Semantics and Domain theory

Exercises 7, Wednesday, March. 26, 2014

1. Show that the untyped λ-term ω $(= \lambda x.x\,x)$ is not typable in PCF. That is: show that there are no τ_1 and τ_2 such that $\vdash \mathbf{fn}\; x : \tau_1.x\; x \quad : \quad \tau_2$.

2. Suppose that the term mult : $\mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}$ defines multiplication in PCF. Give a PCF term that defines the faculty function fac : $\mathbf{nat} \to \mathbf{nat}$.

3. Show that, if $Q \Downarrow_\tau V$, then $(\mathbf{fn}\; x : \tau.\mathbf{fn}\; y : \tau.y)PQ \Downarrow V$. (NB. V denotes an arbitrary *value*.)

4. To prove that PCF evaluation is deterministic, we prove (in Proposition 5.4.1) that the following set is closed under the rules of Fig.3

   $$\{(M, \tau, V) | M \Downarrow_\tau V \wedge \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')\}$$

   Show this for the cases of the rules $(\Downarrow_{\text{if1}})$ and $(\Downarrow_{\text{cbn}})$.
   (An alternative way of looking at this is to prove the following:

   $$M \Downarrow_\tau V \Rightarrow \forall V'(M \Downarrow_\tau V' \Rightarrow V = V')$$

   by induction on the derivation of $M \Downarrow_\tau V$. Do only the cases when the last applied rule is $(\Downarrow_{\text{if1}})$ or $(\Downarrow_{\text{cbn}})$.)

5. (a) Give a type τ, a term M, values V, V' and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \Downarrow_\tau V' \neq C[V]$.
   (b) Give a type τ, a term M, a value V and a context $C[-]$ such that $M \Downarrow_\tau V$ but $C[M] \not\Downarrow_\tau$ ($C[M]$ has no value.)
   (c) Give a type τ, a term M, a value V and a context $C[-]$ such that $M \not\Downarrow_\tau$ but $C[M] \Downarrow_\tau V$

6. Given the definition of plus (Exercise 5.6.3.)

   $$\begin{aligned} \text{plus} \quad = \quad & \mathbf{fix}(\mathbf{fn}\; p : \mathbf{nat} \to \mathbf{nat} \to \mathbf{nat}.\mathbf{fn}\; x : \mathbf{nat}.\mathbf{fn}\; y : \mathbf{nat}. \\ & \;\mathbf{if}\; \mathbf{zero}(y)\; \mathbf{then}\; x\; \mathbf{else}\; \mathbf{succ}(p\, x\, \mathbf{pred}(y))) \end{aligned}$$

   Prove (by induction) that

   $$\forall m, n(\text{plus}\; \mathbf{succ}^m(0)\; \mathbf{succ}^n(0) \quad \Downarrow_{\mathbf{nat}} \quad \mathbf{succ}^{m+n}(0))$$

7. (If we come to this topic at the lecture!) Prove that the following terms M and N are not contextually equivalent.

   $$\begin{aligned} M \quad &= \quad \mathbf{if}\; x\; \mathbf{then}\; 0\; \mathbf{else}\; 1 \\ N \quad &= \quad \mathbf{if}\; y\; \mathbf{then}\; 0\; \mathbf{else}\; 1 \end{aligned}$$