Huygens College Reflection

Assignment 7, Tuesday, Jan. 12, 2016

Remember the definitions of **true** $:= \lambda xy.x (\equiv \mathbf{K})$ and **false** $= \lambda xy.y (=_\beta \mathbf{KI})$, which are also written (on the slides of the course) as $\mathbf{T} := \lambda xy.x$ and $\mathbf{F} := \lambda xy.y$.

Exercise 1

(i) Show that there is a λ-term G that checks whether a term is an abstraction, that is:

$$\begin{aligned} G\ulcorner x\urcorner &= \mathbf{F} \\ G\ulcorner PQ\urcorner &= \mathbf{F} \\ G\ulcorner \lambda x.P\urcorner &= \mathbf{T}. \end{aligned}$$

(See also the last exercise of the previous week.)

(ii) Show that there is a term R such that

$$\begin{aligned} R\ulcorner M\urcorner &= \mathbf{T} \text{ if } M \text{ is a redex} \\ R\ulcorner M\urcorner &= \mathbf{F} \text{ if } M \text{ is not a redex} \end{aligned}$$

(A redex is a term of the shape $(\lambda x.P)Q$.)

(iii) Show that there is a λ-term Norm that checks if a term is in *normal form*, so

$$\begin{aligned} \text{Norm}\ulcorner M\urcorner &= \mathbf{T} \text{ if } M \text{ is in normal form} \\ \text{Norm}\ulcorner M\urcorner &= \mathbf{F} \text{ if } M \text{ is not in normal form} \end{aligned}$$

(A term is *in normal form* if it contains no redex.)

(iv) You may need to define a conjunction & on Booleans first satisfying:

$$\begin{aligned} \&\ \mathbf{T}\,\mathbf{T} &= \mathbf{T} \\ \&\ \mathbf{T}\,\mathbf{F} &= \mathbf{F} \\ \&\ \mathbf{F}\,\mathbf{T} &= \mathbf{F} \\ \&\ \mathbf{F}\,\mathbf{F} &= \mathbf{F} \end{aligned}$$

Exercise 2

(i) Show that there is no term H' satisfying

$$\begin{aligned} H'\ulcorner M\urcorner\ulcorner N\urcorner &= \mathbf{T} \text{ if } M\,N \text{ has a normal form} \\ H'\ulcorner M\urcorner\ulcorner N\urcorner &= \mathbf{F} \text{ if } M\,N \text{ has no normal form} \end{aligned}$$

Hint: Show that, if H' exists, then we can also define the term B that solves the "blank tape" problem. (See course slides.)

(ii) Show that there is no term T satisfying

$$\begin{array}{rcl} T\ulcorner M\urcorner & = & \mathbf{T} \text{ if } M\,N \text{ has a normal form for all } N \\ T\ulcorner M\urcorner & = & \mathbf{F} \text{ if } M\,N \text{ has no normal form for some } N \end{array}$$

Hint: Again, we can reduce B to T.

Exercise 3

(* Meaning: slightly more difficult) Show that there is a term C that contracts the *left-most redex* in a term M, so

$$C\ulcorner M\urcorner = \ulcorner N\urcorner$$

if N arises from M by contracting the left-most redex, if M contains a redex, and otherwise $N \equiv M$.