(3.2) THEOREM. *Any non-degenerate monotone image of an interval is homeomorphic with the interval.*

Proof. Let $f(J) = E$ be monotone, where $J = (0, 1)$ and E is non-degenerate. Let $I(1/2)$ be the closure of the interval $J - f^{-1}f(0) - f^{-1}f(1)$ and let $x(1/2)$ be its mid point. Similarly let $I(1/2^2)$ and $I(3/2^2)$ be the closures of the left and right intervals remaining on the deletion of $f^{-1}f[x(1/2)]$ from $I(1/2)$ and let $x(1/2^2)$ and $x(3/2^2)$ be their respective mid points. Likewise $I(1/2^3), I(3/2^3), I(5/2^3), I(7/2^3)$, are the closures of the intervals into which $I(1/2^2)$ and $I(3/2^2)$ are divided by removing $f^{-1}f[x(1/2^2)]$ and $f^{-1}f[x(3/2^2)]$ ordered from left to right, and so on indefinitely. In this way we define a collection of intervals $I(m/2^n)$ and their mid points $x(m/2^n)$ for all dyadic rational numbers $m/2^n$, $0 \leqq m \leqq 2^n$, so that the length of $I(m/2^n)$ is $\leqq 1/2^{n-1}$.

Now for any dyadic rational $m/2^n$ on J we define $h(m/2^n) = f[x(m/2^n)]$. We next show that h is uniformly continuous. Let $\epsilon > 0$ be given. By uniform continuity of f there exists a $\delta > 0$ such that for any interval H in J of length $<\delta$, $f(H)$ is of diameter $<\epsilon/2$. Let n be chosen so that $1/2^{n-1} < \delta$. Then if t_1 and t_2 are points of the set D of dyadic rationals with $|t_1 - t_2| < 1/2^n$, there is at least one point $t = j/2^n$ such that for each i $(i = 1, 2)$, t is an end point of an interval T_i of the nth dyadic subdivision of J containing t_i. If S_1 and S_2 are the corresponding intervals to T_1 and T_2 in the set $I(m/2^{n+1})$, since each is of length $<\delta$ we have

$$h(t_i) + h(t) \subset f(S_i), \quad i = 1, 2, \quad \text{and} \quad \delta[f(S_1) + f(S_2)] < \epsilon$$

since $\delta[f(S_i)] < \epsilon/2$, $i = 1, 2$. Accordingly, $\rho[h(t_1), h(t_2)] < \epsilon$ and h is uniformly continuous on D.

Let h be extended continuously to $\bar{D} = J$. Then $h(J) = f(J) = E$, because for each n the union of the intervals $I(m/2^n)$ maps onto E under f, so that the images of the mid points of all these intervals for all n, i.e., the set $h(D)$, is dense in E. Finally, h is (1–1). For if x_1 and x_2 are distinct points of J, there exists a point $t = j/2^n$ between x_1 and x_2 with $h(t) \neq h(x_1)$. Then if H_1 and H_2 are the closed intervals into which J is divided by $f^{-1}h(t)$ where $x_1 \varepsilon H_1$, $f(H_1) \cdot f(H_2) = h(t)$ by monotoneity of f and thus $h(H_1) \cdot h(H_2) = h(t)$ since $h(H_i) \subset f(H_i)$ by definition of h. Accordingly $h(x_1)$ non ε $h(H_2)$ so that $h(x_1) \neq h(x_2)$. Thus $h(J) = E$ is a homeomorphism.

4. Arcwise connectedness. Accessibility. A set T homeomorphic with a straight line interval is called a *simple arc*. If a and b are the points of such an arc T which correspond to the end points of the interval under the homeomorphism, then a and b are called the end points of the simple arc T and T is said to *join* a and b. The arc T is written ab, and the set $T - (a + b)$ is written $\widehat{ab}$ or (ab).