Reduction strategies for CBN and CBV

Structural operational style and Felleisen style

Bas Broere

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Introduction

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- A language that emphasises *what* needs to be calculated, instead of *how* this needs to be done, i.e. what does it mean to calculate something?

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- "A language that takes functions seriously"
- A language that emphasises *what* needs to be calculated, instead of *how* this needs to be done, i.e. what does it mean to calculate something?

Important: It supports passing functions as arguments, returning them as value etc.
Basically, you can do everything with them.

Setting up an FP language

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   - Terms: $M, N ::= x \mid \lambda x.M \mid M\,N$;
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3. Add some stuff like primitive data-types and operations;
4. Make an execution model.

Evaluation strategies

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Difference evaluation and reduction strategies

An evaluation strategy tells you *when* you need to *evaluate* an expression.
A reduction strategy tells you *how* to *reduce* a complex expression.

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Call-by-value (CBV)

The argument expression is evaluated, and the resulting value is bound to the corresponding variable in the function.

Call-by-name (CBN)

The arguments to a function are not evaluated before the function is called.

Structural operation style (SOS)

One-step reductions:

$$(\lambda x.M)\, v \to M[x := v] \quad (\beta_v)$$

$$\frac{M \to M'}{M\, N \to M'\, N}\ (\text{app-l}) \qquad \frac{N \to N'}{v\, N \to v\, N'}\ (\text{app-r})$$

Examples

Recution steps

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- We cannot reduce under λ-abstractions (weak reduction). So this is **NOT** a thing:

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- It is deterministic, i.e. for every M there is at most one M' s.t. $M \to M'$.

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   Example: $\omega\,2 \to 2\,2 \not\to$.

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Reduction contexts (felleisen style)

Head reductions:

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$$\frac{M \to_\epsilon M'}{E[M] \to E[M']}\ (\text{context}),$$

with $E ::= [\,] \mid E\, b \mid v\, E$ terms with a hole $[\,]$ in it.

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Equivalence SOS and Felleisen

Remember the earlier example:

$$\frac{\dfrac{(\lambda x.x)\,1 \to 1}{(\lambda x.\lambda y.y\,x)\,((\lambda x.x)\,1) \to (\lambda x.\lambda y.y\,x)\,1}}{(\lambda x.\lambda y.y\,x)\,((\lambda x.x)\,1)\,(\lambda x.x) \to (\lambda x.\lambda y.y\,x)\,1\,(\lambda x.x)}$$

This corresponds to the context:

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Equivalence

SOS and Felleisen approach are one-to-one!

Result

From this equivalence, it follow that Felleisen style also has the previously discussed properties: Left-to-right, Call-by-value, Weak reduction, Deterministic*.

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*: The determinism is also a consequence of the following theorem:

Unique decomposition theorem

For all terms M, there exists at most one reduction context E and one term N, s.t. $M = E[N]$ and N can reduce by head-reduction.

Result

From this equivalence, it follow that Felleisen style also has the previously discussed properties: Left-to-right, Call-by-value, Weak reduction, Deterministic*.

*: The determinism is also a consequence of the following theorem:

Unique decomposition theorem

For all terms M, there exists at most one reduction context E and one term N, s.t. $M = E[N]$ and N can reduce by head-reduction.

Proof: Induction on the terms.

Reduction strategies for CBN

Call-by-name

The arguments to a function are not evaluated before the function is called (perform β-reduction asap).

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Examples

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CBV vs CBN

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CBN in CBV language

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Introduce functions $\lambda x.M$ to delay computation of M (thunks).

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3. $[\![M\,N]\!] = [\![M\,]\!](\lambda x.[\![N]\!])$,

where $()$ is a constant of type *unit* and $x \notin FV(N)$.

Example

Rules

$$[\![x]\!] \stackrel{1}{=} x\,(), \qquad [\![\lambda x.M]\!] \stackrel{2}{=} \lambda x.[\![M]\!] \qquad [\![M\,N]\!] \stackrel{3}{=} [\![M\,]\!](\lambda x.[\![N]\!])$$

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\end{aligned}$$

So we need to evaluate M twice now, just as in CBN.

Types

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Effect on types:

1. $M : A$, then $[\![M]\!] : [\![A]\!]$;
2. $M : A \to B$, then $[\![M]\!] : (\mathit{unit} \to [\![A]\!]) \to [\![B]\!]$;
3. $[\![\mathit{basetype}]\!] = \mathit{basetype}$.

CBV in CBN language

This is hard!

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We will read about it

:)