Computability
Exam 2021–2022
1 April 2022
12:45–14:45

Before you continue, write your name and student number on your answer sheet. There are 9 exercises for which you can get 10 points each, the first 10 points are free, and the mark for this exam is the number of points divided by 10.

Turing machines always have to be given as a state diagram with circles and arrows and/or macros, and *not* as a table. In all Turing machines auxiliary symbols may be used.

Good luck!

1. Give an implementation of the macro CPY_1 (see the list on page 3) as a state diagram, i.e., without using other macros.
2. Give a deterministic 2-tape Turing machine M_2 with alphabet $\Sigma = \{a, b\}$ that reverses its input. For example $M_2(aab) = baa$.

   An input of length n should be reversed in at most $3n + 3$ steps.
3. Let be given a numeric function g that has arity 2, and a macro $\rightarrow\boxed{G}\rightarrow$ that computes this function. We define the unary function f by:

   $$f(x) := \mu n.\, g(n, x)$$

   Give a numeric Turing machine that computes the function f. You may use the macros on page 3.
4. Consider a primitive recursive function χ. Associated with this function we define a language:

   $$L_\chi := \{1^{n+1} \mid \chi(n) \neq 0\}$$

   Is L_χ recursive? Explain your answer.
5. Consider the Turing machine M_5 with code:

   $$R(M_5) = 0001010101010010111010111011000$$

   Describe the language $L(M_5)$.

   See page 4 for a relevant fragment from the book by Sudkamp.
6. The problem P_6 is defined as follows:

- input: a code of a Turing machine M
- question: does there exist a word $w \in \{0,1\}^*$ such that $M(w) = w$?

Show that P_6 is undecidable.

7. Which of the following implications hold for all unary functions f and g:

$$\begin{array}{rcl} f \circ g \text{ is total} & \Longrightarrow & f \text{ is total} \\ f \circ g \text{ is total} & \Longrightarrow & g \text{ is total} \end{array}$$

Explain your answers.

8. We define the unary function f_8 by:

$$f_8 := \textbf{primrec}\big(c_1^{(0)}, \mathsf{exp} \circ (c_2^{(2)}, p_2^{(2)})\big)$$

Give the recursion equations of f_8 and compute $f_8(3)$.

9. We define the unary functon f_9 by:

$$f_9(x) := \text{the least } n \text{ with } 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} \geq x$$

We have for example $f_9(0) = 0$, $f_9(1) = 1$, $f_9(2) = 4$ and $f_9(3) = 11$.

Show that f_9 is μ-recursive (it is also primitive recursive, but you do not need to show that). You may use that the functions on page 4 are primitive recursive.

Hint: Remember that in this course we only have natural numbers. Therefore, you should multiply with a sufficiently large number if you want to prevent the primive recursive version of division from truncating.

Macros for numeric Turing machines

$\longrightarrow\boxed{\text{S}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\quad BB$

$\underset{\uparrow}{B}\,\overline{n+1}\,B$

$\longrightarrow\boxed{\text{Z}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\quad\ B$

$\underset{\uparrow}{B}\,\overline{0}\ldots B$

$\longrightarrow\boxed{\text{E}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\,B$

does not terminate

$\longrightarrow\boxed{\text{P}_i^{(k)}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$\underset{\uparrow}{B}\,\overline{n_i}\ \ B\ldots\quad\ldots B$

$\longrightarrow\boxed{\text{C}_m^{(k)}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$\underset{\uparrow}{B}\,\overline{m}\ \ B\ldots\quad\ldots B$

$\longrightarrow\boxed{\text{A}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\,B\,\overline{m}\qquad B$

$\underset{\uparrow}{B}\,\overline{n+m}\,BBB$

$\longrightarrow\boxed{\text{D}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\qquad\quad B$

$\underset{\uparrow}{B}\,\overline{n\dot{-}1}\ldots B$

$\longrightarrow\boxed{\text{SUB}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\,B\,\overline{m}\quad\ B$

$\underset{\uparrow}{B}\,\overline{n\dot{-}m}\ \ldots B$

$\longrightarrow\boxed{\text{MULT}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\,B\,\overline{m}\,B\ldots$

$\underset{\uparrow}{B}\,\overline{n\cdot m}\,B\ldots$

$\longrightarrow\boxed{\text{MR}_k}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$B\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,\underset{\uparrow}{B}$

$\longrightarrow\boxed{\text{ML}_k}\longrightarrow$

$B\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,\underset{\uparrow}{B}$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$\longrightarrow\boxed{\text{FR}}\longrightarrow$

$\underset{\uparrow}{B}B^i\,\overline{n}\,B\qquad(i\geq 0)$

$B^i\underset{\uparrow}{B}\,\overline{n}\,B$

$\longrightarrow\boxed{\text{FL}}\longrightarrow$

$B\,\overline{n}\,B^i\underset{\uparrow}{B}\qquad(i\geq 0)$

$\underset{\uparrow}{B}\,\overline{n}\,B^iB$

$\longrightarrow\boxed{\text{E}_k}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$\underset{\uparrow}{B}\ldots\qquad\quad\ldots B$

$\longrightarrow\boxed{\text{T}}\longrightarrow$

$\underset{\uparrow}{B}B^i\,\overline{n}\,B\qquad(i\geq 0)$

$\underset{\uparrow}{B}\,\overline{n}\,B^iB$

$\longrightarrow\boxed{\text{INT}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n}\,B\,\overline{m}\,B$

$\underset{\uparrow}{B}\,\overline{m}\,B\,\overline{n}\,B$

$\longrightarrow\boxed{\text{CPY}_k}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B\ldots\qquad\quad\ldots B$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$\longrightarrow\boxed{\text{CPY}_{k,i}}\longrightarrow$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B\,\overline{n_{k+1}}\,B\ldots B\,\overline{n_{k+i}}\,B\ldots\qquad\quad\ldots B$

$\underset{\uparrow}{B}\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B\,\overline{n_{k+1}}\,B\ldots B\,\overline{n_{k+i}}\,B\,\overline{n_1}\,B\ldots B\,\overline{n_k}\,B$

$$\begin{tikzcd} {} \arrow[r] & \boxed{\text{BRN}} \arrow[r, "n=0"] \arrow[d, "n>0"] & {} \\ & {} & \end{tikzcd}$$

$\underset{\uparrow}{B}\,\overline{n}\,B$

$\underset{\uparrow}{B}\,\overline{n}\,B$

Coding transitions

Symbol	Encoding
0	1
1	11
B	111
q_0	1
q_1	11
$\vdots$	$\vdots$
q_n	1^{n+1}
L	1
R	11

Let $en(x)$ denote the encoding of a symbol x. A transition $\delta(q_i, x) = [q_j, y, d]$ is encoded by the string

$$en(q_i)0en(x)0en(q_j)0en(y)0en(d).$$

Primitive recursive functions

$$\begin{aligned}
\mathsf{id}(x) &= x \\
z(x) &= 0 \\
s(x) &= x+1 \\
p_i^{(k)}(x_1, \ldots, x_k) &= x_i \\
c_n^{(k)}(x_1, \ldots, x_k) &= n
\end{aligned}$$

$\mathsf{pred}(y) = y \dot{-} 1$

$\mathsf{add}(x, y) = x + y$

$\mathsf{mult}(x, y) = x \cdot y$

$\mathsf{sub}(x, y) = x \dot{-} y$

$\mathsf{exp}(x, y) = x^y$

$\mathsf{fact}(x) = x!$

$\mathsf{sg}(x) =$ if $x \neq 0$ then 1 else 0

$\mathsf{cosg}(x) =$ if $x \neq 0$ then 0 else 1

$\mathsf{lt}(x, y) =$ if $x < y$ then 1 else 0

$\mathsf{gt}(x, y) =$ if $x > y$ then 1 else 0

$\mathsf{le}(x, y) =$ if $x \leq y$ then 1 else 0

$\mathsf{ge}(x, y) =$ if $x \geq y$ then 1 else 0

$\mathsf{eq}(x, y) =$ if $x = y$ then 1 else 0

$\mathsf{ne}(x, y) =$ if $x \neq y$ then 1 else 0

$\mathsf{max}(x, y) =$ the maximum of x and y

$\mathsf{min}(x, y) =$ the minimum of x and y

$\mathsf{quo}(x, y) =$ if $y \neq 0$ then $\lfloor x/y \rfloor$ else 0

$\mathsf{rem}(x, y) =$ if $y \neq 0$ then x mod y else x

$\mathsf{divides}(x, y) =$ if $y \neq 0$ and $y \,|\, x$ then 1 else 0

$\mathsf{even}(x) =$ if x is even then 1 else 0

$\mathsf{prime}(x) =$ if x is prime then 1 else 0

$\mathsf{pn}(x) =$ the xth prime number
(so $\mathsf{pn}(0) = 2$, $\mathsf{pn}(1) = 3$, etc.)