Formal Reasoning 2025
Solutions Test Block 1: Propositional and Predicate Logic
(25/09/22)

Propositional logic

1. Consider the English sentence:

   You don't get wet unless it rains, in which case you only get wet if you're outside.

   We want to formalize the meaning of just the first part

   'You don't get wet unless it rains'

   as a formula of propositional logic. For the dictionary, we use:

R	it rains
W	you get wet

   A good formalization of this part of the sentence is:

   (a) $R \to W$
   (b) $\neg R \to \neg W$ — (b) is correct
   (c) $(R \to W) \land (\neg R \to \neg W)$
   (d) $(R \to W) \lor (\neg R \to \neg W)$

   Hint: In general, the sentence 'You don't get wet unless it rains' is ambiguous. However, reading the provided context can help you determine what the meaning to be formalized is in this case.

   Answer (b) is correct.

   The most common interpretation of 'A unless B' is 'A if not B', which is equivalent to 'if not B then A', which in turn leads to the formula '$\neg B \to A$'. In this situation, A stands for 'You don't get wet', so for $\neg W$. And B stands for 'it rains', hence for R. Plugging this into $\neg B \to A$ gives $\neg R \to \neg W$.

   However, there are also situations, where 'A unless B' seems more like an 'if B then not A' which in this case would lead to $R \to W$.

   Now the hint says that we have to look at the original sentence: 'You don't get wet unless it rains, in which case you only get wet if you're outside.' Now if we would allow $R \to W$ be a (part of a) proper translation, then we have a contradiction, as apparently the obligatory requirement for getting wet from the rain of being outside is no longer obligatory.

   This means that $R \to W$ and $(R \to W) \land (\neg R \to \neg W)$ aren't correct. And $(R \to W) \lor (\neg R \to \neg W)$ isn't correct either, as this does not for sure state that you don't get wet if it doesn't rain.

2. Consider the truth table of the formula of propositional logic:

   $$\neg a \to b \leftrightarrow c$$

   For each column in the truth table, give the number of zeroes and ones.

   As an example of how to give these numbers, for the formula $a \lor \neg b$ the answer would be:

```
a: 2 zeroes and 2 ones
b: 2 zeroes and 2 ones
~b: 2 zeroes and 2 ones
a \/ ~b: 1 zero and 3 ones
```

Hint: So don't waste time on submitting the full truth table!

This is the corresponding truth table:

a	b	c	$\neg a$	$\neg a \to b$	$\neg a \to b \leftrightarrow c$
0	0	0	1	0	1
0	0	1	1	0	0
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	0	1	1
1	1	1	0	1	0

This table leads to the following answer:

```
a: 4 zeroes and 4 ones
b: 4 zeroes and 4 ones
c: 4 zeroes and 4 ones
~a: 4 zeroes and 4 ones
~a -> b: 2 zeroes and 6 ones
~a -> b <-> c: 4 zeroes and 4 ones
```

3. Which formula of propositional logic is logically equivalent to $\neg(a \to b)$?

   (a) $\neg a \to \neg b$

   (b) $\neg b \to \neg a$

   (c) $a \wedge \neg b$ — (c) is correct

   (d) $\neg a \wedge b$

   Answer (c) is correct.

   We use logical laws to prove the equivalence.

$$
\begin{array}{rcll}
\neg(a \to b) & \equiv & \neg(\neg a \vee b) & \text{material implication} \\
 & \equiv & \neg\neg a \wedge \neg b & \text{De Morgan} \\
 & \equiv & a \wedge \neg b & \text{double complement}
\end{array}
$$

   Using truth tables, it is also easy to see that the others are not equivalent as their columns are not the same.

a	b	$a \to b$	$\neg(a \to b)$	$\neg a$	$\neg b$	$\neg a \to \neg b$	$\neg b \to \neg a$	$a \wedge \neg b$	$\neg a \wedge b$
0	0	1	0	1	1	1	1	0	0
0	1	1	0	1	0	0	1	0	1
1	0	0	1	0	1	1	0	1	0
1	1	1	0	0	0	1	1	0	0

4. Is it the case that for all propositional formulas f and g, that if $\not\models f$ and $\not\models g$, then also $\not\models f \vee g$?

(a) Yes, that is the case. Take for example $f := a$ and $g := b$. Then we find that $\not\models a$ and $\not\models b$, and indeed also $\not\models a \vee b$.

(b) Yes, that is the case, as is clear from the following table:

$\models f$	$\models g$	$\models f \vee g$
0	**0**	**0**
0	1	1
1	0	1
1	1	1

(c) No, that is not the case. In fact, the conclusion $\not\models f \vee g$ does not hold for *any* f and g that satisfy $\not\models f$ and $\not\models g$.

(d) is correct

(d) No, that is not the case. The conclusion $\not\models f \vee g$ indeed holds for *some* f and g that satisfy $\not\models f$ and $\not\models g$, but there are counterexamples.

Answer (d) is correct.

If we take $f := a$ and $g := \neg a$, then clearly $\not\models f$ and $\not\models g$. However, $f \vee g = a \vee \neg a$, and hence $\models f \vee g$ does hold.

The example given above where $f := a$ and $g := b$ shows that there are formulas for which the claim $\not\models f \vee g$ does hold.

Predicate logic

5. Formalize as a formula of predicate logic the meaning of the English sentence:

   Happiness is being loved.

   Use for the dictionary:

B	the domain of beings
$H(x)$	x is happy
$L(x, y)$	x loves y

   This statement is expressed by this formula:

$$\forall x \in B \ \ [H(x) \leftrightarrow \exists y \in B \ \ [L(y, x)]]$$

   or in the syntax according to the official grammar

$$(\forall x \in B \ \ (H(x) \leftrightarrow (\exists y \in B \ L(y, x))))$$

6. Consider the formula of predicate logic:

$$\forall x \in B \, [L(x, k) \wedge H(k)]$$

   and consider the following two statements:

   - The form of this formula, according to the official grammar from the course notes is:
$$(\forall x \in B \, \underline{(} L(x, k) \wedge H(k) \underline{)})$$
   - The underlined parentheses in the formula in the previous statement belong to the syntax of the universal quantifier.

What is the case?

(a) Both statements are correct.

(b) is correct (b) Only the first statement is correct.

(c) Only the second statement is correct.

(d) Neither statement is correct.

Answer (b) is correct.

Let us repeat the formula:

$$(\forall x \in B \underline{(}L(x,k) \wedge H(k)\underline{)})$$

For this formula, it holds that it is indeed written according to the official grammar. However, it is not the case that the underlined parentheses belong to the universal quantifier, as these parentheses belong to the conjunction ($\wedge$).

7. There are formulas of predicate logic that can only be true in infinite models. An example of such a formula is:

$$[\forall x, y, z \in D\,(R(x,y) \wedge R(y,z) \rightarrow R(x,z))] \wedge$$
$$[\forall x \in D\, \neg R(x,x)] \wedge$$
$$[\forall x \in D\, \exists y \in D\, R(y,x)]$$

We consider here models where D is interpreted as the infinite set of natural numbers $\mathbb{N}$. Which interpretation in a model makes this formula true?

(a) $(\mathbb{N}, <)$ where $R(x,y)$ is interpreted as $x < y$

(b) is correct (b) $(\mathbb{N}, <)$ where $R(x,y)$ is interpreted as $y < x$

(c) $(\mathbb{N}, \leq)$ where $R(x,y)$ is interpreted as $x \leq y$

(d) $(\mathbb{N}, \leq)$ where $R(x,y)$ is interpreted as $y \leq x$

Answer (b) is correct.

The formula actually states that there are three properties:

- $\forall x, y, z \in D\,(R(x,y) \wedge R(y,z) \rightarrow R(x,z))$
  This property states that the relation R is *transitive*. And it is a well known fact that both $<$ and $\leq$ are transitive relations on $\mathbb{N}$. In this course, it is not necessary to know this name. However, you can easily check that it holds: if $3 < 7$ and $7 < 11$ then $3 < 11$. And there is nothing specific about the 3, the 7, and the 11. So this property does not exclude any of the four options.
- $\forall x \in D\, \neg R(x,x)$
  This property states that the relation R is not *reflexive*. And it is a well known fact that $<$ is not reflexive on $\mathbb{N}$, as $3 < 3$ is not true, whereas $\leq$ is reflexive, because $x \leq x$ is true for all natural numbers x. So this excludes the two options with $\leq$.

- $\forall x \in D\, \exists y \in D\, R(y, x)$
  This property doesn't have a specific name. It states that for every x we can pick a y such that the relation $R(y, x)$ holds. Do you see the weird order of y and x in this formula? It is essential later on! Now as we already discarded the $\leq$ options, we only have to check the $<$ options.
  - If $R(x, y)$ is defined as $x < y$, then the formula becomes $\forall x \in D\, \exists y \in D\, y < x$ and this is clearly not true. Take $x := 0$ and then it becomes impossible to choose $y \in \mathbb{N}$ such that $y < 0$.
  - If $R(x, y)$ is defined as $y < x$, then the formula becomes $\forall x \in D\, \exists y \in D\, x < y$ and this is indeed true. For every $x \in \mathbb{N}$ we can simply choose $y := x + 1$ and then $y \in \mathbb{N}$ and $x < y$.

So the only solution is: $(\mathbb{N}, <)$ where $R(x, y)$ is interpreted as $y < x$.

8. We want to formalize the following English sentence as a formula of predicate logic:

   Only Sharon is not happy.

   We use for the dictionary:

B	the domain of beings
s	Sharon
$H(x)$	x is happy

   What is a good formalization for this?

   (a) $\neg H(s)$
   (b) $\forall x \in B\, [\neg H(x) \to x = s]$
   (c) $\neg \exists x \in B\, [\neg H(x) \wedge x \neq s]$
   (d) none of the above

   (d) is correct

   Hint: Keep in mind that according to the sentence, Sharon is not happy.

   Answer (d) is correct.

   The default pattern to state this would be:

$$\neg H(x) \wedge \forall x \in B\, [\neg H(x) \to x = s]$$

   However, this is not one of the options.

   So let us have a closer look at the three suggested formulas:

   - $\neg H(s)$
     This formula does state that Sharon is not happy, but it doesn't exclude that there are other beings that are also not happy.
   - $\forall x \in B\, [\neg H(x) \to x = s]$
     This is the second part of the default solution, so it does exclude that there are other people who are not happy. However, it doesn't state that Sharon herself is not happy.

- $\neg \exists x \in B\, [\neg H(x) \wedge x \neq s]$
  Let us rewrite this formula using logical laws:

  $$\begin{array}{lll} \neg \exists x \in B\, [\neg H(x) \wedge x \neq s] & & \\ \quad \equiv & \forall x \in B\, [\neg(\neg H(x) \wedge x \neq s)] & \text{De Morgan for quantifiers} \\ \quad \equiv & \forall x \in B\, [\neg\neg H(x) \vee \neg(x \neq s)] & \text{De Morgan for } \wedge \\ \quad \equiv & \forall x \in B\, [H(x) \vee x = s] & \text{two times double negation} \end{array}$$

  So this means that each being is happy or equal to Sharon. However, as it is an inclusive 'or', it doesn't exclude that Sharon herself is happy.

So none of the suggestions are correct.