# Grammars and Context Free Languages

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# Outline

# Organisational

Next week is different:

- Test over part I (lectures 1-4)
- Make sure you are registered for the course in Blackboard!
- Time: 13:30 – 15:30, make sure you are there at 13:15.
- Locations:
  - Names A – G : HG00.307
  - Names H – M : HG00.071
  - Names N – Z : LIN 3
  - Extra time students: HG00.086

# Generating words

Example of a grammar $G$:

$$\begin{array}{lcl} S & \rightarrow & O \mid E \\ E & \rightarrow & \lambda \mid aEa \mid bEb \\ O & \rightarrow & a \mid b \mid aOa \mid bOb \end{array}$$

Productions, always start with S

$S \Rightarrow E \Rightarrow aEa \Rightarrow abEba \Rightarrow abba$

$S \Rightarrow E \Rightarrow bEb \Rightarrow baEab \Rightarrow babEbab \Rightarrow babaEabab \Rightarrow babaabab$

$S \Rightarrow O \Rightarrow bOb \Rightarrow bab$

$S \Rightarrow O \Rightarrow bOb \Rightarrow baOab \Rightarrow babab$

We can generate exactly the set of words $w$ with $w = w^R$
($w$ is a palindrome)

# Context-free grammar

Let $\Sigma$ be a finite alphabet.

**Definition** A context-free grammar (CFG) $G = \langle V, S, P \rangle$ over $\Sigma$ consists of

- $V$, a set of non-terminal symbols,
- $S \in V$, a start symbol,
- $P$, a set of production rules of the form

$$X \to w,$$

where $X \in V$ and $w \in (V \cup \Sigma)^*$.

Notation:

$$E \to \lambda \mid aEa \mid bEb$$

is shorthand for three rules:

$$E \to \lambda, \quad E \to aEa, \quad E \to bEb.$$

# Context-free language

Using $G$, a language is generated using a relation $\Rightarrow$ ('produces') defined as follows (where $u, v, w$ are arbitrary elements of $(\Sigma \cup V)^*$)

$$X \to w \quad \text{implies} \quad uXv \Rightarrow uwv$$
$$u \Rightarrow v, v \Rightarrow w \quad \text{implies} \quad u \Rightarrow w$$

**Definition** The language generated by $G$, is

$$\mathcal{L}(G) = \{w \in \Sigma^* \mid S \Rightarrow w\}.$$

A language $L$ is context-free if $L = \mathcal{L}(G)$ for some context-free grammar $G$.

# Non-terminal symbols have a meaning

When we write down a grammar, each non-terminal symbol has a meaning.
This helps in explaining the grammar and in proving that it satisfies a property.

Back to our first example grammar $G$:

$$\begin{array}{lcl} S & \to & O \mid E \\ E & \to & \lambda \mid aEa \mid bEb \\ O & \to & a \mid b \mid aOa \mid bOb \end{array}$$

- Words produced by $E$ are palindromes of even length,
- Words produced by $O$ are palindromes of odd length,
- Words produced by $S$ are palindromes of even or odd length.
- So: $\mathcal{L}(G) \subseteq \{w \in \{a,b\}^* \mid w = w^R\}$.

# Leftmost derivations

There can be many ways to produce a word $w \in \mathcal{L}(G)$.

G: $S \to aSB \mid \lambda, \quad B \to b$

Derivations of $aabb$:

$$d_1 : S \Rightarrow aSB \Rightarrow aaSBB \Rightarrow aaBB \Rightarrow aabB \Rightarrow aabb$$
$$d_2 : S \Rightarrow aSB \Rightarrow aSb \Rightarrow aaSBb \Rightarrow aaSbb \Rightarrow aabb$$
$$d_3 : S \Rightarrow aSB \Rightarrow aSb \Rightarrow aaSBb \Rightarrow aaBb \Rightarrow aabb$$

Derivation $d_1$ is leftmost, $d_2$ is rightmost, $d_3$ is neither.

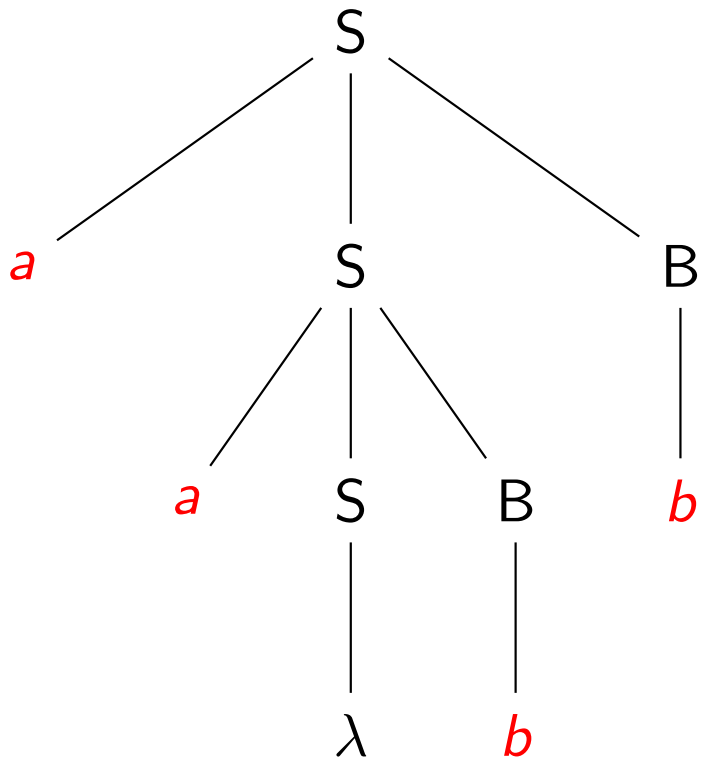
**Definition** A derivation is leftmost if in each step a rule is applied to the leftmost non-terminal.

**Lemma** For all grammars $G$ and words $w$, $w \in \mathcal{L}(G)$ iff there is a leftmost derivation of $w$.

# Parse trees / Derivation trees

G: $S \to a\,S\,B \mid \lambda, \quad B \to b$

A parse tree (or derivation tree) for $a\,a\,b\,b$ is:



# Ambiguity

**Definition** A context-free grammar $G$ is unambiguous if for each $w \in \mathcal{L}(G)$ there exists a unique leftmost derivation of $w$ in $G$. Otherwise $G$ is ambiguous.

Equivalently: A context-free grammar is ambiguous if there is a word with two different parse trees.

**Example** "dangling else"

$$
\begin{array}{lcl}
\mathsf{S} & \to & \texttt{print}\ E \\
 & | & \texttt{if}\ E\ \texttt{then}\ S\ \texttt{else}\ S \\
 & | & \texttt{if}\ E\ \texttt{then}\ S \\
\mathsf{E} & \to & a \mid b \mid \cdots
\end{array}
$$

# Some context-free languages

Is there a grammar $G_2$ such that

$$\mathcal{L}(G_2) = \{a^n b^n \mid n \geq 0\}?$$

$G_2$: $\boxed{S \rightarrow \lambda \mid aSb}$

All possible derivations

$$\begin{array}{ccccccccccc} S & \Rightarrow & aSb & \Rightarrow & aaSbb & \Rightarrow & \cdots & \Rightarrow & a^nSb^n & \Rightarrow & \cdots \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \lambda & & ab & & aabb & & \cdots & & a^nb^n & & \end{array}$$

What is a grammar $G_3$ such that

$$\mathcal{L}(G_3) = \{a^n b^n \mid n > 0\}?$$

$G_3$: $\boxed{S \rightarrow ab \mid aSb}$

# More context-free languages

Let $\Sigma = \{a, b, c\}$

**Claim:** $L = \{a^n b^m c^{2n+1} \mid m, n \geq 0\}$ is context-free.

Let us first show that $L' = \{a^n c^{2n+1} \mid m, n \geq 0\}$ is context-free.

For $L'$ use

$$\begin{array}{lcl} S & \rightarrow & c \mid aScc \end{array}$$

For $L$ use

$$\begin{array}{lcl} S & \rightarrow & Bc \mid aScc \\ B & \rightarrow & \lambda \mid bB \end{array}$$

**Fact** $\{a^n b^n c^n \mid n \geq 0\}$ is *not* context-free.

This can be proved using the Pumping Lemma for Context Free Languages

# Regular languages

**Theorem** Every regular language is context-free.

**Proof:** Let $M = \langle Q, q_0, \delta, F \rangle$ be an NFA (or DFA, or NFA$_\lambda$) that accepts $L \subseteq \Sigma^*$. Define a context-free grammar $G_M$ as follows:

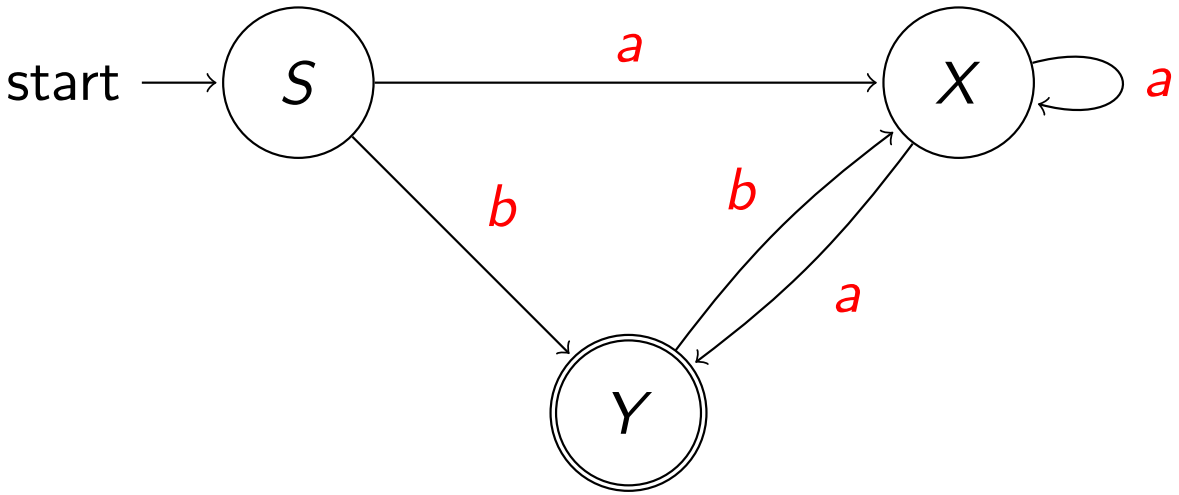
$$V = Q \qquad \text{non-terminals are states}$$
$$S = q_0$$
$$P = \{q \to aq' \mid q' \in \delta(q, a)\} \cup \{q \to \lambda \mid q \in F\}$$

$\mathcal{L}(G_M) = L(M)$, since there is a 1-1 correspondence between computations and derivations:

$$M : q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_n \in F$$

$$G_M : q_0 \Rightarrow a_1 q_1 \Rightarrow a_1 a_2 q_2 \Rightarrow \cdots \Rightarrow a_1 \cdots a_n q_n \Rightarrow a_1 \cdots a_n \lambda$$

# Regular languages: example




Corresponding grammar:

$$\begin{array}{lcl} S & \to & aX \mid bY \\ X & \to & aX \mid aY \\ Y & \to & bX \mid \lambda \end{array}$$

Observe: The context free grammars we construct from an automaton are of a specific (simple) form.

# Regular Grammars

**Definition** A (right) regular grammar is a context-free grammar $G = \langle V, S, P \rangle$ in which all rules have the form

$$X \to a\,Y \quad \text{or} \quad X \to \lambda$$

where $X, Y \in V$ and $a \in \Sigma$.

**Example** Consider the regular language $L := \mathcal{L}(a(aa + b)^*(bb)^*)$. A regular grammar for this language is found by first also allowing rules $X \to w\,Y$ for $w \in \Sigma^*$, and then refining.

$$\begin{array}{lcl} S & \to & a\,X \mid a\,Y \\ X & \to & a\,a\,X \mid b\,X \mid a\,a\,Y \mid b\,Y \\ Y & \to & b\,b\,Y \mid \lambda \end{array}$$

$$\begin{array}{lcl} S & \to & a\,X \mid a\,Y \\ X & \to & a\,U \mid b\,X \mid b\,Y \\ U & \to & a\,X \mid a\,Y \\ Y & \to & b\,V \mid \lambda \\ V & \to & b\,Y \end{array}$$

# Regular Languages and Regular Grammars

**Theorem** Let $L \subseteq \Sigma^*$.
$L$ is regular $\iff$ $L = \mathcal{L}(G)$ for some regular grammar $G$.

**Proof:**
For $\Longrightarrow$: From an NFA accepting $L$, we define a regular grammar accepting $L$. (As shown before.)
For $\Longleftarrow$: Given $G$, build an NFA $M_G = \langle Q, \delta, q_0, F \rangle$ as follows:

- State set $Q = V$, $q_0 = S$,
- $F = \{X \in V \mid (X \to \lambda) \in P\}$.
- $\delta(X, a) = \{Y \in V \mid (X \to a\,Y) \in P\}$

$\mathcal{L}(M_G) = L(G)$, since derivations correspond to computations:

$$G : S \Rightarrow a_1 X_1 \Rightarrow a_1 a_2 X_2 \Rightarrow \cdots \Rightarrow a_1 \cdots a_n X_n \;\Rightarrow\; a_1 \cdots a_n \lambda$$

$$M_G : S \xrightarrow{a_1} X_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} X_n \in F$$

# Parsing

A parser is an algorithm that determines for given $G$ and $w$ whether $w \in \mathcal{L}(G)$?

In practice we also want the parse tree.

The well-known Cocke-Younger-Kasami (CYK) algorithm requires $G$ to be in Chomsky normal form.
For simplicity we assume $\lambda \notin \mathcal{L}(G)$.

**Definition** A CFG $G$ is in Chomsky normal form if all its productions are of the form:

$$X \to YZ \quad \text{or} \quad X \to a$$

where $X, Y, Z \in V$ and $a \in \Sigma$

**Lemma** Every CFG $G$ (with $\lambda \notin \mathcal{L}(G)$) can be transformed into an equivalent CFG in Chomsky normal form.

# Chomsky normal form and the CYK Algorithm

Example of a grammar in Chomsky normal form:

| | | |
|---|---|---|
| S | $\rightarrow$ | $AT \mid AB$ |
| T | $\rightarrow$ | $SB$ |
| A | $\rightarrow$ | $a$ |
| B | $\rightarrow$ | $b$ |

Accepts $L := \{a^n b^n \mid n \geq 1\}$

Advantages of grammar in Chomsky normal form:

- A derivation of a word $w$ is at most $2|w| - 1$ steps long.
- A parse tree is a binary tree,
- so a parse tree of a word $w$ has maximum depth $|w|$.

Idea of CYK algorithm: For each substring $u$ of input word $w$, compute the set of non-terminals that can produce $u$.

# Pumping Lemma for context-free languages

**Theorem** Let $L$ be a context-free language over $\Sigma^*$
Then there exists a number $k > 0$
such that every word $z \in L$ with $|z| > k$
can be written as $z = u_1\, v_1\, w\, v_2\, u_2$ such that

- $|v_1\, w\, v_2| \leq k$
- $|v_1| + |v_2| > 0$
- $u_1\, v_1^i\, w\, v_2^i\, u_2 \in L$ for all $i \geq 0$

**Proofsketch** If there is a sufficiently large word $z \in L$
then in the derivation of $z$ one has

$$\begin{array}{lcll} S & \Rightarrow & u_1\, A\, u_2 & \\ A & \Rightarrow & v_1\, A\, v_2 & \text{a grammar loop!} \\ A & \Rightarrow & w & \end{array}$$

So: $S \Rightarrow u_1\, v_1\, w\, v_2\, u_2 = z$ and also

$$S \Rightarrow u_1\, A\, u_2 \Rightarrow u_1\, v_1\, A\, v_2\, u_2 \Rightarrow \cdots u_1\, v_1^i\, A\, v_2^i\, u_2 \Rightarrow u_1\, v_1^i\, w\, v_2^i\, u_2 \in L$$

# Applications of the pumping lemma for context-free languages

- $L_1 := \{a^k b^k c^k \mid k \geq 0\}$ is not context-free
  because it violates the pumping lemma for CF languages.
- $L_2 := \{ww \mid w \in \{a, b\}^*\}$ is not context-free
  because it violates the pumping lemma for CF languages.
- NB: $L_3 := \{a^k b^k \mid k \geq 0\}$ is context-free hence satisfies the pumping lemma for CF languages.