Semantics and Domain theory
Exercises 11

Recall the following terms in untyped λ-calculus

- c_n denotes the n-th Church numeral, so in particular $c_0 = \lambda f\,x.x$, $c_1 = \lambda f\,x.f\,x$ and in general $c_n = \lambda f\,x.f^n(x)$.
- $\mathbf{K} = \lambda x\,y.x$, $\mathbf{I} = \lambda x.x$

1. (a) Prove that the equation $c_0 = c_1$ is inconsistent in untyped λ calculus.
   (b) (More challenging) Prove that $c_0 = c_{n+1}$ is inconsistent for any $n \in \mathbb{N}$ and conclude that $c_n = c_m$ is inconsistent for any $n, m \in \mathbb{N}$ with $n \neq m$.
2. The applicative structure (A, App), with $A = \mathbb{N}$ (the natural numbers) and $\mathsf{App} = *$ (multiplication) cannot be made into a (consistent) model of the untyped λ-calculus. We prove this in the following steps:

   Consider $(\mathbb{N}, *)$ and assume that there is an interpretation $[\![-]\!]$ satisfying the definition that we have given in the lecture (see Definition 59 of Berline). Now:

   (a) Show that $[\![\mathbf{K}\,\mathbf{I}]\!] = [\![\mathbf{I}\,\mathbf{K}]\!]$.
   (b) Conclude that $d = e$ for all $d, e \in \mathbb{N}$.

   So: contradiction.
3. We consider the model definition as explained in the lecture. (See Definition 57 of Berline; so we assume that the interpretation $[\![-]\!]$ is well-defined.) Assume $G \circ A = \mathrm{id}_M$. Show that the η-rule holds in the model. ($\lambda x.N\,x = N$ for all N with $x \notin \mathrm{FV}(N)$.)
4. Prove that the theory that equates all λ-terms that don't have a normal form is inconsistent by showing that the following equation is inconsistent in untyped λ calculus:
   $$\lambda\,x\,y.x\,y\,\Omega = \lambda\,x\,y.y\,x\,\Omega.$$
5. Which of the following sets are complete lattices.
   (a) The set of flat natural numbers $\mathbb{N}_\perp$.
   (b) The set $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ of *finite subsets* of $\mathbb{N}$.
   (c) The set Ω ($= \mathbb{N} \cup \{\omega\}$, with the ordering we have seen before).
   (d) The set of monotone functions from $\mathbb{B}_\perp^\top$ to $\mathbb{B}_\perp^\top$.
   (Remember that the set of flat booleans with a top element added, $\mathbb{B}_\perp^\top$, is a complete lattice.)
6. Complete the proof of Proposition 3.1.7.
   That is, show that in a complete lattice $(D, \sqsubseteq)$, if we define
   $$\bigsqcap X := \bigsqcup \{y \in D \mid y \sqsubseteq X\},$$
   then $\bigsqcap X$ is indeed the *greatest lower bound* (also called the *inf*) of X.