Type Theory and Coq

Radboud University Nijmegen, The Netherlands

Lecture 3

The Church-Rosser Property and Principal Types

Today's lecture

What do we want to prove about type systems? So: what about the meta theory of type theory

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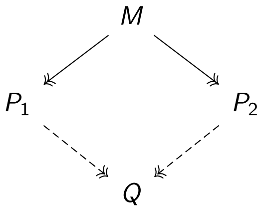
What do we want to prove about type systems? So: what about the meta theory of type theory

- Church-Rosser (confluence) of reduction
- Type inference (inferring principal types)

More properties are of interest, such as (strong) normalization, but that is not for today

Part 1: Church-Rosser property, CR



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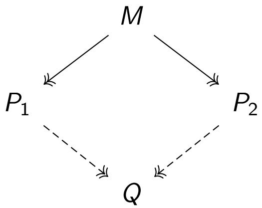

CHURCH-ROSSER THEOREM for β-reduction, CR_β.
If $M \twoheadrightarrow_\beta P_1$ and $M \twoheadrightarrow_\beta P_2$, then $\exists Q(P_1 \twoheadrightarrow_\beta Q \wedge P_2 \twoheadrightarrow_\beta Q)$

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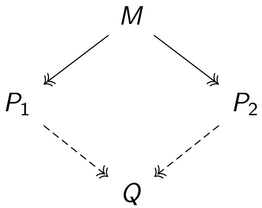

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$M \twoheadrightarrow P$ iff there is a multi-step (0 or more) reduction from M to P.

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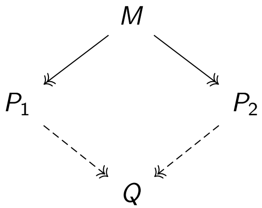

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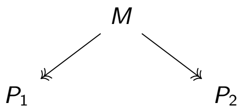
We will prove the Church-Rosser Theorem for β-reduction of the untyped λ-calculus in this lecture.

Church-Rosser (for β) example

$$(\lambda x. y\, x\, x)(\mathbf{I}\,\mathbf{I})$$

Corollaries of the Church-Rosser property

THEOREM CR($\rightarrow_R$) implies UN($\rightarrow_R$) (Uniqueness of Normal forms)



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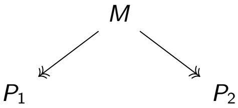

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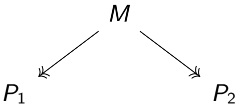

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PROOF: To decide $a =_R b$, just rewrite a and b until you find their normal forms a' and b'. Due to UN (which follows form CR), we have $a =_R b$ iff $a' = b'$.

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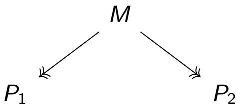

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Foreshadowing: decidability of $=_\beta$ is crucial for decidability of type checking! We will see the conversion rule (next lecture):

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}\ A =_\beta B$$

Parallel reduction in untyped λ-calculus

We prove CR(β) using parallel reduction, a method due to Tait and Martin-Löf and refined by Takahashi.

Parallel reduction $M \Longrightarrow P$ allows to contract several redexes in M in one step. It can be defined inductively.

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Definition

$$\frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{(\lambda x.M)P \Longrightarrow M'[x := P']}\,(\beta) \qquad \frac{M \Longrightarrow M' \qquad P \Longrightarrow P'}{M\,P \Longrightarrow M'\,P'}\,(\text{app})$$

$$\frac{M \Longrightarrow M'}{\lambda x.M \Longrightarrow \lambda x.M'}\,(\lambda) \qquad \frac{}{x \Longrightarrow x}\,(\text{var})$$

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Examples:

$$(\lambda x.y\,x\,x)(\mathbf{I}\,\mathbf{I}) \qquad (\lambda x.x\,(x\,\mathbf{I}))(\mathbf{I}\,\mathbf{I})$$

Properties of parallel reduction

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3. If $M \Longrightarrow P$, then $M \twoheadrightarrow_\beta P$.
   The proof is by induction on the derivation.

Parallel reduction satisfies a strong Diamond Property (I)

Theorem

$$\forall M \, \exists Q \, \forall P \, (\text{if } M \Longrightarrow P \text{ then } P \Longrightarrow Q).$$

This immediately implies DP($\Longrightarrow$) (and thereby CR(β)).

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DEFINITION

$$\begin{array}{rcl} x^* & := & x \\ (\lambda x.M)^* & := & \lambda x.M^* \\ ((\lambda x.P)\, N)^* & := & P^*[x := N^*] \\ (M\, N)^* & := & M^*\, N^* \text{ if } M \neq \lambda x.P \ (M \text{ is not a } \lambda\text{-abstraction}) \end{array}$$

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PROOF by induction on the derivation of $M \Longrightarrow P$. There are 4 cases. We treat 3 of them.

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Then indeed $x \Longrightarrow x^*$ (because $x^* = x$).

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We have $(\lambda x.M)^* = \lambda x.M^*$.
$\lambda x.M' \Longrightarrow \lambda x.M^*$ follows immediately from IH and the definition of $\Longrightarrow$.

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PROOF continued

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To prove this we need a separate

SUBSTITUTION LEMMA If $M \Longrightarrow M'$ and $P \Longrightarrow P'$, then $M[x := P] \Longrightarrow M'[x := P']$.

This is proved by induction on the structure of M.

DP($\Longrightarrow$) implies CR(β)

The proof that DP($\Longrightarrow$) implies CR(β) follows from the properties we have established:

1. If $M \to_\beta P$, then $M \Longrightarrow P$.
2. If $M \Longrightarrow P$, then $M \twoheadrightarrow_\beta P$.
3. If $M \Longrightarrow P$, then $P \Longrightarrow M^*$.

Example

$$\begin{aligned} x^* &:= x \\ (\lambda x.M)^* &:= \lambda x.M^* \\ (M\,N)^* &:= P^*[x := N^*] \text{ if } M = \lambda x.P \\ &:= M^*\,N^* \text{ otherwise.} \end{aligned}$$

$$(\lambda x.y\,x\,x)(\mathbf{I}\,\mathbf{I})$$

$$(\lambda z.z\,z)\,(\mathbf{I}\,(\mathbf{I}\,x))$$

This is a flexible proof of Church-Rosser

- Methods works for proving CR for reduction in Combinatory Logic
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- Method extends to typed lambda calculus with data types, for example natural numbers:

$$M, N := x \mid M\,N \mid \lambda x.M \mid 0 \mid \textbf{suc}\,M \mid \textbf{nrec}\,M\,N\,P$$

with

$$\begin{array}{rcl} \textbf{nrec}\,M\,N\,0 & \to & M \\ \textbf{nrec}\,M\,N\,(\textbf{suc}\,P) & \to & N\,P\,(\textbf{nrec}\,M\,N\,P) \end{array}$$

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- Method extends to η-reduction:

$$\lambda x.M\,x \to_\eta M \qquad \text{if } x \notin \mathrm{FV}(M)$$

Part 2: Principal Typing

Why do programmers want types?

- Types give a (partial) specification
- Typed terms can't go wrong (Milner)
  Subject Reduction property: If $M : A$ and $M \twoheadrightarrow_\beta N$, then $N : A$.
- Typed terms always terminate
- The type checking algorithm detects (simple) mistakes

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But:

- The compiler should compute the type information for us! (Why would the programmer have to type all that?)
- This is called a type assignment system, or also typing à la Curry:
- For M an untyped term, the type system assigns a type σ to M (or not)

Simple Type Theory à la Church and à la Curry

$\lambda\!\to$ (à la Church):

$$\frac{x{:}\sigma \in \Gamma}{\Gamma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma\to\tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \qquad \frac{\Gamma, x{:}\sigma \vdash P : \tau}{\Gamma \vdash \lambda x{:}\sigma.P : \sigma\to\tau}$$

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Type Assignment systems

- With typed assignment also called typing à la Curry, we assign types to untyped λ-terms
$$\lambda x.x : \alpha \to \alpha$$
- As a consequence:
  - Terms do not have unique types,
  - A principal type can be computed using unification.

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  - Terms do not have unique types,
  - A principal type can be computed using unification.
- Example:

$$\lambda x.\lambda y.y(\lambda z.x)$$

  can be assigned the types
  - $\alpha \rightarrow ((\beta \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$
  - $(\alpha \rightarrow \alpha) \rightarrow ((\beta \rightarrow \alpha \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma$
  - ...

  with $\alpha \rightarrow ((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma$ being the principal type

Example of computing a principal type

Consider $\lambda x.\lambda y.y\,(\lambda z.y\,x)$

Example of computing a principal type

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2. Assign type vars to all applicative subterms: $y\,x$ and $y(\lambda z.y\,x)$:

$$\lambda x^{\alpha}.\lambda y^{\beta}.\underbrace{y^{\beta}(\lambda z^{\gamma}.\overbrace{y^{\beta}x^{\alpha}}^{\delta})}_{\varepsilon}$$

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4. Find a most general unifier (a substitution) for the type vars that solves the equations: $\alpha := \gamma \to \varepsilon,\ \beta := (\gamma \to \varepsilon) \to \varepsilon,\ \delta := \varepsilon$

Example of computing a principal type

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4. Find a most general unifier (a substitution) for the type vars that solves the equations: $\alpha := \gamma \to \varepsilon,\ \beta := (\gamma \to \varepsilon) \to \varepsilon,\ \delta := \varepsilon$
5. The principal type of $\lambda x.\lambda y.y(\lambda z.yx)$ is now

$$(\gamma \to \varepsilon) \to ((\gamma \to \varepsilon) \to \varepsilon) \to \varepsilon$$

Example of computing a principal type (II)

$$\lambda x.\lambda y.x\,(y\,x)$$

Which of these terms is typable?

- $M_1 := \lambda x.x\,(\lambda y.y\,x)$
- $M_2 := \lambda x.\lambda y.x\,(x\,y)$
- $M_3 := \lambda x.\lambda y.x\,(\lambda z.y\,x)$

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- $M_2 := \lambda x.\lambda y.x\,(x\,y)$
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Poll:

A M_1 is not typable, M_2 and M_3 are typable.
B M_2 is not typable, M_1 and M_3 are typable.
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Principal Types: Definitions

- A type substitution (or just substitution) is a map S from type variables to types with a finite domain such that variables that occur in the range of S are not in the domain of S.
- A substitution S is written as $[\alpha_1 := \sigma_1, \ldots, \alpha_n := \sigma_n]$ where
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Principal Types: solving a list of equations

All notions generalize to lists of equations

$$\mathcal{E} = \langle \sigma_1 = \tau_1, \ldots, \sigma_n = \tau_n \rangle$$

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Computability of most general unifiers

THEOREM There is an algorithm U that, given a list of equations $\mathcal{E} = \langle \sigma_1 = \tau_1, \ldots, \sigma_n = \tau_n \rangle$ outputs

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Principal type

DEFINITION σ is a principal type for the untyped closed λ-term M if

- $\vdash M : \sigma$ in $\lambda\!\rightarrow$ à la Curry
- for all types τ, if $\vdash M : \tau$, then $\tau = \sigma\, T$ for some substitution T.

Principal Type Theorem

THEOREM There is an algorithm PT that, when given an (untyped) closed λ-term M, outputs

- A principal type σ for M if M is typable in $\lambda\!\rightarrow$ à la Curry.
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PROOF In the algorithm we

- first label the bound variables and all applicative sub-terms with type variables, and we give the candidate type τ,
- then we generate the equations that need to hold for the term to be typable,
- then we compute the mgu of this set of equations and we obtain the substitution S or "Fail",
- then we have as output the principal type $\tau\,S$ or "Fail".

The proof that this output indeed correctly computes the principal type can be found in the literature.

Conclusion

Today we saw

- A proof of the Church-Rosser property for the untyped λ-calculus
- Key technique: **parallel reduction**
- We also an algorithm to assign types to untyped λ-terms
- Important: this algorithm finds **principal types**