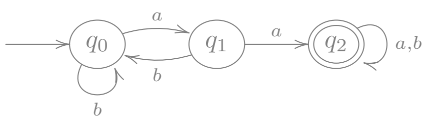
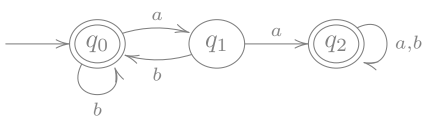
Formal Reasoning 2025
Solutions Test Block 4: Automata and Modal Logic
(18/12/25)

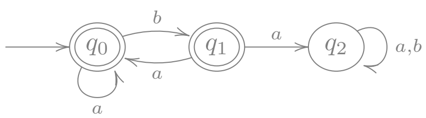
Automata

1. Which deterministic finite automaton accepts the language

$$L_1 := \{w \in \{a,b\}^* \mid w \text{ does not contain } aa\}$$

(a)


(b)


(c)


(d) is correct

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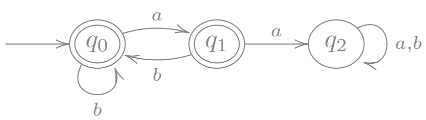
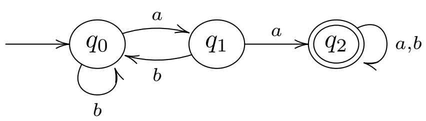
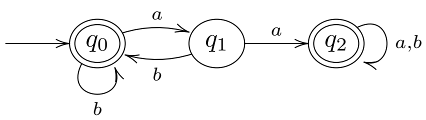


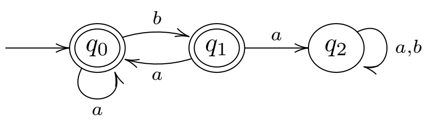
Answer (d) is correct.

(a)


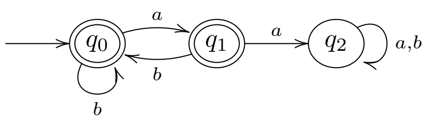
This automaton accepts aa.

(b)


This automaton accepts aa.

(c)


This is not a DFA as q_1 has two outgoing a transitions.

(d)


This is the correct one. Note that it is the complement of the one that accepts words that have aa in it.

2. Consider the context-free grammar

$$G_2 : \quad S \to aa \mid SbS$$

Give a deterministic finite automaton M_2 such that $\mathcal{L}(M_2) = \mathcal{L}(G_2)$. Provide the automaton by copying the list below and completing it by typing your answers in ASCII:

```
Set of states:
Initial state:
Set of final states:
delta(q0, a):
delta(q0, b):
```

Take for instance:

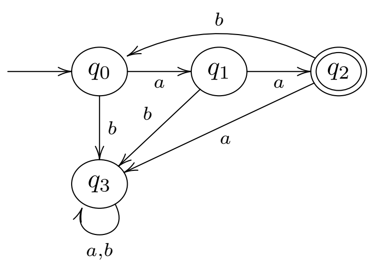
$$M_2 := \langle \{a, b\}, \{q_0, q_1, q_2, q_3\}, q_0, \{q_2\}, \delta \rangle$$

with

$$\begin{array}{cccc} \delta(q_0, a) = q_1 & \delta(q_1, a) = q_2 & \delta(q_2, a) = q_3 & \delta(q_3, a) = q_3 \\ \delta(q_0, b) = q_3 & \delta(q_1, b) = q_3 & \delta(q_2, b) = q_0 & \delta(q_3, b) = q_3 \end{array}$$

This automaton has three states, including one final state.

Represented as a diagram, this gives:



3. How many different deterministic finite automata M_3 over the alphabet $\Sigma := \{a, b\}$ exist, such that

- M_3 has at most two states,
- $\lambda \notin \mathcal{L}(M_3)$, and
- $ab \in \mathcal{L}(M_3)$.

(a) 0

(b) 4

(c) is correct — (c) 8

(d) 12

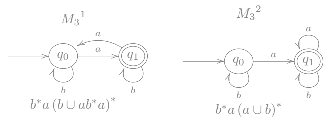
Answer (c) is correct.

The first claim is that from the last two requirements, it follows that the automaton should have *exactly* two states. Because if it has only one state which is not a final state, then $ab \notin \mathcal{L}(M_3)$, which is not allowed. And if the only state is a final state, then $\lambda \in \mathcal{L}(M_3)$, which is also not allowed. Let us call the initial state q_0 and the second state q_1.

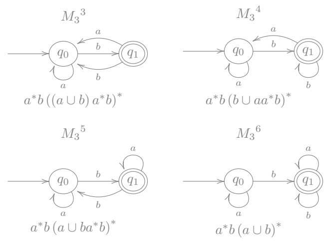
The second claim is that q_0 is not a final state, that there is at least one transition from q_0 to q_1 and q_1 is a final state. We have already seen that q_0 cannot be a final state. And if no states are final, then $ab \notin \mathcal{L}(M_3)$ which is not allowed. So q_1 must be a final state. And if there is no transition from q_0 to q_1, then again $ab \notin \mathcal{L}(M_3)$ which is not allowed.

For the transition(s) from q_0 to q_1 there are three mutually exclusive options.

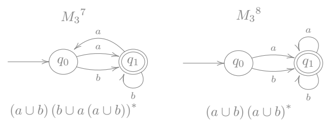
- There is a single transition labeled with a. So $\delta(q_0, a) = q_1$ and $\delta(q_0, b) = q_0$. As ab must end in final state q_1 this implies that $\delta(q_1, b) = q_1$. So the only thing we can actually choose in this case is whether $\delta(q_1, a) = q_0$ or $\delta(q_1, a) = q_1$, as both will do. This gives two different automata.



- There is a single transition labeled with b. So $\delta(q_0, a) = q_0$ and $\delta(q_0, b) = q_1$. This automatically implies that ab ends in final state q_1. So it doesn't matter what $\delta(q_1, a)$ and $\delta(q_1, b)$ are. This gives four different automata.

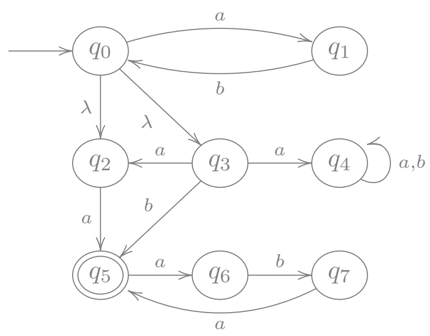


- There are two transitions labeled with a and b. So $\delta(q_0, a) = q_1$ and $\delta(q_0, b) = q_1$. As ab must end in final state q_1 this implies that $\delta(q_1, b) = q_1$. Again, the only thing we can choose in this case is whether $\delta(q_1, a) = q_0$ or $\delta(q_1, a) = q_1$, as both will do. This gives two different automata.



So in total, eight different automata comply with these requirements.

4. Consider the following non-deterministic finite automaton M_4:



What is $\mathcal{L}(M_4)$?

(a) $\mathcal{L}((ab)^*(a \cup b)(aba)^*)$
(b) $\mathcal{L}((ab)^*(\lambda \cup a)(a \cup b)(aba)^*)$
(c) is correct (c) $\mathcal{L}((ab)^*(a \cup aa \cup b)(aba)^*)$
(d) None of the above.

Answer (c) is correct.

Note that the sink doesn't add anything, as an NFA typically stops in a deadlock instead. The loop between the states q_0 and q_1 accepts all strings $(ab)^k$ for $k \geq 0$. The loop between the states q_5, q_6, and q_7 accepts all strings $(aba)^l$ for $l \geq 0$. And there are only three ways to get from q_0 to q_5:

- $q_0 \xrightarrow{\lambda} q_2 \xrightarrow{a} q_5$
- $q_0 \xrightarrow{\lambda} q_3 \xrightarrow{a} q_2 \xrightarrow{a} q_5$
- $q_0 \xrightarrow{\lambda} q_3 \xrightarrow{b} q_5$

So this leads to the language

$$L_4 := \{(ab)^k x (aba)^l \mid x \in \{a, aa, b\}\}$$

And using regular expressions, this is

$$L_4 := \mathcal{L}\left((ab)^*(a \cup aa \cup b)(aba)^*\right)$$

Modal logic

5. Consider the sentence

Always when it rains, I get wet.

If we translate this sentence to a formula using the modality of 'time' and the dictionary

R	it rains
W	I get wet

which formula is the best option?

(a) is correct (a) $\Box(R \to W)$

(b) $\Box R \to W$
(c) $R \to \Box W$
(d) $\Box R \to \Box W$

Answer (a) is correct.

If we translate the formulas back into English, while stressing the difference between $\Box f$ and f as in 'now and ever after f' respectively 'now f', we get:

(a) $\Box(R \to W)$ It is now and ever after the case that if it rains at that moment, I get wet at that moment.
(b) $\Box R \to W$ If it now and ever after rains, then I get wet now.
(c) $R \to \Box W$ If it rains now, then I get wet now and ever after.
(d) $\Box R \to \Box W$ If it now and ever after rains, then I get wet now and ever after.

Clearly, the first option makes the most sense.

6. In the logic T, the axiom schemes $\Box f \to f$ and $\Box f \to \Diamond f$ both hold for all formulas f. Is this logic appropriate for deontic logic?

(a) No, because it is possible to require something that is not forbidden.
(b) is correct
(b) No, because it is possible not to do things that are obligatory.
(c) Yes, because it is not possible not to do things that are obligatory.
(d) Yes, because it is not possible to require something that is forbidden.

Answer (b) is correct.

Deontic logic is about obligation. So the first axiom, named T, means *If f ought to be done, then f is done*, which is certainly not the case due to the reason explained in the second option.

In particular, this also implies that the third option can't be correct.

The second axiom, named D, means *If f ought to be done, then f is permissible*, which typically holds in deontic logic. This is actually what the fourth option states, however, as that option also states 'yes', it clearly can't be correct.

And the first option has no relation to the axioms mentioned, and therefore it makes no sense.

7. Provide a Kripke model $\mathcal{M}_7$ with at most three worlds $\{x_0, x_1, x_2\}$, which is serial, not reflexive, and such that the formula $\Box a \leftrightarrow \Diamond a$ holds, or more precisely, such that

$$\mathcal{M}_7 \vDash \Box a \leftrightarrow \Diamond a$$

Provide your answer by filling in the following table, which combines the definition of the model with a proof of correctness that your model satisfies the claim $\mathcal{M}_7 \vDash \Box a \leftrightarrow \Diamond a$.

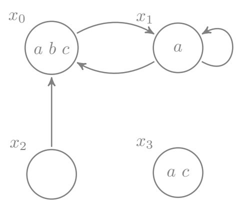
Do not forget to add a conclusion about what the table actually shows.

```
worlds ||     R     |     V     | ||- a  | ||- []a  | ||- <>a  | ||- []a <-> <>a
-----------------------------------------------------------------------------
x0     ||           |           |        |          |          |
x1     ||           |           |        |          |          |
x2     ||           |           |        |          |          |
       ||           |           |        |          |          |
```

As Ans doesn't like 'verbatim' text, the table is given as a Python code block, but don't worry about the syntax highlighting, just type your answers in ASCII.

Please keep the bars to separate the columns, as that way we can still parse the table if its alignment is not perfect. Note that if your model has fewer than three worlds, you can just leave out the worlds that you don't need. And if it has more than three worlds, add rows, even though your model is not allowed to have more than three worlds.

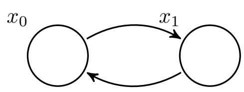
Hint: Below you can find an example of a Kripke model and what such a table should look like.



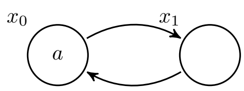
worlds	R	V	$\Vdash a$	$\Vdash b$	$\Vdash c$	$\Vdash \Box a$	$\Vdash b \vee c$	$\Vdash \Diamond(b \vee c)$	$\Vdash \Box a \leftrightarrow \Diamond(b \vee c)$
x_0	$\{x_1\}$	$\{a, b, c\}$	1	1	1	1	1	0	0
x_1	$\{x_0, x_1\}$	$\{a\}$	1	0	0	1	0	1	1
x_2	$\{x_0\}$	$\{\}$	0	0	0	1	0	1	1
x_3	$\{\}$	$\{a, c\}$	1	0	1	1	1	0	0

A Kripke model that is serial, needs to have an outgoing arrow for each world. A Kripke model that is not reflexive, is not allowed to have loops on the worlds. This means that it is impossible to have such a model with only one state, as the 'obligatory outgoing arrow' needs to go to a different world.

The simplest Kripke frame that meets this requirement looks like this:



Now let's turn this frame into a model by adding a valuation,



and see whether the claim holds or not by completing the table.

worlds	R	V	$\Vdash a$	$\Vdash \Box a$	$\Vdash \Diamond a$	$\Vdash \Box a \leftrightarrow \Diamond a$
x_0	x_1	a	1	0	0	1
x_1	x_0		0	1	1	1
x_2						

As each world in the model has a 1 in the final column, it follows that indeed

$$\mathcal{M}_7 \vDash \Box a \leftrightarrow \Diamond a$$

If we look at the formula and think a little bit about it, that is not a surprise. The formula $\Box a \leftrightarrow \Diamond a$ compares a $\Box$ with a $\Diamond$, or in other words, it compares a 'for all accessible worlds' with a 'there exists an accessible world'. In general, these properties are different. But as we have a model where each world has exactly one outgoing transition, both mean the same! So no matter which valuation is chosen for this frame, the formula will always hold in the whole model!

8. Consider the sentence

> *Some day after tomorrow it will always be true that it never rains for three consecutive days.*

Which LTL formula represents this sentence best, assuming that the time between x_i and x_{i+1} is a day? Use as a dictionary

R it rains

(b) is correct

(a) $\mathcal{F}\mathcal{X}\mathcal{G}\,((R \wedge \mathcal{X}R) \to \neg\mathcal{X}\mathcal{X}R)$
(b) $\mathcal{X}\mathcal{F}\mathcal{X}\mathcal{G}\,((R \wedge \mathcal{X}R) \to \mathcal{X}\mathcal{X}\neg R)$
(c) $\mathcal{G}\mathcal{X}\mathcal{F}\,((R \wedge \mathcal{X}R) \to \mathcal{X}\mathcal{X}\neg R)$
(d) $\mathcal{G}\mathcal{X}\mathcal{X}\mathcal{F}\,((\neg R \wedge \neg\mathcal{X}R) \vee \neg\mathcal{X}\mathcal{X}R)$

Answer (b) is correct.

In order to specify 'some day after tomorrow', we need to have two times $\mathcal{X}$ and one time $\mathcal{F}$, the order is not important. The second $\mathcal{X}$ is needed as otherwise the event could already take place tomorrow, instead of after tomorrow.

And in order to specify 'there is never a series of three consecutive days of rain' we use the $\mathcal{G}$ to indicate the 'never' and $R \wedge \mathcal{X}R$ as a check that it rains two consecutive days, and then the third day in the series, it should not rain, so $\mathcal{X}\mathcal{X}\neg R$.

Two of the options are lacking one $\mathcal{X}$, so these are clearly wrong.

And in one of the two options with the proper number of $\mathcal{X}$'s, the implication in $(R \wedge \mathcal{X}R) \to \neg\mathcal{X}\mathcal{X}R$ is replaced by the equivalent $(\neg\,(R \wedge \mathcal{X}R)) \vee \neg\mathcal{X}\mathcal{X}R$, but in the last step, De Morgan has been applied incorrectly as this formula is equivalent to $(\neg R \vee \neg\mathcal{X}R) \vee \neg\mathcal{X}\mathcal{X}R$, which is not the same as $(\neg R \wedge \neg\mathcal{X}R) \vee \neg\mathcal{X}\mathcal{X}R$.