# Formal Reasoning 2025
# Exam
**(12/01/26)**

There are six sections, with together 16 multiple choice questions and 6 open questions. Each multiple choice question is worth 3 points, and each open question is worth 7 points. The mark for this test is the number of points divided by ten, The first 10 points are free. Good luck!

## Propositional logic

1. Give a formula of propositional logic that expresses that the weekend is Saturday and Sunday. Use for the dictionary:

| | |
|---|---|
| WE | it is weekend |
| Sat | it is Saturday |
| Sun | it is Saturday |

   (a) $\mathsf{WE} \leftrightarrow \mathsf{Sat} \wedge \mathsf{Sun}$
   (b) $\mathsf{WE} \leftrightarrow \mathsf{Sat} \vee \mathsf{Sun}$
   (c) $\mathsf{WE} \equiv \mathsf{Sat} \wedge \mathsf{Sun}$
   (d) $\mathsf{WE} \equiv \mathsf{Sat} \vee \mathsf{Sun}$

2. Give the number of zeroes and ones for each of the columns of the truth table of the propositional formula:

$$(\neg a \vee b) \wedge (a \vee \neg b)$$

   For each column, both give the formula at the top of that column, as well as the number of zeroes and ones in it. Make sure you do not forget the columns with the atoms at the start of the table.

   As Ans doesn't like 'verbatim' text, the template for the table is given as a Python code block, but don't worry about the syntax highlighting, just type your answers in ASCII.

   Add rows or remove rows as needed. Please keep the bars to separate the columns, as that way we can still parse the table if its alignment is not perfect.

3. Consider the following two formulas of propositional logic:

$$f_1 := (a \wedge b) \vee (\neg a \wedge c)$$
$$f_2 := (a \to b) \wedge (\neg a \to c)$$

   What is the case?

   (a) $f_1 \equiv f_2$, which amounts to $f_1 \models f_2$ and $f_2 \models f_1$
   (b) $f_1 \models f_2$, but $f_2 \not\models f_1$
   (c) $f_1 \not\models f_2$, but $f_2 \models f_1$
   (d) $f_1 \not\models f_2$ and $f_2 \not\models f_1$

## Predicate logic

4. Formalize the following sentence as a formula of predicate logic:

   *January is in the winter.*

   Use for the dictionary:

   | | |
   |---|---|
   | $D$ | the domain of days |
   | $J(x)$ | $x$ is in January |
   | $W(x)$ | $x$ is in the winter |

5. Consider the formula of predicate logic:

   $$f := \forall x, y, z \in N\, [R(x, y) \to R(y, z) \to R(x, z)]$$

   What does *not* hold?

   (a) $((\mathbb{N}, <), I) \vDash f$, with I mapping $N$ to $\mathbb{N}$ and $R(x, y)$ to $x < y$
   (b) $((\mathbb{N}, \leq), I) \vDash f$, with I mapping $N$ to $\mathbb{N}$ and $R(x, y)$ to $x \leq y$
   (c) $((\mathbb{N}, =), I) \vDash f$, with I mapping $N$ to $\mathbb{N}$ and $R(x, y)$ to $x = y$
   (d) $((\mathbb{N}, \neq), I) \vDash f$, with I mapping $N$ to $\mathbb{N}$ and $R(x, y)$ to $x \neq y$

6. We want to formalize the following *true* sentence as a formula of predicate logic with equality:

   *Only zero is divisible by all natural numbers.*

   We use for the dictionary:

   | | |
   |---|---|
   | $N$ | the domain of natural numbers |
   | $\mathsf{zero}$ | the number zero |
   | $\mathsf{divides}(x, y)$ | $y$ is a multiple of $x$ |

   Which of the following formulas is a correct formalization of this?

   (a) $\forall x \in N\, [\, \mathsf{divides}(x, \mathsf{zero})\, ]$
   (b) $\forall x \in N\, [\, \mathsf{divides}(\mathsf{zero}, x)\, ]$
   (c) $\forall x \in N\, [\forall y \in N\, \mathsf{divides}(x, y) \leftrightarrow x = \mathsf{zero}\, ]$
   (d) $\forall x \in N\, [\forall y \in N\, \mathsf{divides}(y, x) \leftrightarrow x = \mathsf{zero}\, ]$

## Discrete mathematics

7. Does there exist a graph that has a Hamiltonian circuit, but has no Eulerian path?

   (a) Yes, there is a graph with three vertices that has this property.
   (b) Yes, there is a graph with four vertices that has this property.
   (c) Yes, but any example has to have at least five vertices.
   (d) No, such a graph does not exist.

8. Consider the graph $G_8 := \langle V_8, E_8 \rangle$, where

$$\begin{array}{lcl} V_8 & := & \{\langle x, y, z\rangle \mid x, y, z \in \{0, 1\}\} \\ E_8 & := & \{(\langle x, y, z\rangle, \langle x', y', z'\rangle) \mid |x - x'| + |y - y'| + |z - z'| = 1\} \end{array}$$

So $\langle 0, 0, 0\rangle$ and $\langle 0, 1, 0\rangle$ are connected by an edge as $|0 - 0| + |0 - 1| + |0 - 0| = 0 + 1 + 0 = 1$, but $\langle 0, 0, 0\rangle$ and $\langle 1, 0, 1\rangle$ are not connected as $|0 - 1| + |0 - 0| + |0 - 1| = 1 + 0 + 1 = 2$.

This graph has eight vertices and twelve edges. What is the case?

(a) $G_8$ is planar and bipartite
(b) $G_8$ is planar, but not bipartite
(c) $G_8$ is bipartite, but not planar
(d) $G_8$ is neither planar, nor bipartite

9. The *Golomb sequence* can be recursively defined as:

$$\begin{array}{rcl} a(1) & = & 1 \\ a(n + 1) & = & 1 + a(n + 1 - a(a(n))) \quad \text{for } n \geq 1 \end{array}$$

What is the value of $a(6)$?

(a) 3
(b) 4
(c) 5
(d) none of the above

10. Prove using induction that $n! \geq 2^n$ for all $n \geq 4$.

Follow the template as closely as possible.

11. The *Catalan numbers* can be defined by:

$$C_n := \frac{1}{n+1}\binom{2n}{n}$$

for $n \geq 0$.

It turns out that these are integers, despite the division by $n + 1$. What is another formula for these numbers?

(a) $C_n = n!$
(b) $C_n = \dfrac{(2n)!}{n!(n+1)!}$
(c) $C_n = n^2 - 2n + 2$
(d) none of the above

## Languages

12. We consider the language $L_{12}$ that is defined with alphabet $\Sigma = \{a, b\}$ by:

$$L_{12} := \left(\overline{\{a\}}\right)^*$$

In this, we of course have:

$$\overline{\{a\}} = \{w \in \{a, b\}^* \mid w \neq a\}$$

What is the case?

(a) $L_{12} = \varnothing$
(b) $L_{12} = \{a\}$
(c) $L_{12} = \overline{\{a\}}$
(d) $L_{12} = \Sigma^*$

13. What is a regular expression for the language:

$$\overline{\{a\}} = \{w \in \{a, b\}^* \mid w \neq a\}$$

(a) $\overline{a}$
(b) $b^*$
(c) $\lambda \cup (aa \cup ab \cup b)(a \cup b)^*$
(d) This is not a regular language, and therefore it can not be described by a regular expression.

14. What is a context-free grammar for the language:

$$\overline{\{a\}} = \{w \in \{a, b\}^* \mid w \neq a\}$$

(a) $S \to SbS \mid aa \mid \lambda$
(b) $S \to AA \mid b \mid \lambda$
$A \to AA \mid a \mid b$
(c) $S \to \overline{A}$
$A \to a$
(d) The complement of a context-free grammar is not context-free, which means this language can not be produced by a context-free grammar.

15. Consider the grammar $G_{15}$:

$$\begin{array}{lcl} S & \to & aA \\ A & \to & B \mid \lambda \\ B & \to & B \mid bB \end{array}$$

Someone claims being able to show that $ba \notin \mathcal{L}(G_{15})$ using the property

$$P_{15}(w) = \text{the word } w \text{ does not contain the subword } ba$$

as an invariant for this language. Is this correct?

(a) Yes, that is correct. There are no words in the language $\mathcal{L}(G_{15})$ that contain the symbol $b$.

(b) Yes, that is correct. It is not possible to have a production starting in $S$ that produces a word containing $ba$.

(c) No, that is not correct. The property $P_{15}$ is not an invariant for the language $L_{15}$.

(d) No, that is not correct. Although the property $P_{15}$ is an invariant for the language $L_{15}$, the word $ba$ does not satisfy it, which means it cannot be used to show that this word is not in the language.

16. Again, consider the grammar:

$$\begin{array}{rcl} S & \to & aA \\ A & \to & B \mid \lambda \\ B & \to & B \mid bB \end{array}$$

For this grammar, give all:

- $\lambda$-rules
- chain rules
- useless symbols

## Automata

17. Give a deterministic finite automaton (DFA) $M_{17}$ for the language:

$$\overline{\{a\}} = \{w \in \{a,b\}^* \mid w \neq a\}$$

Either create an image in `TikZiT` and copy-paste it into Ans, or give the automaton as a tuple $\langle \Sigma, Q, q_0, F, \delta \rangle$, by writing:

```
M17 = <Sigma, Q, q0, F, delta>
Sigma = { ... }
Q = { q0, ... }
F = { ... }
delta(q0,a) = ...
...
```

18. Does the language

$$\overline{\{a\}} = \{w \in \{a,b\}^* \mid w \neq a\}$$

have a right linear context-free grammar?

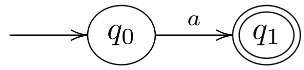
(a) Yes, this language has a right linear context-free grammar, but there are also context-free languages that do not have a right linear context-free grammar.

(b) Yes, all context-free languages also have a right linear context-free grammar.

(c) No, but it does have a context-free grammar that is not right linear.

(d) No, it does not have a context-free grammar.

19. The language $\{a\}$ is recognized by a non-deterministic finite automaton (NFA) with only two states:




Does that mean that its complement

$$\overline{\{a\}} = \{w \in \{a,b\}^* \mid w \neq a\}$$

is also recognized by a non-deterministic finite automaton with two states?

(a) Yes, one can transform an NFA to recognize the complement of its language by flipping all states between final and non-final.

(b) Yes, but that automaton is not what one gets from the above automaton by flipping the states between final and non-final.

(c) No, two states are not enough, but this language *is* recognized by a non-deterministic finite automaton with more than two states.

(d) No, this language is not recognized by any non-deterministic finite automaton.

## Modal logic

20. In doxastic logic, $\Box f$ means that you believe that $f$ is true, while $\Box\neg f$ means that you believe that $f$ is not true. But which formula formalizes the third option, where you are agnostic about the truth of $f$, i.e., where you believe that $f$ might be true but that it also might be false.

(a) $\Diamond f$

(b) $\Diamond\neg f$

(c) $\neg\Box f$

(d) $\neg\Box f \wedge \Diamond f$

21. In a world $x_i$ of a Kripke model, one can consider three formulas: $a$, $\Diamond a$ and $\Box a$. These all can be either true or false, which gives eight possible combinations. Which of these combinations is not possible?

(a) $x_i \nVdash a$, $x_i \Vdash \Diamond a$, $x_i \Vdash \Box a$

(b) $x_i \Vdash a$, $x_i \nVdash \Diamond a$, $x_i \Vdash \Box a$

(c) $x_i \nVdash a$, $x_i \nVdash \Diamond a$, $x_i \Vdash \Box a$

(d) All of the above combinations are possible.

22. Explain why in LTL the formulas $\mathcal{GF}a$ and $\mathcal{FG}a$ mean something different. You may do this informally using words, or you may define an LTL Kripke model (by defining the valuation $V(x_i)$ for all $i \in \mathbb{N}$) in which one of these formulas is true, but the other is not.