Formal Reasoning
Exercises

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This file will be updated every week with the latest solutions!

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Contents

Chapter 1

Propositional logic

Exercise 1.A
Form sentences in our formal language that correspond to the following English sentences:

(i) It is neither raining, nor is the sun shining.

> Note that if in the exercise it is not explicitly stated that you should use the official notation for formulas, you may omit superfluous parentheses.

Solution: $\neg R \wedge \neg S$: It is not raining and the sun is not shining.

(ii) The sun shines unless it rains.
Solution: The meaning of the word 'unless' can be a bit unclear. Often it is read as 'The sun shines, but not when it is raining,' but other interpretations can be given. Specifically, the following interpretations are not logically equivalent (see Definition 1.13):

- $R \to \neg S$: When it rains, the sun does not shine. Notice that nothing is said about the situation in which the sun does shine, but does this fit with the meaning of 'unless'?
- $\neg R \to S$: When it is not raining, the sun shines.
- $S \leftrightarrow \neg R$: The sun shines if and only if it is not raining.
- $\neg S \to R$: If the sun doesn't shine, it must rain.
- $S \to \neg R$: When the sun shines, it cannot rain.

Which one of these interpretations do you think matches the original sentence the best?

(iii) Either the sun shines, or it rains. (But not both simultaneously.)
Solution: Here, too, are multiple possibilities, though in this case they are logically equivalent. The first resembles the original English sentence more, though.

- $(S \vee R) \wedge \neg(S \wedge R)$: the sun shines or it rains, and not both at the same time,
- $(S \wedge \neg R) \vee (\neg S \wedge R)$: either the sun shines and it doesn't rain, or the sun doesn't shine and it rains.

Notice that the formulas $S \leftrightarrow \neg R$ and $\neg(S \leftrightarrow R)$ have the same truth table as these solutions. But because the form of these formulas doesn't resemble the original English sentence much, we think these formulas are less good translations.
Simply $S \vee R$ is wrong, because it doesn't account for the 'but not both simultaneously' part.

(iv) There is only a rainbow if the sun is shining and it is raining.
Solution: This solution seems best: $RB \to (S \wedge R)$: if there is a rainbow, then the sun must be shining and it must be raining. (Because only when the sun is shining and it is raining, there can be a rainbow.)

The following solution seems less good, but might be defensible: $\mathrm{RB} \leftrightarrow (\mathrm{S} \wedge \mathrm{R})$: there is a rainbow if and only if the sun shines and it is raining. In this translation, the word 'only' is somewhat unusually interpreted.
Furthermore, it is clear that $(\mathrm{S} \wedge \mathrm{R}) \to \mathrm{RB}$ is not a good translation of the sentence. Because this would correspond to 'There is a rainbow if the sun is shining and it is raining,' leaving away the 'only' as if it meant nothing special.

(v) If I'm outside, I get wet, but only if it rains.
Solution: There are many acceptable solutions in this case. Although we know that typically sentences like 'A, but B' are translated to '$A \wedge B$,' and usually sentences like 'A only if B' are translated to '$A \to B$', it is quite unclear how 'A, but only if B' should be translated. We could interpret 'A, but only if B' for instance as:

i) 'A if (also) B', or

ii) 'A only if B'.

Furthermore, in the original sentence it is unclear where the implicit parentheses are located. So this gives another two possibilities of interpreting the sentence:

I) ((If I'm outside, I get wet), but only if it rains.)

II) (If I'm outside, (I get wet, but only if it rains).)

If we combine these options, we get the following interpretations of the sentence and the corresponding formulas:

i+I) "I get wet when I'm outside, if it (also) rains." $\mathrm{R} \to (\mathrm{Out} \to \mathrm{W})$ or "I get wet when I'm outside and it (also) rains." $(\mathrm{Out} \wedge \mathrm{R}) \to \mathrm{W}$

ii+I) "If I get wet wet when I'm outside, then it must be raining." $(\mathrm{Out} \to \mathrm{W}) \to \mathrm{R}$

> Note that formulas of the form $(A \to B) \to C$ usually have a more subtle meaning than you expect and should only be used with great care.

i+II) "If I'm outside, then I get wet if it (also) rains." $\mathrm{Out} \to (\mathrm{R} \to \mathrm{W})$

ii+II) "If I'm outside, then I get wet only if it rains." $\mathrm{Out} \to (\mathrm{W} \to \mathrm{R})$

Note that the bi-implication $(\mathrm{Out} \to \mathrm{W}) \leftrightarrow \mathrm{R}$ is not in this list although both its parts $(\mathrm{Out} \to \mathrm{W}) \to \mathrm{R}$ and $\mathrm{R} \to (\mathrm{Out} \to \mathrm{W})$ are. The reason we don't consider this a really good solution is that both directions of this bi-implication are accepted under different interpretations!

Exercise 1.B
Can you also express $f \leftrightarrow g$ using the other connectives? If so, show how.

Solution: The formula $f \leftrightarrow g$ can also be expressed as $(f \to g) \wedge (g \to f)$ or as $(f \vee g) \to (f \wedge g)$. Also $(f \wedge g) \vee (\neg f \wedge \neg g)$ is logically equivalent.

Exercise 1.C
Translate the following formal sentences into English:

(i) $\mathrm{R} \leftrightarrow \mathrm{S}$
Solution: It rains if and only if the sun shines. (Or you could say something as 'it rains exactly when the sun shines.') Note that the sentence 'If it rains then the sun shines and if the sun shines, it rains' is perfectly well a translation of the formula $(\mathrm{R} \to \mathrm{S}) \wedge (\mathrm{S} \to \mathrm{R})$, which is in turn logically equivalent to $\mathrm{R} \leftrightarrow \mathrm{S}$, but nevertheless it is not so good of a translation of $\mathrm{R} \leftrightarrow \mathrm{S}$.

(ii) $\mathrm{RB} \to (\mathrm{R} \wedge \mathrm{S})$
Solution: If there is a rainbow, then it rains and the sun shines.

(iii) Out → ¬ In
Solution: If I'm outside, I'm not inside.

(iv) Out ∨ In
Solution: I'm outside or inside (or both).

Exercise 1.D

Draw the parse trees and give the truth tables for:

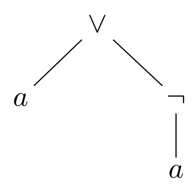
(i) $a \vee \neg a$

Make sure that your truth tables are created properly:

- Always start with a column for each atomic proposition.
- Each connective in the formula should have its own column.
- Columns are separated by lines.
- Rows are also separated by lines, but if your sheet has already horizontal lines, you may use these existing lines.
- There should be a row for each possible valuation.
- The rows should be ordered in such a way that if you take the 0's and 1's to be bits in a bit string, the first row coincides with the value 0 and the last row coincides with the value $2^n - 1$, where n is the number of atomic propositions.
- If it is easier for you to repeat columns, this is allowed, but this is not obligatory.
- Also not obligatory, but you may add parentheses to make the structure of the formula easier to parse (read correctly).

Solution:

The parse tree:

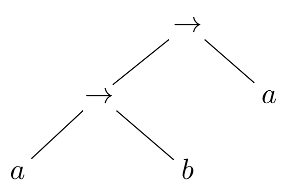


And the truth table:

a	$\neg a$	$a \vee \neg a$
0	1	1
1	0	1

(ii) $(a \to b) \to a$
Solution:

The parse tree:

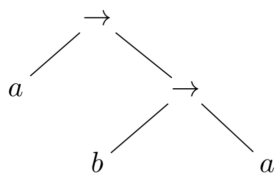


And the truth table:

a	b	$a \to b$	$(a \to b) \to a$
0	0	1	0
0	1	1	0
1	0	0	1
1	1	1	1

(iii) $a \to (b \to a)$
Solution:

The parse tree:

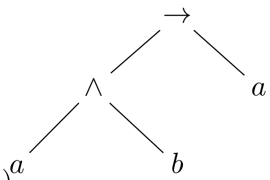


And the truth table:

a	b	$b \to a$	$a \to (b \to a)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

(iv) $a \wedge b \to a$
Solution:

The parse tree:

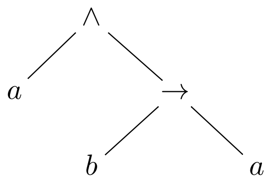


And the truth table:

a	b	$(a \wedge b)$	$(a \wedge b) \to a$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

(v) $a \wedge (b \to a)$
Solution:

The parse tree:

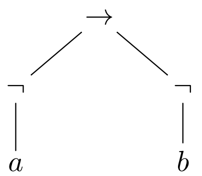


And the truth table:

a	b	$b \to a$	$a \wedge (b \to a)$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	1	1

(vi) $\neg a \to \neg b$
Solution:

The parse tree:



And the truth table:

a	b	$\neg a$	$\neg b$	$\neg a \to \neg b$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	1
1	1	0	0	1

Exercise 1.E
Which of the following propositions are logically true?

> You can find out whether a proposition is true by writing out its truth table. If the proposition's column is filled only with 1's, then the formula is logically true. If not, then the formula is not logically true.
> *Do not only write out the truth table as your answer. Always make sure to directly answer what was asked, by explicitly indicating which property of the truth table you have used to derive this conclusion!*

(i) $a \vee \neg a$

Solution: From the truth table of Exercise 1.D (i) you can conclude that $a \vee \neg a$ is logically true, because the last column contains only 1's.

(ii) $a \to (a \to a)$

Solution: Logically true. Note how the last column of its truth table is filled only with 1's:

a	$a \to a$	$a \to (a \to a)$
0	1	1
1	1	1

(iii) $a \to a$

Solution: Logically true. Note how the second column of the truth table above contains only 1's.

(iv) $(a \to b) \to a$

Solution: From the 0's in the corresponding truth table of Exercise 1.D (ii) you can conclude that $(a \to b) \to a$ is not logically true. Note that having only one 0 already implies that the formula is not logically true.

(v) $a \to (b \to a)$

Solution: From the truth table of Exercise 1.D (iii) you can conclude that $a \to (b \to a)$ is logically true, because the last column contains only 1's.

(vi) $a \wedge b \to a$

Solution: From the truth table of Exercise 1.D (iv) you can conclude that $a \wedge b \to a$ is logically true, because the last column contains only 1's.

(vii) $a \vee (b \to a)$

Solution: Not logically true. This follows from the 0 on the second row of the last column of its truth table:

a	b	$b \to a$	$a \vee (b \to a)$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	1	1

(viii) $a \vee b \to a$

Solution: Not logically true. Again, this follows from the 0 on the second row of the last column of its truth table:

a	b	$a \vee b$	$(a \vee b) \to a$
0	0	0	1
0	1	1	0
1	0	1	1
1	1	1	1

Exercise 1.F

Let f and g be *arbitrary* propositions. Find out whether the following statements hold. Explain your answers.

Note that all these statement are of the form 'if..., then...'.

- If the statement holds, you should give a general explanation why. In this explanation you may not assume anything about the structure of the arbitrary formulas f and g. In particular you don't know which atomic propositions are used within these formulas, so you don't know how many rows the corresponding truth tables have, so you cannot *write out* these truth tables. But you can *reason about* these truth tables!

- A proof for an 'if..., then...' statement works by first assuming the 'if' part, and then showing that the 'then' part must necessarily hold.

- If the statement does not hold, you can simply give a counterexample by choosing specific formulas f and g. In this case it is also possible to give a general explanation why the statement cannot be true, but that is usually much more difficult to formulate than giving an explanation about a specific counterexample.
  - To give a counterexample, you should pick some specific f and g in such a way that the 'if' part holds for these f and g, but the 'then' part fails to hold.

(i) If $\models f$ and $\models g$, then $\models f \wedge g$.
Solution: True.
Assume that $\models f$ and $\models g$. Which means that for every model v it holds that $v(f) = 1$ and $v(g) = 1$. Then for any such model v, you can calculate $v(f \wedge g)$ according to the truth table for $\wedge$, and you will find that $v(f \wedge g) = 1$. So apparently, given our assumptions, we have that for every model v, $v(f \wedge g) = 1$. Which then means that, by definition, $\models f \wedge g$. So the statement is true.

(ii) If not $\models f$, then $\models \neg f$.
Solution: Not true.
To find a counterexample for which we have 'not $\models f$,' we have to pick an f for which at least one model v_1 has $v_1(f) = 0$. For this v_1 it then automatically holds that $v_1(\neg f) = 1$. But to be sure that '$\models \neg f$' does not hold, we also need a model v_2 for which $v_2(\neg f) = 0$. This can only be the case if $v_2(f) = 1$. So, we can only find a counterexample if f is true in certain models and false in others. Luckily, there are plenty such f. For example, simply take $f = a$. Then we have 'not $\models a$', and yet the conclusion '$\models \neg a$' does not hold. So, we have found a counterexample, proving that the statement in question is not true.

(iii) If $\models f$ or $\models g$, then $\models f \vee g$.
Solution: True.
Assume that $\models f$ or $\models g$. So, we actually have three distinct cases, and for each of these cases we have to prove the same conclusion, namely $\models f \vee g$.

(1) $\models f$ is true. This means that for every v we have $v(f) = 1$. But for each such v, we then also have $v(f \vee g) = 1$, because the truth of f is enough to cause the truth of the composition $f \vee g$. So then we have $\models f \vee g$.

(2) $\models g$ is true. This case is exactly the same as the previous case (but the other way around).

(3) Both $\models f$ and $\models g$ are true. Then in particular, we have $\models f$, and in the case where we prove $\models f \vee g$ from $\models f$ we of course never relied on whether $\models g$ holds or does not hold. So we can use the same proof again.

So, in all three cases, the conclusion holds. So, the conclusion holds under the composite assumption '$\models f$ or $\models g$'. So, the statement holds.

(iv) If (if $\models f$, then $\models g$), then $\models f \to g$.
Solution: Not true.
So, we need a counterexample for which '(if $\models f$, then $\models g$)' holds. This sub-statement is again of the 'if..., then...' form and can thus be made true in two ways:

(1) By the falsity of $\models f$.

(2) By the truth of $\models f$ as well as $\models g$.

If in one of these situations we can show that $\models f \to g$ does not hold, we have found a counterexample.

Let us take $f = a$ and $g = b$. Now if we look at the models of $a \to b$, we see that there are four such models: $v_1(a) = 0$ and $v_1(b) = 0$, $v_2(a) = 0$ and $v_2(b) = 1$, $v_3(a) = 1$ and $v_3(b) = 0$, $v_4(a) = 1$ and $v_4(b) = 1$.

In a truth table:

model	a	b	$a \to b$
v_1	0	0	1
v_2	0	1	1
v_3	1	0	0
v_4	1	1	1

But because for example $v_1(a) = 0$, we immediately have that $\models a$ does not hold. So, for this specific choice for f and g we have '(if $\models f$, then $\models g$)'. However, if we compute $v_3(a \to b)$, we find $v_3(a \to b) = 0$. Which means that $\models f \to g$ does not hold. And thus we have indeed found a counterexample. So, the statement does not hold.

(v) If $\models \neg f$, then not $\models f$.

Solution: True.

This time over, we will give the proof without even using the word 'model', but by describing what happens in the truth tables, without writing them out in full. Because we have to the prove the statement for every f, you actually can't even write out the full truth table, because you don't know what f might look like. This doesn't matter, however, because you can reason about the 1's and 0's in the tables, and that's enough. (If the statement holds, of course.)

Assume that $\models \neg f$. Then the column of $\neg f$ contains only 1's. So, the column for f must show only 0's, and thus by definition $\models f$ does not hold. And thus, the statement holds.

(vi) If $\models f \vee g$, then $\models f$ or $\models g$.

Solution: Not true.

We will try to find a counterexample such that $\models f \vee g$, but not $\models f$ and also not $\models g$. Take $f = a$ and $g = \neg a$. Then there are only two different models: v_1 with $v_1(a) = 0$ and v_2 with $v_2(a) = 1$. It follows immediately that $v_1(\neg a) = 1$ and $v_2(\neg a) = 0$. It follows from $v_1(a) = 0$ that $\models a$ does not hold. And from $v_2(\neg a) = 0$ it also follows that $\models \neg a$ does not hold. However, $v_1(a \vee \neg a) = 1$ and $v_2(a \vee \neg a) = 1$, so we do have $\models f \vee g$. So, this is a valid counterexample. So, the statement does not hold.

(vii) If $\models f \to g$, then (if $\models f$, then $\models g$).

Solution: True.

Assume $\models f \to g$. Then for every model v we have $v(f \to g) = 1$. By looking at the way that $\to$ works, we see that there are two ways in which $v(f \to g) = 1$ can happen:

(1) $v(f) = 0$ (and it doesn't matter what $v(g)$ is)

(2) $v(f) = 1$ and $v(g) = 1$.

For both cases, we must show that 'if $\models f$, then $\models g$.'

(1) Suppose that we have a model v such that $v(f) = 0$. Then that means immediately that $\models f$ cannot be true, and thus the statement 'if $\models f$, then $\models g$' holds.

(2) So now we only have to look at the situation in which all models give $v(f) = 1$. But then, it follows from the case distinction that we made, that $v(g) = 1$ holds for all models. And thus we have 'if $\models f$, then $\models g$.'

Because in both cases, the conclusion 'if $\models f$, then $\models g$' holds, the original statement is true.

(viii) If $\models f \leftrightarrow g$, then ($\models f$ if and only if $\models g$).

Solution: True.

A proof with truth tables again.

Assume that $\models f \leftrightarrow g$. This means that in the truth table, in every row, the value in the column of f and the value in the column of g are the same. (Either they are both 0, or both 1.) If we assume furthermore that $\models f$, then f has only 1's in its column. But then, the same holds for g because their values were the same. And thus we can conclude that $\models g$. This last part works equally well the other way around: if $\models g$ is true, then g has all 1's in its column, and thus f has all 1's too, and thus $\models f$ is true. So, the whole statement is true.

(ix) If ($\models f$ if and only if $\models g$), then $\models f \leftrightarrow g$.
Solution: Not true.
Take $f = a$ and $g = b$. We then have four different models again, as in item (iv). In particular, we have $v_1(a) = 0$ and $v_1(b) = 0$. So, $\models a$ does not hold, and also $\models b$ does not hold. But, we do have the truth of the statement '$\models f$ if and only if $\models g$'! When we look at v_3, we see that $v_3(a \leftrightarrow b) = 0$, and thus $\models f \leftrightarrow g$ is false. So, we have found a counterexample. So, the statement is false.

Exercise 1.G
For each of the following couples of propositions, show that they are logically equivalent to each other.

Solution: To see whether two propositions are logically equivalent, we can inspect their truth tables. The propositions are logically equivalent in the case that their columns are exactly the same.

(i) $(a \wedge b) \wedge c$ and $a \wedge (b \wedge c)$
Solution: These propositions are logically equivalent because the last two columns are exactly the same:

a	b	c	$a \wedge b$	$b \wedge c$	$(a \wedge b) \wedge c$	$a \wedge (b \wedge c)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

(ii) $(a \vee b) \vee c$ and $a \vee (b \vee c)$
Solution: These propositions are logically equivalent too, because the last two columns are exactly the same:

a	b	c	$a \vee b$	$b \vee c$	$(a \vee b) \vee c$	$a \vee (b \vee c)$
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	1	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Exercise 1.H
Let f and g be propositions. Is the following statement true? $f \equiv g$ if and only if $\models f \leftrightarrow g$.

Solution: Let's split the statement up in two parts:

1. First we show that 'If $f \equiv g$ then $\models f \leftrightarrow g$' holds. If we assume that $f \equiv g$ holds, then f and g have identical columns. But then, the column of $f \leftrightarrow g$ has only 1's in it, and so $\models f \leftrightarrow g$ is true.

2. Then we show that 'If $\models f \leftrightarrow g$, then $f \equiv g$' holds. If we assume that $\models f \leftrightarrow g$ is true. Then the column of $f \leftrightarrow g$ is filled with only 1's. But then that means that, for every model, the truth value for f is the same as the truth value for g. And that is exactly what $f \equiv g$ expresses, so we have the truth of $f \equiv g$.

Both parts hold, and thus the original statement holds as well.

Exercise 1.I

Are the following statements true?

Solution: Like before, we can easily solve this exercise by writing out the truth tables for the propositions involved, and then checking the definition of 'logical entailment.'

(i) $a \wedge b \models a$

Solution: $a \wedge b \models a$ is true, because in every model where $a \wedge b$ holds, a holds as well.

(ii) $a \vee b \models a$

Solution: $a \vee b \models a$ is not true; the statement says that in every model in which $a \vee b$ is true, also a is true. But this is not the case, and we can find models in which $a \vee b$ but not a. For example, take the model v with $v(a) = 0$ and $v(b) = 1$.

(iii) $a \models a \vee b$

Solution: $a \models a \vee b$ is true: if a is true in a model, then $a \vee b$ must always also be true in that model.

(iv) $a \wedge \neg a \models b$

Solution: $a \wedge \neg a \models b$ is true: if $a \wedge \neg a$ is true in a model, then b must necessarily also be true in that model. This seems like a weird and untrue statement, for example because $a \wedge \neg a$ is never true in any model. However, the 'if. . . , then. . . ' construction makes the statement true, exactly because of the fact that the 'if' part is never true. It is what is called a 'vacuous truth.' You can also look at it the other way around: if the statement $a \wedge \neg a \models b$ would not be true, then you must be able to find a model in which b is not true, and $a \wedge \neg a$ is. And that, of course, would be impossible.

Chapter 2

Predicate logic

Exercise 2.A
Give two possible translations for the following sentence.

Sharon loves Maud; a nice man loves this intelligent character.

Solution: This can be a complicated exercise, if you are willing to bend your mind around the (somewhat forced) ambiguities in this English sentence. It isn't entirely clear what is meant by *this intelligent character*. (And also not entirely clear what is meant by *a nice man* ....) Which is to say, you won't be able to translate the sentence without giving your own interpretation of its meaning. Though of course there is a difference between an acceptable interpretation and a simply wrong one: by which we mean to say, your interpretation should be *defensible*. As far as we see (or intended) it, the possible ambiguities are as follows:

- Interpreting *a nice man loves ...* as either:
  1. *there is some nice man who loves ...*, or as
  2. *a nice man would love ...* (which is the same as saying *any nice man would love ...*).
- Interpreting *loves this intelligent character* as either:
  3. *loves Maud*, or
  4. *loves Sharon*, or
  5. *loves Maud, who is intelligent*, or
  6. *loves Sharon, who is intelligent*, or
  7. *loves Maud's intelligent character* (which would translate to *loves any woman who is as intelligent as Maud*), or
  8. *loves Sharon's intelligent character* (which would translate to *loves any woman who is as intelligent as Sharon*), or
  9. *loves this intelligent character* (which would translate to *loves any woman who is intelligent*), or

...leading to a whole range of possible translations, of which we will here list only a few:

(i) Interpreting as per (2) and (9):

$$L(s,m) \wedge \forall x \in M\, [N(x) \rightarrow \forall w \in W\, [I(w) \rightarrow L(x,w)]]$$

(ii) Interpreting as per (2) and (6):

$$L(s,m) \wedge \forall x \in M\, [N(x) \rightarrow (L(x,s) \wedge I(s))]$$

(iii) Interpreting as per (1) and (5):

$$L(s,m) \land \exists x \in M\,[N(x) \land (L(x,m) \land I(m))]$$

(iv) Interpreting as per (1) and (9):

$$L(s,m) \land \exists x \in M\,[N(x) \land \forall w \in W\,[I(w) \to L(x,w)]]$$

(v) Interpreting as per (2) and (8):

$$L(s,m) \land \forall x \in M\,[N(x) \to \forall w \in W\,[(I(w) \leftrightarrow I(s)) \to L(x,w)]]$$

Note that this indeed does not say anything about the intelligence of Sharon, but just that any man would love any woman *as intelligent as* Sharon. (Which would be up to the nice man's opinion, probably.)

(vi) Interpreting as per (1) and (3):

$$L(s,m) \land \exists x \in M\,[N(x) \land L(x,m)]$$

Exercise 2.B
Translate the following sentences to English.

Solution: This kind of translation exercises are best done in two steps. First, you quite literally translate the sentence to English, but then the English sentence will probably be a bit ugly. So then secondly, you play around with the English connectives to make the sentence more readable. But do make sure that this second English sentence still means the same as the original one!

(i) $\exists x \in M \Big[T(x) \land \exists w \in W\,[B(w) \land I(w) \land L(x,w)]\Big]$
Solution:

- There is a tall man and there is a beautiful intelligent woman, and this man loves this woman.
- There is a tall man and there is a beautiful intelligent woman, whom is loved by this man.
- There is a tall man who loves this beautiful and intelligent woman. (*Note the use of "this" to specify that this is a specific woman we are talking about, and not any/every beautiful and intelligent woman.*)

(ii) $\exists x \in M \Big[T(x) \land \exists w \in W\,[B(w) \land \neg I(w) \land L(x,w)] \land \exists w' \in W[I(w') \land L(w',x)]\Big]$
Solution:

- There is a tall man, and there is a beautiful but not intelligent woman whom he loves, and there is another woman who is intelligent and loves him.
- There is a tall man who loves this beautiful but unintelligent woman, and who is loved back by another woman, who is intelligent.
  Note that only saying $\exists w \in W$ and $\exists w' \in W$ doesn't exclude the possibility that w and w' are actually the very same person. However, we know that they are distinct people, because one of them is said to be intelligent, and the other not.

Exercise 2.C
Formalize the following sentence.

Sharon is beautiful; there is a guy who feels good about himself whom she loves.

Here, we will treat feeling good about oneself as being in love with oneself.

Solution: Using the dictionary (see page 13 and page 14) this can be formalized as:

$$B(s) \land \exists x \in M\;[L(x,x) \land L(s,x)]$$

Exercise 2.D

Formalize the following sentences:

(i) *For every two persons we have: the first one loves the second one only if the first one feels good about him- or herself.*

Solution: Note that we have chosen x to represent 'the first one' and y to represent 'the second one' within this exercise. We could have used different variables, but this choice seems more reasonable than taking, for instance, w for 'the first one' and v for 'the second one'. Note also that 'only if' doesn't mean the same thing as 'if and only if'. So, $\forall x, y \in W \cup M[L(x,y) \to L(x,x)]$ is correct, but $\forall x, y \in W \cup M[L(x,y) \leftrightarrow L(x,x)]$ is not. (If you want to stress that it is really about 'two distinct' persons, you should add this requirement, for instance like this: $\forall x, y \in W \cup M[\neg(x = y) \wedge L(x,y) \to L(x,x)]$. But this construction with equalities will only be explained in Section 2.4, so don't worry about it now.)

(ii) *For every two persons we have: the first one loves the second one if this second person feels good about him- or herself.*

Solution: $\forall x, y \in W \cup M[L(y,y) \to L(x,y)]$

(iii) *For every two persons we have: the first one loves the second one exactly in the case that the second one feels good about him- or herself.*

Solution: $\forall x, y \in W \cup M[L(x,y) \leftrightarrow L(y,y)]$ or $\forall x, y \in W \cup M[L(y,y) \leftrightarrow L(x,y)]$, but because of the matching word order in the sentence and the first option, we consider that one slightly better.

(iv) *There is somebody who loves everyone.*

Solution: $\exists x \in W \cup M\,[\forall y \in W \cup M[L(x,y)]]$

Exercise 2.E

(i) Verify that F_2 does not hold in M_1 under the interpretation I_1. But does F_1 hold?
Solution: Recall the definitions of F_1 and F_2:

$$F_1 \;:=\; \forall x \in D \exists y \in D\; K(x,y)$$
$$F_2 \;:=\; \exists x \in D \forall y \in D\; K(x,y)$$

So, independent of the structure and interpretation used, F_1 says that for every object x in the structure, there is some object y such that the relation $K(x,y)$ holds. And F_2 says that there is some object x in the structure, such that the relation $K(x,y)$ holds for every y in the structure. And remember: y may also be the same object as x.
So now let's first turn to the truth of F_2 within the structure M_1 and under the interpretation of I_1:

D	all students in the lecture hall
$K(x,y)$	x has a student number lower than y

We have been told already that F_2 will not hold. So, we need to prove that F_2 does indeed not hold. Because F_2 is of the form $\exists x \in D[f]$, such a proof amounts to showing that for every x, the statement f is not true. And because f has the form $\forall y \in D[g]$, proving that f is not true amounts to demonstrating that there is some y such that g is not true. Putting these two together, we have to prove that for all x, there is at least one y, such that $K(x,y)$ is not true.
Because there are but a finite number of students in the lecture hall (let's say n students), we can sort them by their unique student numbers and give them an index ranging from 1 to n. So let's name the students s_1, s_2 up to s_n, where the student s_1 has the lowest student number, and the student s_n has the highest.

To turn things around, let's see if we can prove F_2. If F_2 were to be true, you'd be able to pick out some student who has the lowest student number of anyone in the lecture hall. Obviously, the only contestant would be s_1, right? So, we take $x = s_1$ in our first step to proving F_2 true. And now it remains to be proved that

$$\forall y \in D\ K(s_1, y).$$

Of course this indeed holds for many students y, but it does not hold for *all students* in the lecture hall, because specifically s_1 is also one of these students, and he/she doesn't have a lower student number than him/herself.
Because s_1 is our only real possibility, but we don't have $K(s_1, s_1)$, we have proved that F_2 is not true (in M_1 under I_1).
What a long proof! Can't we write this down shorter? Yes, we can, for example:

> Sort the students by their unique student number in ascending order: s_1 to up s_n. (So, s_n has the highest student number and s_1 the lowest.) Suppose that F_2 is true. Then there are two possible cases for x:
>
> - If $x = s_i$, for $i \in \{2, \ldots n\}$, then $K(s_i, s_1)$ does not hold, so certainly not $\forall y \in D\ K(x, y)$. And so F_2 does not hold for $x = s_i$.
> - If $x = s_1$, then $K(s_1, s_1)$ does not hold, so indeed $\forall y \in D\ K(x, y)$ does not hold either, because $s_1 \in D$ too. And so F_2 does not hold for $x = s_1$.
>
> We have seen that F_2 does not hold for any x, so F_2 does not hold in this structure.

Now the question whether F_1 holds in this structure. In other words: is it true that for every student x we can find some student y with a higher student number? Well, obviously not. Because F_1 is of the shape $\forall x \in D\ [\exists y \in D\ K(x, y)]$, we are satisfied with proving the formula as soon as we can pick out some x such that no y exists for which $K(x, y)$.
So we pick $x = s_n$, the student with the highest student number, and indeed there is no y with a higher student number than x. Thus, we have completed the proof, that F_1 indeed does not hold in structure M_1 under the interpretation I_1.

(ii) Verify that F_1 holds in M_1 under the interpretation I_2. Does F_2 hold as well?
Solution: Recall interpretation I_2:

D	all students in the lecture hall
$K(x, y)$	x isn't older than y

We have to demonstrate the truth of F_1 in the interpretation I_2. This means, that for every x, we must show that we can find a y, such that $K(x, y)$. In other words: we must give a method, or algorithm, by which, for any x, we can choose a y in such a way that $K(x, y)$ holds.
Which is in this case surprisingly simple: choose y to be $y = x$ itself. Then indeed $K(x, y)$, because one is never older than oneself. So indeed, F_1 holds under interpretation I_2.
In a similar way we can show that F_2 holds in structure M_1 under the interpretation I_2. To do that, we must pick a suitable x. Before, we have ordered students according to their student number. Now, let's order the students according to their age. But note that, unlike student numbers, the ages of students need not be unique. (Even if you'd measure age in seconds or an even smaller timescale, instead of years.) So, we are not able to speak of the *oldest* or the *youngest* student, because there may be multiple such students with exactly the same age. But even though we may not have a youngest student, we do have a subset of youngest students, and that's enough for our current purposes. Let's take x to be any of these youngest students (if there are multiple, or else

just the single youngest person). Then indeed for all students y, it holds that $K(x, y)$, or, x is not older than y. So we have proved F_2 in M_1 under I_2.

(iii) Check whether F_1 in M_1 is true under the interpretation of I_3, by looking around in class. And check whether F_2 is true or not, as well.

Solution: Recall I_3:

D	all students in the lecture hall
$K(x, y)$	x is sitting next to y

At the time of writing this document, it can't be stated whether F_1 is true or not, because it depends on the specific situation in the lecture hall. If someone is sitting alone from the rest, he/she would be a counterexample, proving the falsity of F_1. Though if everyone is sitting next to someone else, then you have a method by which for every x you can pick a y such that $K(x, y)$ holds. Namely, pick one of his/her neighbors.

However, F_2 is never true in M_1 under interpretation I_3. Because it is never the case that some student is sitting next to all others. Even if you'd arrange the chairs in such a way that all other students sit next to the person in question, this person would not be sitting next to him- or herself. So, F_2 is definitely false under interpretation I_3.

Exercise 2.F

Verify that G_2 is indeed true in structure M_4 under the interpretation I_8, but not in structure M_3 under the interpretation I_7. Stated differently: verify that $((\mathbb{Q}, <), I_8) \models G_2$ and verify that $((\mathbb{N}, <), I_7) \not\models G_2$.

Solution:

(i) Interpretation I_7 gives G_2 the meaning that for every $x \in \mathbb{N}$ there is a $y \in \mathbb{N}$ such that $y < x$. But this is not true. Counterexample: take $x = 0$, and then there will be no y such $y < 0$.

(ii) Interpretation I_8 gives G_2 the meaning that for every $x \in \mathbb{Q}$ there is a $y \in \mathbb{Q}$ such that $y < x$. This time the statement is true, because one can always take for example $y = x - 1$ in $\mathbb{Q}$.

Exercise 2.G

Define the interpretation I_9 as:

D	$\mathbb{N}$
$K(x, y)$	$x = 2 \cdot y$

Are the formulas G_1 and/or G_2 true under this interpretation?

Solution:

(i) Under interpretation I_9 the formulas G_1 states that for every $x \in \mathbb{N}$ there is a $y \in \mathbb{N}$ such that $x = 2 \cdot y$. Not true: take for example $x = 3$, and then there will be no such y.

(ii) Under interpretation I_9 the formulas G_2 states that for every $x \in \mathbb{N}$ there is a $y \in \mathbb{N}$ such that $y = 2 \cdot x$. True: for any x we can take $y = 2 \cdot x$.

Exercise 2.H

Define the interpretation I_{10} as:

D	$\mathbb{Q}$
$K(x, y)$	$x = 2 \cdot y$

Are the formulas G_1 and/or G_2 true under this interpretation?

Solution:

(i) Under the interpretation I_{10} the formula G_1 states that for every $x \in \mathbb{Q}$ there is a $y \in \mathbb{Q}$ such that $x = 2 \cdot y$. True: for any x we can take $y = \frac{x}{2}$.

(ii) Under the interpretation I_{10} the formula G_2 states that for every $x \in \mathbb{Q}$ there is a $y \in \mathbb{Q}$ such that $y = 2 \cdot x$. True: for any x we can take $y = 2 \cdot x$.

Exercise 2.I

We take as structure the countries of Europe, and the following interpretation I_{11}:

E	the set of countries of Europe
n	The Netherlands
g	Germany
i	Ireland
$B(x, y)$	x borders y
$T(x, y, z)$	x, y, and z share a tripoint (where the borders of all three countries meet)

(i) Formalize the sentence "The Netherlands and Germany share a tripoint."

Solution: This would be any of the equivalent formulas $\exists x \in E\ [T(n, g, x)]$, or $\exists x \in E\ [T(n, g, x)]$, or $\exists x \in E\ [T(x, g, n)]$, or $\exists y \in E\ [T(y, n, g)]$, etc.

(ii) Which of the following formulas are true in this structure and under this interpretation?

(1) $G_3 := \forall x \in E\ \exists y \in E\ [B(x, y)]$

Solution: G_3 states that every country x borders at least some country y. But this is not true, for example for island countries such as Iceland and Malta.

(2) $G_4 := \forall x, y \in E\ [(\exists z \in E\ T(x, y, z)) \to B(x, y)]$

Solution: G_4 states that for every two countries x and y, if they share a tripoint with some country z, then x and y border each other as well. True, if they share a tripoint, then in particular they indeed share a border.

(3) $G_5 := \forall x \in E\ [B(i, x) \to \exists y \in E\ [T(i, x, y)]]$.

Solution: G_5 states that if a country borders Ireland, then it shares a tripoint with Ireland as well. Not true: take the United Kingdom, which borders Ireland (via Northern-Ireland), but does not share a tripoint with Ireland. (However, if i would have been interpreted to mean Iceland, the G_5 would have been *vacuously true*, because $B(i, x)$ is never true for any x, and therefore the implication is true.)

Exercise 2.J

Find a structure M_5 and an interpretation I_{12} such that this formula holds:

$$(M_5, I_{12}) \models \forall x \in D\ \exists y \in E\ [R(x, y) \wedge \neg R(y, x) \wedge \neg R(y, y)]$$

Solution: Two possible solutions for M_5 and I_{12} are:

- The structure $(\mathbb{N}, <)$ with interpretation I_{12}

D	$\mathbb{N}$
E	$\mathbb{N}$
$R(x, y)$	$x < y$

(Verify that the formula holds by taking y to be $x + 1$.)

- The structure

Domain(s)	all people
Relation(s)	being a father of

with interpretation I_{12}

D	all people
E	all people
$R(x, y)$	y is the father of x

Verify that this would not work if $R(x, y)$ would be interpreted as x being y's father.

Note that the domains D and E must be related. We will not further discuss this in this course, but in the course "Logic and Applications" it will be explained that every relation has a *type*, for example $\mathbb{N} \to \mathbb{Q} \to \{0, 1\}$, meaning that x must be an element of $\mathbb{N}$ and y an element of $\mathbb{Q}$, and that the relation $R(x, y)$ then results in a 0 or a 1 depending on the interpretation. In this exercise, however, we write $R(x, y)$ as well as $R(y, x)$, meaning that then we must have $x \in D$, $x \in E$, $y \in D$, as well as $y \in E$! So, defining a structure in which D is the set of cars, E is the set of bicycles, and defining $R(x, y)$ to mean that 'x is faster than y' would be a bit weird because cars are not bicycles and neither the other way around. Though of course we could define the superset V to be the set of vehicles and let R be of type $V \to V \to \{0, 1\}$, and things would work correctly again. If you don't understand this remark, don't worry, it will be treated in full in the course "Logic and Applications."

Exercise 2.K

Consider the interpretation I_{14}:

H	domain of all human beings
$F(x)$	x is female
$P(x, y)$	x is parent of y
$M(x, y)$	x is married to y

Formalize the following sentences into formulas of predicate logic with equality:

(i) *Everyone has exactly one mother.*

Solution:

$$\begin{aligned} \forall x \in H \quad [\quad & \exists y \in H \quad [\ F(y) \wedge P(y, x)\ ] \\ & \wedge \\ & \forall y, z \in H \quad [\ (F(y) \wedge P(y, x) \wedge F(z) \wedge P(z, x)) \to y = z\ ] \\ ] \quad & \end{aligned}$$

In this formula, x is the "everyone", and y and z play the role of mother. The first y is used to state that indeed x has at least one mother, (namely y), and the second y, together with z, is used to state that x has at most one mother, by stating that for any two people who would both be x's mother, these people are the same person. Note: there is no obligation to use the same variable for the second y as we did for the first y; we could equally well have called this second y for example w. Also note the *scope* of x: in the second part (at most 1 mother) we must be able to refer back to x.

This alternative solution would have worked as well:

$$\begin{aligned} \forall x \in H \quad [\quad \exists y \in H \quad [\quad & F(y) \wedge P(y, x) \\ & \wedge \\ & \forall z \in H [(F(z) \wedge P(z, x)) \to y = z] \\ ] \quad & \\ ] \quad & \end{aligned}$$

Here, the y is bound only once, slightly more elegantly stating that this is the only mother that x has. (And directly stating that any mother z of x is indeed the same mother.)

(ii) *Everybody has exactly two grandmothers.*

Solution:

$$\begin{array}{llll}
\forall x \in H & [\quad \exists y, z \in H & [& y \neq z \ \wedge \\
& & & \exists u \in H[F(y) \wedge P(y,u) \wedge P(u,x)] \wedge \\
& & & \exists u \in H[F(z) \wedge P(z,u) \wedge P(u,x)] \\
& &] & \\
& \quad \wedge & & \\
& \quad \forall y, z, v \in H & [& (\\
& & & y \neq z \ \wedge \\
& & & \exists u \in H[F(y) \wedge P(y,u) \wedge P(u,x)] \wedge \\
& & & \exists u \in H[F(z) \wedge P(z,u) \wedge P(u,x)] \wedge \\
& & & \exists u \in H[F(v) \wedge P(v,u) \wedge P(u,x)] \\
& & &) \\
& & \quad \rightarrow & \\
& & & (v = y \vee v = z) \\
& &] & \\
&] & &
\end{array}$$

Here again, x plays the role of "everybody", and now y and z play the role of the two distinct grandmothers, and u plays the role of the father and/or mother of x, simultaneously being the child of y, resp. z.

(iii) *Every married man has exactly one spouse.*

Solution: Note that a spouse can be both the husband or the wife. So in the given solution x must be a man, but y, z_1 and z_2 may be both male or female.

$$\begin{array}{ll}
\forall x \in H & [\quad \neg F(x) \wedge \exists y \in H\ [M(x,y)] \\
& \quad \rightarrow \\
& \quad (\forall z_1, z_2 \in H\ [M(x,z_1) \wedge M(x,z_2) \rightarrow z_1 = z_2]) \\
&]
\end{array}$$

Exercise 2.L

Use the interpretation I_{14} of Exercise 2.K to formalize the following properties.

(i) $C(x,y)$: x and y have had a child together.
Solution: Define

$$C(x,y) := x \neq y \wedge \exists z \in H[P(x,z) \wedge P(y,z)]$$

Then, x and y are indeed two distinct people, and there is a child z who has parents x and y. (If we don't include the $x \neq y$ the sentence is not an exact translation.)

(ii) $B(x,y)$: x is a brother of y (take care: refer also to the next item).
Solution: Define

$$B(x,y) := \neg F(x) \wedge x \neq y \wedge \exists q, r \in H[q \neq r \wedge P(q,x) \wedge P(q,y) \wedge P(r,x) \wedge P(r,y)]$$

So, x is a brother of y if x is male, and not the same person as y, and there are two distinct parents who are the parents of x as well as of y.

(iii) $S(x,y)$: x is a step-sister to y.
Solution: A first try:

$$\begin{array}{l}
S(x,y) := F(x) \wedge x \neq y \wedge \\
\quad \exists o_1, o_2, o_3 \in H[o_1 \neq o_2 \wedge o_2 \neq o_3 \wedge o_3 \neq o_1 \wedge \\
\quad P(o_1,x) \wedge \neg P(o_1,y) \wedge P(o_2,x) \wedge P(o_2,y) \wedge \neg P(o_3,x) \wedge P(o_3,y)]
\end{array}$$

So, x is a step-sister to y if x is female, and there are three distinct parents o_1, o_2, o_3, the first two of which are the parents of x, and the last two of which are the parents of y (so x and y share exactly one parent o_2.)
Unfortunately this is not a step-sister, but a half-sister! A step-sister is

> A daughter of one's step-parent by a marriage other than with one's own parent.

And a step-parent is defined as

> A person who is married to one's parent, but is not one's parent.

If we combine these two definitions we get the following formalization for x is a step-sister of y:

$$S(x,y) := F(x) \wedge \\ \exists o_1 \in H[P(o_1,y) \wedge \neg P(o_1,x) \wedge \exists o_2 \in H[P(o_2,x) \wedge \neg P(o_2,y) \wedge M(o_2,o_1)]]$$

This formula expresses that x is a step-sister of y if

- x is female,
- o_1 is a parent of y, but is not a parent of x,
- o_2 is a parent of x, but is not a parent of y,
- and o_2 is married to o_1.

Note that we didn't specify that $x \neq y$, but that follows automatically from $P(o_1,y) \wedge \neg P(o_1,x)$. And from $M(o_2,o_1)$ it follows that $o_2 \neq o_1$, because you cannot be married to your self. Similarly it follows that x, y, o_1 and o_2 are all different persons.

Translate the following formulas back to English.

(iv) $\exists x{\in}H \forall y{\in}H\ P(x,y)$. And is this true?
Solution: "There is a person who is everyone's parent." This is obviously not true.

(v)

$$\forall z_1 \in H \forall z_2 \in H \quad [\quad \exists x{\in}H \exists y_1{\in}H \exists y_2{\in}H \quad [P(x,y_1) \wedge P(y_1,z_1) \wedge P(x,y_2) \wedge P(y_2,z_2)] \rightarrow \neg(\exists w{\in}H\ [P(z_1,w) \wedge P(z_2,w)]) \quad]$$

And is this true?
Solution: "Every two people who share a common grandparent, do not share a child." This is also not true. Note that these two people z_1 and z_2 can be the same person! Furthermore, it is also not unlikely that there are cousins within a family who do share a child.

Exercise 2.M
Given the interpretation I_{15}:

D	$\mathbb{N}$
$A(x,y,z)$	$x + y = z$
$M(x,y,z)$	$x \cdot y = z$

Formalize the following:

Solution: In this exercise we will need formulas that formalize the numbers $x = 0$ and $x = 1$, so let's first solve these.

- We can capture the character of x being 0 neatly by stating that $a \cdot x = x$ for all a. So, define
$$x = 0 := \forall a \in D[M(a, x, x)]$$
and $x \neq 0 := \neg(x = 0)$.

- Similarly, for $x = 1$ we have the defining property that $x \cdot a = a$ holds for all a. So, define
$$x = 1 := \forall a \in D[M(x, a, a)]$$
and $x \neq 1 := \neg(x = 1)$.

(i) $x < y$.
Solution: $x < y$ holds precisely in the case that $x + r = y$ and $r \neq 0$.
So, define:
$$x < y := \exists r \in D[A(x, r, y) \wedge \neg(r = 0)]$$

(ii) $x \mid y$ (x divides y).
Solution: $x|y$ (x divides y) whenever there is some number z such that $x \cdot z = y$. So, define:
$$x|y := \exists z \in D[M(x, z, y)]$$

(iii) x is a prime number.
Solution: x is a prime number when x is not 1, and has no other factors other than 1 and itself. So, define:
$$x \text{ is a prime number} := x \neq 1 \wedge \neg\exists y, z \in D[M(y, z, x) \wedge y \neq x \wedge z \neq x \wedge y \neq 1 \wedge z \neq 1]$$
We could also simply use the $x|y$ which we already defined above.
$$x \text{ is a prime number} := x \neq 1 \wedge \forall y \in D[(y|x) \rightarrow (y = 1 \vee y = x)]$$