Formal Reasoning 2024
Test Blocks 1, 2 and 3: Additional Test
(09/01/25)

There are six multiple choice questions and two open questions (questions 3 and 7). Each multiple choice question is worth 10 points, and the open questions are worth 15 points each. The mark for this test is the number of points divided by ten, and the first ten points are free. Good luck!

1. Consider the English sentence:

   If it doesn't rain, then I don't get wet.

   Which of the following formulas of propositional logic corresponds to the meaning of this sentence? We use the dictionary:

R	it rains
W	I get wet

   Hint: Use logical laws, or use a truth table to check the correctness of your answer.

   (a)
   $$\neg(R \wedge W)$$
   (b)
   $$\neg(R \wedge \neg W)$$
   (c)
   $$\neg(\neg R \wedge W)$$
   (d)
   $$\neg(\neg R \wedge \neg W)$$

2. Consider the interpretation I_2, that maps N to the set of natural numbers $\mathbb{N}$, and $L(x, y)$ to $x \leq y$. Which of the following statements does *not* hold?

   (a)
   $$((\mathbb{N}, \leq), I_2) \vDash \forall x \in N\, \exists y \in N\, L(x, y)$$
   (b)
   $$((\mathbb{N}, \leq), I_2) \vDash \forall x \in N\, \exists y \in N\, L(y, x)$$
   (c)
   $$((\mathbb{N}, \leq), I_2) \vDash \exists x \in N\, \forall y \in N\, L(x, y)$$
   (d)
   $$((\mathbb{N}, \leq), I_2) \vDash \exists x \in N\, \forall y \in N\, L(y, x)$$

3. Consider the language

   $$L_3 := \overline{\mathcal{L}\big(a^*b^*\big)} \cap \mathcal{L}\big((ab)^*\big)$$

   with alphabet $\Sigma = \{a, b\}$. Give one word that is in L_3, give one word that is in $\overline{\mathcal{L}\big(a^*b^*\big)}$ but not in $\mathcal{L}\big((ab)^*\big)$, and give one word that is in $\mathcal{L}\big((ab)^*\big)$ but not in $\overline{\mathcal{L}\big(a^*b^*\big)}$.

   Make sure that the length of each of these three words is at least two and at most four.

4. Consider the following context-free grammar G_4:

$$S \to aSb \mid \lambda$$

The language $L_4 := \mathcal{L}(G_4)$ is context-free but not regular.

Which of the following properties is *not* an invariant that can be used to show that $abba \notin L_4$?

(a)
$$P_4(w) := \big[w = S \text{ or } w = \lambda \text{ or } w \text{ ends with } b\big]$$

(b)
$$P_4(w) := \big[w \text{ is of the form } a^n u b^n \text{ with } n \in \mathbb{N} \text{ and } u \in \{S, \lambda\}\big]$$

(c)
$$P_4(w) := \big[w \text{ does not contain } ba\big]$$

(d)
$$P_4(w) := \big[w \in \mathcal{L}\big(a^*(S \cup \lambda)b^*\big)\big]$$

5. Consider all deterministic finite automata with alphabet $\Sigma = \{a\}$ and exactly two states. How many different languages do these automata accept?

(a) 2 or less
(b) 4 or less, but more than 2
(c) 8 or less, but more than 4
(d) more than 8

6. What is the number of Hamiltonian cycles in the graph K_n with $n \geq 3$?

(a) $\frac{1}{2}n(n-1)$
(b) $(n-1)!$
(c) $n!$
(d) None of the above.

7. We define the binomial coefficients recursively by

$$\begin{array}{llcllcl} \text{(i)} & \binom{0}{0} & = & 1 & \text{(iii)} & \binom{0}{k+1} & = & 0 \\ \text{(ii)} & \binom{n+1}{0} & = & 1 & \text{(iv)} & \binom{n+1}{k+1} & = & \binom{n}{k} + \binom{n}{k+1} \end{array}$$

where in all these equations n and k range over the natural numbers. Note that the definitions are labeled with Roman numerals for easy reference.

We want to prove from this that

$$\binom{n}{1} = n$$

for all natural numbers n using induction on n.

Below we give a partial proof where we included all the administrative steps but left open the two steps where you (presumably) have to do some thinking.

Provide these two missing steps. Make sure to separate the two steps clearly in your answer. And if you use one of the four definitions provided above, make sure that you clearly reference it, preferably by its number.

0 **Proposition:**

$\binom{n}{1} = n$ for all $n \geq 0$.

1 **Proof** by induction on n.

We first define our predicate P as:

2 $$P(n) \;:=\; \left[\binom{n}{1} = n\right]$$

3 **Base Case.** We show that $P(0)$ holds, i.e. we show that

$$\binom{0}{1} = 0$$

4 This indeed holds, because ...

5 **Induction Step.** Let k be any natural number such that $k \geq 0$.

6 Assume that we already know that $P(k)$ holds, i.e. we assume that

$\binom{k}{1} = k$ (Induction Hypothesis IH)

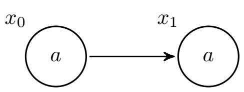
7 We now show that $P(k+1)$ also holds, i.e. we show that

$$\binom{k+1}{1} = k+1$$

8 This indeed holds, because ...

9 Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

8. Consider the following Kripke model $\mathcal{M}_8$:



Does the following hold?

$$x_0 \Vdash \Box\Diamond a$$

(a) Yes, x_1 has no successors, which means that in that world any formula of the form $\Box f$ holds.

(b) Yes, because a holds in all worlds of the model.

(c) No, because $x_1 \nVdash \Box a$.

(d) No, because $x_1 \nVdash \Diamond a$.