Formal Reasoning 2025
Solutions Test Block 2: Discrete mathematics
(16/10/25)

Discrete mathematics

1. Does each connected graph in which each vertex has even degree have an Eulerian circuit?

    (a) Yes, that is exactly the statement about Eulerian circuits in Euler's theorem.

    (b) Yes, but it also has an Eulerian circuit in the case when at most two vertices have an odd degree.

    (c) No, the Petersen graph is a counterexample.

    (d) No, the graph K_1 is a counterexample.

    (d) is correct

    Answer (d) is correct.

    It looks like the statement on Eulerian circuits in Euler's theorem, however, the general condition that the graph should have at least two vertices is omitted.

    In particular, this observation leads to the counterexample K_1, which is just a single vertex with no edges at all. So the degrees of all vertices in this connected graph are (is?) zero. However, as it doesn't have edges, it doesn't have circuits, as circuits need to have at least three edges.

    So all 'yes' answers are certainly wrong.

    Note that the Petersen graph is not a counterexample, as this is a graph where each vertex has an odd degree.

2. Is every tree bipartite?

    (a) No, because a graph can be bipartite and have a cycle (for example, the cube graph), which means there are bipartite graphs that are not trees.

    (b) No, because a graph can be bipartite and not be connected (for example, a graph with two vertices and no edges), which means there are bipartite graphs that are not trees.

    (c) Yes, because all connected graphs are bipartite, which means that all trees are bipartite.

    (d) Yes, because all graphs without a cycle are bipartite, which means that all trees are bipartite.

    (d) is correct

    Answer (d) is correct.

    It is quite difficult to prove why the correct answer is correct.

    Fortunately, the other options are clearly incorrect.

    It is true that graphs can be bipartite and have a cycle, like the cube graph. However, the question is not about general graphs, but about trees. And any tree is without cycles, so the fact that some graphs that are certainly not trees are bipartite is not helping to solve the question.

It is also true that graphs can be bipartite and not be connected, like a graph with two vertices and no edges. However, again, the question is not about general graphs, but about trees. And any tree is connected, so the fact that some graphs that are certainly not trees are bipartite is again not helping to solve the question.

Note that it is certainly not true that all connected graphs are bipartite. For instance, the graph K_3 is connected, but cannot be colored with only two colors, so it is not bipartite.

So the first three answers are incorrect. So the fourth answer must be correct.

As not giving an argument why the correct answer is indeed correct, does not seem very convincing, we try to give the intuition behind the correct answer.

First, every graph without cycles is a forest. And a forest is a graph where each component is a tree. Obviously, as the components are not connected, we only have to look at the individual trees in the forest. So let us take an arbitrary tree from the forest. And let us take an arbitrary vertex in this tree. We color this vertex red and give it the special name p_1. We have seen in the exercises that between every pair p and q of vertices in a tree, there is exactly one path from p to q. Now let us focus on the (unique) paths from p_1 to all other vertices in the tree. Let q be a vertex different from p_1. Now if the length of the unique path from p_1 to q is odd, we color q blue, and if it is even, we color q red. As there is only a single path between p_1 and q, by construction, this gives that the chromatic number of this tree is 2, which implies that the tree is bipartite. So all trees in the forest are bipartite. So the entire forest and hence the entire graph is bipartite. And in particular, as a tree is just a forest of a single tree, it also holds that each tree is bipartite.
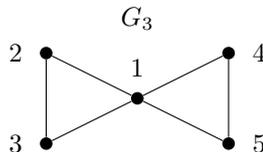
3. Give a graph that has an Eulerian circuit, but does not have a Hamiltonian circuit.

   Write the graph as $\langle V, E \rangle$ by giving V and E using set notation in the style:

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   V = { ... }
   E = { ... }
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   Take for instance this graph G_3:

   Here, the set of vertices V is $\{1, 2, 3, 4, 5\}$ and the set of edges E is $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{4, 5\}\}$.

   Now the graph G_3 has an Eulerian circuit:

   $$1 \to 2 \to 3 \to 1 \to 4 \to 5 \to 1$$

However, G_3 has no Hamiltonian circuit. If it would have such a circuit then the edges that go in and out of vertices with degree two have to be in the circuit. So $(1, 2)$, $(2, 3)$, $(3, 1)$, $(1, 4)$, $(4, 5)$, and $(5, 1)$ all have to be in the circuit. No matter in which order you put these edges to create a circuit, the vertex 1 is visited two times, which is not allowed in a Hamiltonian circuit.

Note that there is a simpler solution that is not connected:
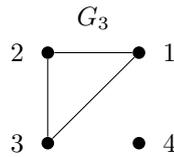
Here, the set of vertices V is $\{1, 2, 3, 4\}$ and the set of edges E is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. It clearly has an Eulerian circuit

$$1 \to 2 \to 3 \to 1$$

and it clearly has no Hamiltonian circuit as there is no circuit containing vertex 4.

4. We define the function a with two arguments using the recursion equations:

$$a(m, 0) = m$$
$$a(m, k + 1) = a(m, k) + 1$$

In this, both m and k range over all natural numbers.

What is the value of $a(3, 4)$?

(a) 3

(b) 4

(c) 7

(d) none of the above

Answer (c) is correct.

We unfold the definition of $a(3, 4)$ by writing it in the form $a(m, k + 1)$:

$$
\begin{aligned}
a(3, 4) &= a(3, 3 + 1) \\
&= a(3, 3) + 1 \\
&= a(3, 2 + 1) + 1 \\
&= a(3, 2) + 1 + 1 \\
&= a(3, 1 + 1) + 1 + 1 \\
&= a(3, 1) + 1 + 1 + 1 \\
&= a(3, 0 + 1) + 1 + 1 + 1 \\
&= a(3, 0) + 1 + 1 + 1 + 1 \\
&= 3 + 1 + 1 + 1 + 1 \\
&= 7
\end{aligned}
$$

3

5. Recall that we defined the function a with two arguments using the recursion equations:

$$a(m, 0) = m$$
$$a(m, k+1) = a(m, k) + 1$$

In this, again, both m and k range over all natural numbers.

Prove by induction that $a(0, n) = n$ for all natural numbers n.

Note: Although this exercise and the previous one are both about the same function, the result of this exercise is not usable for solving the previous exercise, because $a(3, 4)$ is not of the form $a(0, n)$.

Proposition:

$a(0, n) = n$ for all $n \geq 0$.

Proof <u>by induction</u> on n.

We first define our predicate P as:
$$P(n) \; := \; [\, a(0, n) = n \,]$$

Base Case. We show that $P(0)$ holds, i.e. we show that
$a(0, 0) = 0$
This indeed holds, because $a(0, 0) = 0$ by definition.

Induction Step. Let k be any natural number such that $k \geq 0$.
Assume that we already know that $P(k)$ holds, i.e. we assume that
$a(0, k) = k$ (<u>Induction Hypothesis</u> IH)
We now show that $P(k+1)$ also holds, i.e. we show that
$a(0, k+1) = k+1$
This indeed holds, because

$$\begin{aligned} a(0, k+1) \;&=\; a(0, k) + 1 \quad \text{by definition} \\ &=\; k + 1 \quad\quad\quad\; \text{by IH} \end{aligned}$$

Hence it follows by induction that $P(n)$ holds for all $n \geq 0$.

6. In the induction step of an induction proof, there occurs both a statement $P(k)$ and a statement $P(k+1)$. What is the case?

(a) The statements $P(k)$ and $P(k+1)$ both may be used in the part of the induction step that starts with 'This indeed holds, because . . .'

(b) The statements $P(k)$ and $P(k+1)$ both have to be shown to be true in the part of the induction step that starts with 'This indeed holds, because . . .'

(c) The statement $P(k)$ may be used, but the statement $P(k+1)$ has to be shown in the part of the induction step that starts with 'This indeed holds, because . . .'

(c) is correct

(d) The statement $P(k)$ has to be shown, but the statement $P(k+1)$ may be used in the part of the induction step that starts with 'This indeed holds, because . . .'

Answer (c) is correct.

Note that $P(k)$ is introduced in step 6: 'Assume that we already know that $P(k)$ holds, i.e. we assume that ...'.

And $P(k+1)$ is introduced in step 7: 'We now have to show that $P(k+1)$ also holds, i.e. we have to show that ...'.

So $P(k)$ can be used, whereas $P(k+1)$ has to be shown.

7. What is the number of edges in the graph K_n, for $n \geq 2$?

    (a) $\binom{n-1}{2}$

    (b) $\binom{n}{2}$

    (c) $\binom{n+1}{2}$

    (d) none of the above

Answer (b) is correct.

We can give both a combinatorial and a computational argument.

Let us start with the combinatorial argument, which is more elegant. By definition, edges connect two vertices. And because this is the complete graph K_n, we know that every pair of two vertices in the graph is connected by an edge. So if we derive the number of ways that we can pick two vertices from the set of n vertices, we also get the number of edges. By definition, picking two objects from a set of n distinguishable objects, where the order doesn't matter, can be done in $\binom{n}{2}$ ways.

The computational argument uses the fact that we have already seen in the course notes that the number of edges in K_n, the complete graph with n vertices, is equal to $\frac{1}{2}n(n-1)$.

So, to figure out which of the four options is correct, we just have to check whether one of the three options gives the same formula.

- The first option gives:

$$\binom{n-1}{2} = \frac{(n-1)!}{(n-1-2)! \cdot 2!} = \frac{(n-1) \cdot (n-2)}{2} = \frac{1}{2}(n-1)(n-2)$$

  So it is not correct.

- The second option gives:

$$\binom{n}{2} = \frac{n!}{(n-2)! \cdot 2!} = \frac{n \cdot (n-1)}{2} = \frac{1}{2}n(n-1)$$

  And hence this is the correct one!

- The third option gives:

$$\binom{n+1}{2} = \frac{(n+1)!}{(n+1-2)! \cdot 2!} = \frac{(n+1) \cdot n}{2} = \frac{1}{2}n(n+1)$$

  So it is not correct.

8. If we know that the number of ways to choose three elements out of five objects is 35, then it must be the case that:

Answer (c) is correct.

We know the formulas for these four situations of choosing k elements out of n objects:

(a) The order does matter and duplicates are allowed:

$$n^k = 5^3 = 125$$

(b) The order does matter, but duplicates are not allowed:

$$\frac{n!}{(n-k)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$$

(c) The order does not matter, but duplicates are allowed:

$$\binom{n+k-1}{k} = \binom{7}{3} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35$$

(d) The order does not matter and duplicates are not allowed:

$$\binom{n}{k} = \binom{7}{3} = \frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$