Huygens College Reflection

Assignment 4, Tuesday, Dec. 8, 2015

In the exercises, $=$ should be read as $=_\beta$ and $\equiv$ should be read as $\equiv_\alpha$.

1. Write down for each item a term $F \in \Lambda$, with a minimal number of variables, satisfying:

$$\begin{array}{rrcl} \text{(i)} & F\,x & = & x\,(x\,x) \\ \text{(ii)} & F\,x & = & x\,\mathbf{I} \\ \text{(iii)} & F\,x\,y & = & y\,x \\ \text{(iv)} & F\,x & = & y\,x \\ \text{(v)} & F\,x & = & x\,F\,\mathbf{I}. \end{array}$$

2. Let $W \equiv \lambda xy.x\,y\,y$. Draw $\mathcal{G}(W\,W\,W)$.
   [Hint. This graph consists of exactly four terms.]

3. (Exercise of [Jan Willem Klop])
   Let $Y = L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L$, where

$$L = \lambda abcdefghijklmnopqstuvwxyzr.r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r).$$

   Show that for all $F \in \Lambda$ one has $Y\,F = F\,(Y\,F)$. (So Y is a fixed point combinator.)

4. (a) Write down precisely a lambda term M such that

$$M\,x = x\,M\,x.$$

   Can you make it satisfy $M\,x \twoheadrightarrow x\,M\,x$?

   (b) (Turing) Consider the term $\Theta := (\lambda xy.y(x\,x\,y))\,(\lambda xy.y(x\,x\,y))$. Show that Θ is a *reducing fixed point combinator*, that is:

$$\Theta\,F \twoheadrightarrow F(\Theta\,F) \text{ for all } F.$$

Answer Info: ....

i. We recall the fixed-point combinator $\mathbf{Y} := \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$. *We define* $F \equiv \lambda mx.xmx$. *By the fixed-point theorem we obtain the term* $M := \mathbf{Y}F$ *such that* $FM = M$. *Now we have*

$$Mx = FMx \equiv (\lambda mx.xmx)Mx \twoheadrightarrow xMx.$$

It is not the case that $Mx \twoheadrightarrow xMx$.

ii. We have to show that $\Theta F \twoheadrightarrow F(\Theta F)$.

$$\begin{aligned} \Theta F &\equiv (\lambda xy.y(xxy))\;(\lambda xy.y(xxy))\;F \\ &\twoheadrightarrow F\;((\lambda xy.y(xxy))\;(\lambda xy.y(xxy))\;F) \\ &\equiv F(\Theta F) \end{aligned}$$

Taking Θ *as fixed point combinator, we would have* $Mx \twoheadrightarrow xMx$ *in teh first part of this exercise.*

End Answer Info ....