Formal Reasoning 2024
Test Block 3: Discrete Mathematics and Modal Logic
(19/12/24)

There are six multiple choice questions and two open questions (questions 2 and 5). Each multiple choice question is worth 10 points, and the open questions are worth 15 points each. The mark for this test is the number of points divided by ten, and the first ten points are free. Good luck!

Discrete Mathematics

1. What is the chromatic number of the Petersen graph?

   (a) 3 or less

   (b) 4

   (c) 5 or more

   (d) The Petersen graph does not have a chromatic number.

2. A recursive definition of the *Ackermann function* $A(n, k)$ is:

$$\begin{aligned} A(0, k) &= k + 1 \\ A(n + 1, 0) &= A(n, 1) \\ A(n + 1, k + 1) &= A(n, A(n + 1, k)) \end{aligned}$$

   In all these equations we have $n, k \geq 0$.

   Compute $A(2, 2)$ and explain your answer by giving other relevant values of $A(n, k)$.

   Hint: First compute $A(0, k)$ and $A(1, k)$ for several $k \geq 0$.

3. A definition of the binomial coefficients consists of the following recursion equations:

$$\begin{aligned} \binom{0}{0} &= 1 & \binom{0}{k+1} &= 0 \\ \binom{n+1}{0} &= 1 & \binom{n+1}{k+1} &= \binom{n}{k} + \binom{n}{k+1} \end{aligned}$$

   In all these equations we have $n, k \geq 0$.

   From this definition, someone proves a version of the binomial theorem using induction on n. What is the induction hypothesis (IH) in this proof, when the induction step starts with 'Let k be any natural number such that $k \geq 0$'?

   (a)
$$(1 + x)^n = \binom{n}{0} x^0 + \cdots + \binom{n}{n} x^n$$

   (b)
$$(1 + x)^{n-1} = \binom{n-1}{0} x^0 + \cdots + \binom{n-1}{n-1} x^{n-1}$$

(c)
$$(1+x)^k = \binom{k}{0}x^0 + \cdots + \binom{k}{k}x^k$$

(d)
$$(1+x)^{k+1} = \binom{k+1}{0}x^0 + \cdots + \binom{k+1}{k+1}x^{k+1}$$

4. In a D&D game, the Dungeon Master states:

   > *I grab 2 of the 10 arrows without looking and fire them, hoping I didn't grab one of the 5 cursed ones.*

   What is the probability that no cursed arrows are fired?

   (a) Half of the arrows are cursed, therefore the probability is $\frac{1}{2} \cdot \frac{1}{2}$.

   (b) After grabbing a non-cursed arrow, there are only four non-cursed arrows left, therefore the probability is $\frac{5}{10} \cdot \frac{4}{10}$.

   (c) There are $\binom{10}{2}$ ways to grab two arrows, but only $\binom{5}{2}$ ways to grab two non-cursed arrows, therefore the probability is $\binom{5}{2}/\binom{10}{2}$.

   (d) None of the above answers gives the correct probability.

Modal Logic

5. Give an example of a formula of doxastic logic that contains both modalities, and give its meaning, using for the dictionary:

In	I am inside
Out	I am outside

6. Do the formulas of axiom scheme D, $\Box f \to \Diamond f$, all follow from those of axiom scheme T, $\Box f \to f$?

   (a) Yes. If we have $\Box f \to f$ and $\Box\neg f \to \neg f$, then because $\Box\neg f \to \neg f \equiv \neg\neg f \to \neg\Box\neg f \equiv f \to \Diamond f$, it follows that $\Box f \to \Diamond f$.

   (b) Yes. All formulas of axiom scheme D are logically equivalent to a formula of axiom scheme T.

   (c) No. But this would be the case if one interchanges T and D.

   (d) No. And this is not true with T and D interchanged either.

7. Give the minimum number of worlds in a Kripke model $\mathcal{M}$ such that:
$$\mathcal{M} \vDash \Diamond a \wedge \neg\Box a$$

   (a) 1

   (b) 2

   (c) 3 or more

   (d) There does not exist a Kripke model with this property.

8. A *Kripke frame* $\langle W, R\rangle$ consists of the first two components of a Kripke model $\langle W, R, V\rangle$. What can be the Kripke frames of an LTL Kripke model?

(a) Any frame $\langle W, R\rangle$ with R a function such that $R(x) \subseteq W$ for all $x \in W$.

(b) The frame $\langle W, R\rangle$ with

$$\begin{aligned} W &= \{x_i \mid i \geq 0\} \\ R(x_i) &= \{x_j \mid j \geq i\} \qquad \text{for } i \geq 0 \end{aligned}$$

(c) The frame $\langle W, R\rangle$ with

$$\begin{aligned} W &= \{x_i \mid i \geq 0\} \\ R(x_i) &= \{x_{i+1}\} \qquad \text{for } i \geq 0 \end{aligned}$$

(d) None of the above answers is correct.