Conversion to Continuation-Passing Style

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Conversion to Continuation-Passing Style

- We wish to apply CPS transforms to the language F_ω+control.
- 4 different evaluation strategies: CBV/CBN and Standard/ML-like.
- ML-like strategies only work for F_ω^-+control.

Transformation of Constructors

$$
\begin{aligned}
|A| &= (A^* \to \alpha) \to \alpha = \neg\neg A^* \qquad (\neg A = A \to \alpha) \\
\alpha^* &= \alpha \\
u^* &= u
\end{aligned}
$$

$$
\begin{aligned}
(A_1 \to A_2)^* &= A_1^* \to |A_2| && \text{(CBV)} \\
(A_1 \to A_2)^* &= |A_1| \to |A_2| && \text{(CBN)}
\end{aligned}
$$

$$
\begin{aligned}
(\forall u{:}K.A)^* &= \forall u{:}K.|A| && \text{(standard)} \\
(\forall u{:}K.A)^* &= \forall u{:}K.A^* && \text{(ML-like)}
\end{aligned}
$$

$$
\begin{aligned}
(\lambda u{:}K.A)^* &= \lambda u{:}K.A^* \\
(A_1 A_2)^* &= A_1^* A_2^*
\end{aligned}
$$

Example

Standard CBV:

$$
\begin{aligned}
&(\forall u{:}\Omega.(u \to u) \to (u \to u))^* \\
&= \forall u{:}\Omega.|(u \to u) \to (u \to u)| \\
&= \forall u{:}\Omega.\neg\neg((u \to u) \to (u \to u))^* \\
&= \forall u{:}\Omega.\neg\neg((u \to u)^* \to |u \to u|) \\
&= \forall u{:}\Omega.\neg\neg((u \to u)^* \to \neg\neg(u \to u)^*) \\
&= \forall u{:}\Omega.\neg\neg((u^* \to |u|) \to \neg\neg(u^* \to |u|)) \\
&= \forall u{:}\Omega.\neg\neg((u \to \neg\neg u) \to \neg\neg(u \to \neg\neg u))
\end{aligned}
$$

Standard CBN: $\forall u{:}\Omega.\neg\neg(\neg\neg(\neg\neg u \to \neg\neg u) \to \neg\neg(\neg\neg u \to \neg\neg u))$
ML-like CBV: $\forall u{:}\Omega.(u \to \neg\neg u) \to \neg\neg(u \to \neg\neg u)$
ML-like CBN: $\forall u{:}\Omega.\neg\neg(\neg\neg u \to \neg\neg u) \to \neg\neg(\neg\neg u \to \neg\neg u)$

Constructor transformation properties

Theorem (Constructor Well-formedness Preservation)

1. *If* $F_\omega \vdash \Delta \triangleright A : K$*, then* $F_\omega \vdash \Delta \triangleright A^* : K$*.*
2. *If* $F_\omega \vdash \Delta \triangleright A : \Omega$*, then* $F_\omega \vdash \Delta \triangleright |A| : \Omega$*.*

Theorem (Constructor Equality Preservation)

1. *If* $F_\omega \vdash \Delta \triangleright A_1 = A_2 : K$*, then* $F_\omega \vdash \Delta \triangleright A_1^* = A_2^* : K$*.*
2. *If* $F_\omega \vdash \Delta \triangleright A_1 = A_2 : \Omega$*, then* $F_\omega \vdash \Delta \triangleright |A_1| = |A_2| : \Omega$*.*

Theorem (Compositionality)

$([A_1/u]A_2)^* = [A_1^*/u]A_2^*$

Standard CPS Transforms

- We introduce a transformation for values, $(-)^*$, and a transformation for terms, $|-|$.
- We also introduce optimized versions, $(-)^\star$ and $|-|_Y$ (relative to continuation Y).
- The optimized version does β-reduction on a known continuation.

Call-by-Value

Definition (CBV CPS Transform for F_ω+control without types)

$$
\begin{aligned}
|\Delta;\Gamma \triangleright V : A| &= \lambda k \quad .kV^* \\
|M_1 M_2 : A| &= \lambda k \quad .|M_1|(\lambda x_1 \qquad . \\
&\qquad |M_2|(\lambda x_2 \quad .x_1 x_2 k)) \\
|M\{A_1\} : [A_1/u]A_2| &= \lambda k \qquad . \\
&\qquad |M|(\lambda x \qquad .x \quad k) \\
|\mathrm{abort}_A(M) : A| &= \lambda k \quad .|M|(\lambda m \quad .m) \\
|\mathrm{callcc}_A(M) : A| &= \lambda k \quad .|M|(\lambda m \qquad . \\
&\qquad m(\quad \lambda l \qquad .l(\lambda x \quad .\lambda k' \quad .kx))k) \\
|M : A'| &= |M|,
\end{aligned}
$$

Call-by-Value

Definition (CBV CPS Transform for F_ω+control with types)

$$
\begin{aligned}
|\Delta;\Gamma \triangleright V : A| &= \lambda k{:}\neg A^*.kV^* \\
|M_1 M_2 : A| &= \lambda k{:}\neg A^*.|M_1|(\lambda x_1{:}(A \to A_2)^*. \\
&\qquad |M_2|(\lambda x_2{:}A_2^*.x_1 x_2 k)) \\
&\text{where } M_1 : A_2 \to A \text{ and } M_2 : A_2 \\
|M\{A_1\} : [A_1/u]A_2| &= \lambda k{:}\neg([A_1/u]A_2)^*. \\
&\qquad |M|(\lambda x{:}(\forall u{:}K_1.A_2)^*.x\{A_1^*\}k) \\
|\mathrm{abort}_A(M) : A| &= \lambda k{:}\neg\alpha^*.|M|(\lambda m{:}\alpha^*.m) \\
|\mathrm{callcc}_A(M) : A| &= \lambda k{:}\neg A^*.|M|(\lambda m{:}((\forall u{:}\Omega.A \to u) \to A)^*. \\
&\qquad m(\Lambda u{:}\Omega.\lambda l{:}\neg(A \to u)^*.l(\lambda x{:}A^*.\lambda k'{:}\neg u^*.kx))k) \\
|M : A'| &= |M|, \text{ where } M : A \text{ and } A = A' : \Omega
\end{aligned}
$$

Call-by-Value

Definition (CBV CPS Transform for F_ω+control cont.)

$$
\begin{aligned}
(x : A)^* &= x \\
(\lambda x{:}A.M : A \to A')^* &= \lambda x{:}A^*.|M| \\
(\Lambda u{:}K.M : \forall u{:}K.A)^* &= \Lambda u{:}K.|M| \\
(V : A')^* &= V^*, \text{ where } V : A \text{ and } A = A' : \Omega
\end{aligned}
$$

Example

$$
\begin{aligned}
&(\Lambda u{:}\Omega.\lambda x{:}u.x)^* \\
&= \Lambda u{:}\Omega.|\lambda x{:}u.x| \\
&= \Lambda u{:}\Omega.\lambda k{:}\neg u^*.k(\lambda x{:}u.x)^* \\
&= \Lambda u{:}\Omega.\lambda k{:}\neg u.k(\lambda x{:}u^*.|x|) \\
&= \Lambda u{:}\Omega.\lambda k{:}\neg u.k(\lambda x{:}u.\lambda k'{:}\neg u^*.k'x^*) \\
&= \Lambda u{:}\Omega.\lambda k{:}\neg u.k(\lambda x{:}u.\lambda k'{:}\neg u.k'x)
\end{aligned}
$$

Theorem (CBV CPS Typing)

If F_ω+*control* $\vdash \Delta; \Gamma \triangleright M : A$*, then* $|M|$ *exists and is a strict CPS value such that* $F_\omega \vdash \Delta; \Gamma^* \triangleright |M| : |A|$*. If* M *is a CBV value, then* M^* *exists and is a strict CPS value such that* $F_\omega \vdash \Delta; \Gamma^* \triangleright M^* : A^*$*.*

Definition (Optimized CBV CPS Transform for F_ω+control)

$$
\begin{aligned}
|\Delta; \Gamma \triangleright V : A|_Y &= Y V^\star \\
|V_1 V_2 : A|_Y &= V_1^\star V_2^\star Y \\
|VN : A|_Y &= |N|_{Y'}, \text{ where } Y' = \lambda n{:}A_2^*. V^\star n Y \text{ and } N : A_2 \\
|MN : A|_Y &= |M|_{Y'}, \text{ where } N : A_2 \\
Y' &= \lambda m{:}(A_2 \to A)^*. |N|_{Y''} \\
Y'' &= \lambda n{:}A_2^*. m n Y \\
|V\{A_1\} : [A_1/u]A_2|_Y &= V^\star\{A_1^*\} Y \\
|M\{A_1\} : [A_1/u]A_2|_Y &= |M|_{Y'}, \text{ where } Y' = \lambda m{:}(\forall u{:}K.A_2)^*. m\{A_1^*\} Y
\end{aligned}
$$

Definition (Optimized CBV CPS Transform cont.)

$$
\begin{aligned}
|\mathrm{abort}_A(M) : A|_Y &= |M|_{\lambda x:\alpha.x} \\
|\mathrm{callcc}_A(M) : A|_Y &= |M|_{Y'} \\
Y' &= \lambda m{:}((\forall u{:}\Omega.A \to u) \to A)^*. \\
&\quad (\lambda n{:}(\forall u{:}\Omega.A \to u)^*.mnY)Y'' \\
Y'' &= \Lambda u{:}\Omega.\lambda l{:}\neg(A \to u)^*.l(\lambda x{:}A^*.\lambda k'{:}\neg u^*.Yx) \\
|M : A'|_Y &= |M|_Y, \text{ where } M : A \text{ and } A = A' : \Omega
\end{aligned}
$$

Definition (Optimized CBV CPS Transform cont.)

$$
\begin{aligned}
(x : A)^\star &= x \\
(\lambda x{:}A_1.M : A_1 \to A_2)^\star &= \lambda x{:}A_1^*.\lambda k{:}\neg A_2^*.|M|_k \\
(\Lambda u{:}K.M : \forall u{:}K.A)^\star &= \Lambda u{:}K.\lambda k{:}\neg A^*.|M|_k \\
(V : A') &= V^\star,\ \text{where } V : A \text{ and } A = A' : \Omega
\end{aligned}
$$

Theorem (Optimized CBV CPS Typing)

1. *If* F_ω*+control* $\vdash \Delta; \Gamma \triangleright M : A$ *and* Y *is a strict CPS value such that* $F_\omega \vdash \Delta; \Gamma \triangleright Y : \neg A^*$*, then* $|M|_Y$ *exists and is a strict CPS term such that* $F_\omega \vdash \Delta; \Gamma^* \triangleright |M|_Y : \alpha$*.*
2. *If* F_ω*+control* $\vdash \Delta; \Gamma \triangleright V : A$*, then* $V^\star$ *exists and is a strict CPS value such that* $F_\omega \vdash \Delta; \Gamma \triangleright V^\star : A^*$*.*

Theorem (Optimization)

$|M|Y \to^*_{\beta_v} |M|_Y$ *and* $V^* \to^*_{\beta_v} V^\star$

Theorem

1. *If M is not a value, then* $|E[M]|_Y = |M|_{|E|_Y}$.
2. *If V is a CBV value, then* $|V|_{|E|_Y} \to^*_{\beta_v} |E[V]|_Y$.

Thus $|M|_{|E|_Y} \to^*_{\beta_v} |E[M]|_Y$, *regardless of whether M is a value.*

Theorem (CBV Simulation)

If P is a program and $P \hookrightarrow_{cbv} Q$, then $|P|_{\lambda x:\alpha.x} \hookrightarrow^*_{\beta} |Q|_{\lambda x:\alpha.x}$. *Moreover, each β-step induces at least one β-step on the converted form.*

Theorem

For any program P,

1. *There exists a unique CBV value V such that* $P \hookrightarrow^*_{cbv} V$.
2. *If* $P \hookrightarrow^*_{cbv} V$ *then* $|P|(\lambda x{:}\alpha.x) \hookrightarrow^*_{\beta} V'$ *where* V' *is such that* $V^* \hookrightarrow^*_{\beta} V'$.

Call-by-Name

- We can reuse almost everything from the CBV case.
- We use the CBN version in the constructor transformation.
- In the CBV transform we modify the application and *callcc* clauses. Also variables are no longer considered values.

Definition (CBN CPS Transform for F_ω+control Modifications)

$$
\begin{aligned}
|\Delta; \Gamma \triangleright x : A| &= x \\
|M_1 M_2 : A| &= \lambda k{:}\neg A^*.|M_1|(\lambda x_1{:}(A_1 \to A_2)^*.x_1 |M_2| k) \\
&\quad \text{where } M_1 : A_2 \to A \text{ and } M_2 : A_2 \\
|callcc_A(M) : A| &= \lambda k{:}\neg A^*.|M|(\lambda m{:}((\forall u{:}\Omega.A \to u) \to A)^*.mYk) \\
Y &= \lambda l{:}\neg(\forall u{:}\Omega.A \to u)^*.l(\Lambda u{:}\Omega.\lambda l{:}\neg(A \to u)^*. \\
&\qquad l(\lambda x{:}|A|.\lambda k'{:}\neg u^*.xk)) \\
(\lambda x{:}A.M : A \to A')^* &= \lambda x{:}|A|.|M|
\end{aligned}
$$

Theorem (CBN CPS Typing)

If F_ω+control $\vdash \Delta; \Gamma \triangleright M : A$, then $|M|$ exists and is a strict CPS value such that $F_\omega \vdash \Delta; |\Gamma| \triangleright |M| : |A|$. If M is a CBN value, then M^* exists and is a strict CPS value such that $F_\omega \vdash \Delta; |\Gamma| \triangleright M^* : A^*$.

Theorem

Let P be a program.

1. *There exists a unique CBN value V such that $P \hookrightarrow^*_{cbn} V$.*
2. *If $P \hookrightarrow^*_{cbn} V$ then $|P|(\lambda x{:}\alpha.x) \hookrightarrow^*_\beta V'$ where V' is such that $V^* \hookrightarrow^*_\beta V'$.*

ML-like CPS Transforms

- In the standard constructor transform, we use the definition $(\forall u{:}K.A)^* = \forall u{:}K.|A|$.
- This means that constructor applications are 'serious' and need a continuation.
- However, in F_ω^-+control the continuation is always invoked with a value.
- To make these computations trivial and ML-like, we use the definition $(\forall u{:}K.A)^* = \forall u{:}K.A^*$.

ML-like CBV

Definition (ML-CBV CPS Transform for F_ω^-+control)

$$
\begin{aligned}
|M\{A\} : [A/u]B| &= \lambda k{:}\neg([A/u]B)^*.|M|(\lambda m{:}(\forall u{:}K.B)^*.k(m\{A^*\})) \\
|\mathsf{callcc}_A(M) : A| &= \lambda k{:}\neg A^*.|M|(\lambda m{:}((\forall u{:}\Omega.A \to u) \to A)^*. \\
&\qquad m(\Lambda u{:}\Omega.\lambda x{:}A^*.\lambda k'{:}\neg u^*.kx)k) \\
(\Lambda u{:}K.V : \forall u{:}K.A)^* &= \Lambda u{:}K.V^*
\end{aligned}
$$

Example

$$
\begin{aligned}
&(\Lambda u{:}\Omega.\lambda x{:}u.x)^* \\
&= \Lambda u{:}\Omega.(\lambda x{:}u.x)^* \\
&= \Lambda u{:}\Omega.(\lambda x{:}u^*.|x|) \\
&= \Lambda u{:}\Omega.(\lambda x{:}u.\lambda k{:}\neg u^*.kx^*) \\
&= \Lambda u{:}\Omega.(\lambda x{:}u.\lambda k{:}\neg u.kx)
\end{aligned}
$$

ML-like CBV

Theorem (ML-CBV Typing)

If F_ω^- *+control* $\vdash \Delta; \Gamma \triangleright M : A$, *then* $|M|$ *exists and is a relaxed CPS value such that* $F_\omega^- \vdash \Delta; \Gamma^* \triangleright |M| : |A|$.

This transformation is essentially a typed version of the usual untyped CBV CPS transform:

Theorem (ML-CBV Simulation)

If F_ω^- *+control* $\vdash \Delta; \Gamma \triangleright M : A$, *then* $|M|^\circ \hookrightarrow_\eta^* |M^\circ|_{ucbv}$.

Alternative ML-like CBN

- The ML-CBN' CPS transformation looks a lot like the standard CBN CPS transformation.
- The constructor transform is slightly different.
- It has the same constructor application and abstraction transform rules as ML-like CBV. Additionally, it has a different rule for *callcc*:

$$|\mathrm{callcc}_A(M) : A| = \lambda k{:}\neg A^*.|M|(\lambda m{:}((\forall u{:}\Omega.A \to u) \to A)^*.mYk)$$
$$Y = \lambda l{:}\neg(\forall u{:}\Omega.A \to u)^*.l(\Lambda u{:}\Omega.\lambda x{:}|A|.\lambda k'{:}\neg u^*.xk)$$

Alternative ML-like CBN

Theorem (ML-CBN' CPS Typing)

If $F_\omega^- + control \vdash \Delta; \Gamma \triangleright M : A$*, then* $|M|$ *exists and is a relaxed CPS value such that* $F_\omega^- \vdash \Delta; |\Gamma| \triangleright |M| : |A|$*.*

Theorem (ML-CBN' Simulation)

If $F_\omega^- + control \vdash \delta; \Gamma \triangleright M : A$*, then* $|M|^\circ \hookrightarrow_\eta^* |M^\circ|_{ucbn}$