Induction scheme

Theorem (or **Proposition** or **Lemma**): 0

$\boxed{\ldots}$ for all $n \geq \boxed{\ldots}$

Proof by induction on n. 1

We first define our induction predicate P as:

$$P(n) \quad := \quad \left[\boxed{\ldots}\right] \qquad 2$$

Base Case. We show that $P(\boxed{\ldots})$ holds, i.e. we show that 3

$\boxed{\ldots \textit{ where } n \textit{ is replaced by } \boxed{\ldots}}$

This indeed holds, because ... 4

Induction Step. Let k be any natural number such that $k \geq \boxed{\ldots}$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6

$\boxed{\ldots \textit{ where } n \textit{ is replaced by } k}$ (Induction Hypothesis)

We now have to show that $P(k+1)$ also holds, i.e. we have to show that 7

$\boxed{\ldots \textit{ where } n \textit{ is replaced by } (k+1)}$

This indeed holds, because ... 8

Hence it follows by induction that $P(n)$ holds for all $n \geq \boxed{\ldots}$. 9

(Note that all blue ... are textually always exactly the same. This also holds for all red ..., except for the given substitutions. The four underlined names should all appear in the proof. The green items mark the only items where you really have to do something else besides just copying.)

Example

Proposition: 0

$1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for all $n \geq 1$.

Proof by induction on n. 1

We first define our induction predicate P as:

$$P(n) \quad := \quad \left[\, 1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1) \,\right]$$ 2

Base Case. We show that $P(1)$ holds, i.e. we show that 3

$1 = \frac{1}{2} \cdot 1 \cdot (1+1)$

This indeed holds, because 4

$$\begin{aligned} \tfrac{1}{2} \cdot 1 \cdot (1+1) &= \tfrac{1}{2} \cdot 1 \cdot 2 \\ &= 1 \end{aligned}$$

Induction Step. Let k be any natural number such that $k \geq 1$. 5

Assume that we already know that $P(k)$ holds, i.e. we assume that 6
$1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$ (Induction Hypothesis IH)

We now have to show that $P(k+1)$ also holds, i.e. we have to show that 7
$1 + 2 + \cdots + (k+1) = \frac{1}{2}(k+1)((k+1)+1)$

This indeed holds, because 8

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &\overset{\text{IH}}{=} \tfrac{1}{2}k(k+1) + (k+1) \\ &= (\tfrac{1}{2}k^2 + \tfrac{1}{2}k) + (k+1) \\ &= \tfrac{1}{2}k^2 + \tfrac{3}{2}k + 1 \\ &= \tfrac{1}{2}(k^2 + 3k + 2) \\ &= \tfrac{1}{2}(k+1)(k+2) \\ &= \tfrac{1}{2}(k+1)((k+1)+1) \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 1$. 9