Huygens College Reflection

Assignment 7, Tuesday, Jan. 12, 2016

Remember the definitions of **true** $:= \lambda xy.x (\equiv \mathbf{K})$ and **false** $= \lambda xy.y (=_\beta \mathbf{KI})$, which are also written (on the slides of the course) as $\mathbf{T} := \lambda xy.x$ and $\mathbf{F} := \lambda xy.y$.

Exercise 1

(i) Show that there is a λ-term G that checks whether a term is an abstraction, that is:

$$\begin{array}{rcl} G\ulcorner x\urcorner & = & \mathbf{F} \\ G\ulcorner PQ\urcorner & = & \mathbf{F} \\ G\ulcorner \lambda x.P\urcorner & = & \mathbf{T}. \end{array}$$

(See also the last exercise of the previous week.)

(ii) Show that there is a term R such that

$$\begin{array}{rcl} R\ulcorner M\urcorner & = & \mathbf{T} \text{ if } M \text{ is a redex} \\ R\ulcorner M\urcorner & = & \mathbf{F} \text{ if } M \text{ is not a redex} \end{array}$$

(A redex is a term of the shape $(\lambda x.P)Q$.)

(iii) Show that there is a λ-term Norm that checks if a term is in *normal form*, so

$$\begin{array}{rcl} \text{Norm}\ulcorner M\urcorner & = & \mathbf{T} \text{ if } M \text{ is in normal form} \\ \text{Norm}\ulcorner M\urcorner & = & \mathbf{F} \text{ if } M \text{ is not in normal form} \end{array}$$

(A term is *in normal form* if it contains no redex.)

(iv) You may need to define a conjunction & on Booleans first satisfying:

$$\begin{array}{rcl} \& \,\mathbf{T}\,\mathbf{T} & = & \mathbf{T} \\ \& \,\mathbf{T}\,\mathbf{F} & = & \mathbf{F} \\ \& \,\mathbf{F}\,\mathbf{T} & = & \mathbf{F} \\ \& \,\mathbf{F}\,\mathbf{F} & = & \mathbf{F} \end{array}$$

Solution: In the last exercise of previous week you were asked to give these terms completely. Now we just show these terms exists, using the Recursion Theorem II of the slides. So we have to give the appropriate A_1, A_2 and A_3.

(i) Take

$$\begin{array}{rcl} A_1 & := & \lambda xh.\mathbf{F} \\ A_2 & := & \lambda xyh.\mathbf{F} \\ A_3 & := & \lambda xh.\mathbf{T} \end{array}$$

(ii) Take (using G of the first item)

$$\begin{array}{lcl} A_1 & := & \lambda xh.\mathbf{F} \\ A_2 & := & \lambda xyh.Gx \\ A_3 & := & \lambda xh.\mathbf{F} \end{array}$$

(iii) Take (using G and R of the first two items)

$$\begin{array}{lcl} A_1 & := & \lambda xh.\mathbf{T} \\ A_2 & := & \lambda xyh.\ \&\ (\ \&\ (h\,x)(h\,y))(\neg(G\,x)) \\ A_3 & := & \lambda xh.h(x\,y) \end{array}$$

where y is some "fresh variable", that is, a variable that doesn't occur anywhere. Here $\neg$ is the negation: $\neg := \lambda x.x\,\mathbf{F}\,\mathbf{T}$.

(iv) Take $\& = \lambda xy.xyx$.
Then $\&\ \mathbf{T} = \lambda y.\mathbf{T}\,y\,\mathbf{T} = \lambda y.y = \mathbf{I}$ and $\&\ \mathbf{F} = \lambda y.\mathbf{F}\,y\,\mathbf{F} = \lambda y.\mathbf{F}$.
So $\&$ works as intended

Another option is $\& = \langle\langle\mathbf{T},\mathbf{F}\rangle,\langle\mathbf{F},\mathbf{F}\rangle\rangle$.

Exercise 2

(i) Show that there is no term H' satisfying

$$\begin{array}{lcl} H'\ulcorner M\urcorner\ulcorner N\urcorner & = & \mathbf{T} \text{ if } M\,N \text{ has a normal form} \\ H'\ulcorner M\urcorner\ulcorner N\urcorner & = & \mathbf{F} \text{ if } M\,N \text{ has no normal form} \end{array}$$

Hint: Show that, if H' exists, then we can also define the term B that solves the "blank tape" problem. (See course slides.)

(ii) Show that there is no term T satisfying

$$\begin{array}{lcl} T\ulcorner M\urcorner & = & \mathbf{T} \text{ if } M\,N \text{ has a normal form for all } N \\ T\ulcorner M\urcorner & = & \mathbf{F} \text{ if } M\,N \text{ has no normal form for some } N \end{array}$$

Hint: Again, we can reduce B to T.

Solution:

(i) Suppose there is a term H' satisfying

$$\begin{array}{lcl} H'\ulcorner M\urcorner\ulcorner N\urcorner & = & \mathbf{T} \text{ if } M\,N \text{ has a normal form} \\ H'\ulcorner M\urcorner\ulcorner N\urcorner & = & \mathbf{F} \text{ if } M\,N \text{ has no normal form} \end{array}$$

Then $B := \lambda m.H'\ulcorner\mathbf{I}\urcorner\,m$ satisfies the specificaton of the Blank Tape problem:

$$\begin{array}{lcl} B\ulcorner M\urcorner = H'\ulcorner\mathbf{I}\urcorner\ulcorner M\urcorner & = & \mathbf{T} \text{ if } \mathbf{I}\,M \text{ has a normal form} \\ B\ulcorner M\urcorner = H'\ulcorner\mathbf{I}\urcorner\ulcorner N\urcorner & = & \mathbf{F} \text{ if } \mathbf{I}\,M \text{ has no normal form} \end{array}$$

(ii) Suppose that there is a term T satisfying

$$\begin{array}{rcl} T\,\ulcorner M\urcorner &=& \mathbf{T} \text{ if } M\,N \text{ has a normal form for all } N \\ T\,\ulcorner M\urcorner &=& \mathbf{F} \text{ if } M\,N \text{ has no normal form for some } N \end{array}$$

Then $B := \lambda m.T(\mathtt{Abs}(\lambda z.m))$ satisfies the specificaton of the Blank Tape problem. (Basically, we first create from $\ulcorner M\urcorner$ a term that throws away its input, returning $\ulcorner M\urcorner$; we feed this term to T.)

$$\begin{array}{rcl} B\,\ulcorner M\urcorner &=& T\,(\mathtt{Abs}(\lambda z.\ulcorner M\urcorner)) = T\,(\ulcorner \lambda z.M\urcorner) = \\ &=& \mathbf{T} \text{ if } M \text{ has a normal form} \\ &=& \mathbf{F} \text{ if } M \text{ has no normal form} \end{array}$$

Exercise 3

(* Meaning: slightly more difficult) Show that there is a term C that contracts the *left-most redex* in a term M, so

$$C\ulcorner M\urcorner = \ulcorner N\urcorner$$

if N arises from M by contracting the left-most redex, if M contains a redex, and otherwise $N \equiv M$.