><th>$\{q_0\}$</th><th>$\{q_1\}$</th><th>$\{q_0, q_1\}$</th></tr>
<tr><td></td><td>$\varnothing$</td><td>$\mathcal{L}(a^*)$</td><td>$\varnothing$</td><td>$\mathcal{L}(a^*)$</td></tr>
<tr><td></td><td>$\varnothing$</td><td>$\mathcal{L}(a^*)$</td><td>$\varnothing$</td><td>$\mathcal{L}(a^*)$</td></tr>
<tr><td></td><td>$\varnothing$</td><td>$\{\lambda\}$</td><td>$\mathcal{L}(aa^*)$</td><td>$\mathcal{L}(a^*)$</td></tr>
<tr><td></td><td>$\varnothing$</td><td>$\mathcal{L}\big((aa)^*\big)$</td><td>$\mathcal{L}\big(a(aa)^*\big)$</td><td>$\mathcal{L}(a^*)$</td></tr>
</table>

   So these are the six different languages that are accepted: $\varnothing$, $\{\lambda\}$, $\mathcal{L}\big((aa)^*\big)$, $\mathcal{L}\big(a(aa)^*\big)$, $\mathcal{L}\big(aa^*\big)$, and $\mathcal{L}\big(a^*\big)$.

6. What is the number of Hamiltonian cycles in the graph K_n with $n \geq 3$?

   (a) $\frac{1}{2}n(n-1)$

(b) $(n-1)!$

(c) is correct (c) $n!$

(d) None of the above.

Answer (c) is correct.

Note that $K_n = \langle\{1, 2, \ldots, n\}, \{\{i, j\} \mid 1 < i < j < n\}\rangle$. A Hamiltonian cycle in K_n is a path $x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$ where there exists a bijective mapping between the sets $\{1, 2, \ldots, n\}$ and $\{x_1, x_2, \ldots, x_n\}$. We have n options for the starting vertex x_1 of the Hamiltonian cycle. For the next vertex x_2 we have $n-1$ options left, because x_1 is connected by an edge to all other vertices as K_n is the complete graph with n vertices. Likewise, for the next vertex x_3 we have $n-2$ options left. And so on, till we get that for vertex x_n we have only 1 option left. So the number of different Hamiltonian cycles in K_n equals $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1 = n!$.

In fact, it is not difficult to understand that the number of bijective functions between $\{1, 2, \ldots, n\}$ and $\{x_1, x_2, \ldots, x_n\}$ is equal to the number of permutations (the number of ways we can order the elements of $\{1, 2, \ldots, n\}$) which is also known to be $n!$.

And note that if $n \leq 2$ the graph K_n has no cycles and hence zero Hamiltonian cycles, which doesn't match with the formula $n!$, so the $n \geq 3$ is essential.

7. We define the binomial coefficients recursively by

$$\text{(i)} \quad \binom{0}{0} = 1 \qquad \text{(iii)} \quad \binom{0}{k+1} = 0$$

$$\text{(ii)} \quad \binom{n+1}{0} = 1 \qquad \text{(iv)} \quad \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

where in all these equations n and k range over the natural numbers. Note that the definitions are labeled with Roman numerals for easy reference.

We want to prove from this that

$$\binom{n}{1} = n$$

for all natural numbers n using induction on n.

Below we give a partial proof where we included all the administrative steps but left open the two steps where you (presumably) have to do some thinking.

Provide these two missing steps. Make sure to separate the two steps clearly in your answer. And if you use one of the four definitions provided above, make sure that you clearly reference it, preferably by its number.

0 **Proposition:**

$\binom{n}{1} = n$ for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \ := \ \left[\binom{n}{1} = n\right]$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that

$\binom{0}{1} = 0$

4 This indeed holds, because ...

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that

$\binom{k}{1} = k$ (Induction Hypothesis IH)

7 We now show that $P(k+1)$ also holds, i.e. we show that

$\binom{k+1}{1} = k+1$

8 This indeed holds, because ...

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

This is the proof of the base case:

4 This indeed holds, because

$$\begin{aligned} \binom{0}{1} &= \binom{0}{0+1} && \text{elementary algebra} \\ &= 0 && \text{by definition (iii) of } \tbinom{0}{k+1} \text{ where } k = 0 \end{aligned}$$

And this is the proof of the inductive step:

8 This indeed holds, because

$$\begin{aligned} \binom{k+1}{1} &= \binom{k+1}{0+1} && \text{elementary algebra} \\ &= \binom{k}{0} + \binom{k}{1} && \text{recursive definition (iv) of } \tbinom{n+1}{k+1} \text{ where } n = k \text{ and } k = 0 \\ &= \binom{k}{0} + k && \text{applying IH} \end{aligned}$$

So if we can prove that $\binom{k}{0} = 1$ for all $k \geq 0$, then it indeed follows that $\binom{k+1}{1} = k+1$.

Now let us prove that indeed $\binom{k}{0} = 1$ for all $k \geq 0$. As $k \geq 0$ it follows that there are two options:

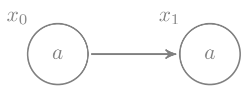
- $k = 0$ and then we get $\binom{k}{0} = \binom{0}{0} = 1$ by definition (i) of $\binom{0}{0}$, or
- $k > 0$, so there exists $k' \geq 0$ such that $k = k' + 1$. Now it follows that $\binom{k}{0} = \binom{k'+1}{0} = 1$ by definition (ii) of $\binom{n+1}{0}$ where $n = k'$.

So in both cases, we get that $\binom{k}{0} = 1$. Hence the conclusion is that

$$\binom{k+1}{1} = \binom{k+1}{0+1} = \binom{k}{0} + \binom{k}{1} = \binom{k}{0} + k = 1 + k = k + 1$$

which is what we had to prove.

8. Consider the following Kripke model $\mathcal{M}_8$:



Does the following hold?

$$x_0 \Vdash \Box\Diamond a$$

(a) Yes, x_1 has no successors, which means that in that world any formula of the form $\Box f$ holds.

(b) Yes, because a holds in all worlds of the model.

(c) No, because $x_1 \nVdash \Box a$.

(d) is correct (d) No, because $x_1 \nVdash \Diamond a$.

Answer (d) is correct.

The statement $x_0 \Vdash \Box\Diamond a$ holds if in all accessible worlds from x_0 the formula $\Diamond a$ holds. As the only accessible world from x_0 is x_1, this means that $x_1 \Vdash \Diamond a$ should hold. In turn, this means that x_1 has an accessible world where a holds. However, as there are no accessible worlds from x_1, there certainly isn't one where a holds. So the fact that $x_1 \nVdash \Diamond a$ causes that $x_0 \nVdash \Box\Diamond a$.

Note that $x_1 \Vdash \Box a$ vacuously holds as x_1 has no accessible worlds.

Of course, we can also create a satisfiability table:

$\Vdash$	a	$\Diamond a$	$\Box\Diamond a$	$\Box a$
x_0	1	1	0	1
x_1	1	0	1	1

The last column is only added to explicitly show that $x_1 \Vdash \Box a$ and hence invalidate the other 'No' option. It is not relevant for determining whether $x_0 \Vdash \Box\Diamond a$ holds or not.