Semantics and Domain theory

Exercises 11

Recall the following terms in untyped λ-calculus

- c_n denotes the n-th Church numeral, so in particular $c_0 = \lambda f\, x.x$, $c_1 = \lambda f\, x.f\, x$ and in general $c_n = \lambda f\, x.f^n(x)$.
- $\mathbf{K} = \lambda x\, y.x$, $\mathbf{I} = \lambda x.x$, $\Omega = (\lambda x.x\, x)(\lambda x.x\, x)$.

1. (a) Prove that the equation $c_0 = c_1$ is inconsistent in untyped λ calculus. (That is: show that, if you assume $c_0 = c_1$, then you can prove $M = N$ for all terms M, N.)

   Answer: ....

   For convenience we also introduce $\mathbf{K}_* = \lambda x\, y.y$. Assume $c_0 = c_1$. Then $c_0\, (\lambda z.\mathbf{K})\, \mathbf{K}_* = c_1\, (\lambda z.\mathbf{K})\, \mathbf{K}_*$. This implies that $\mathbf{K}_* = (\lambda f\, x.x)\, (\lambda z.\mathbf{K})\, \mathbf{K}_* = (\lambda f\, x.f\, x)\, (\lambda z.\mathbf{K})\, \mathbf{K}_* = \mathbf{K}$ and so $M = N$ for all terms M, N.

   End answer..

   (b) Prove that $c_0 = c_{n+1}$ is inconsistent for any $n \in \mathbb{N}$.

   Answer: ....

   The same trick as in Exercise 1a works: Assume $c_0 = c_{n+1}$. Then $\mathbf{K}_* = c_0\, (\lambda z.\mathbf{K})\, \mathbf{K}_* = c_{n+1}\, (\lambda z.\mathbf{K})\, \mathbf{K}_* = \mathbf{K}$.

   End answer..

   (c) Prove that $c_n = c_m$ is inconsistent for $n, m \in \mathbb{N}$ with $n \neq m$.

   Answer: ....

   Let $n > m$ and assume $c_n = c_m$. We could apply the predecessor $n - m$-times to c_n and c_m and then use Exercise 1b. But then we have to define predecessor for the Church numerals first (which is remarkably tricky!) So we proceed as follows.

   We may assume that $m \geq 1$ (otherwise we can use Exercise 1b immediately and we are done). Note that we have (+) $c_p\, \mathbf{K}\, Q = \lambda x_1, \ldots, x_p.Q$ for all $p \in \mathbb{N}$. (This can be shown by proving $\mathbf{K}^p(Q) = \lambda x_1, \ldots, x_p.Q$ by induction on p.) Then from $c_n = c_m$ we get $c_n\, \mathbf{K}\, \mathbf{K}\, \mathbf{I} \ldots \mathbf{I} = c_m \mathbf{K}\, \mathbf{K}\, \mathbf{I} \ldots \mathbf{I}$ (a sequence of $\mathbf{I}$'s of length n), from which we derive, using (+): $\mathbf{K} = \mathbf{K}\, \mathbf{I} \ldots \mathbf{I}$, a sequence of $\mathbf{I}$'s of length $n - m$. Now, if $n - m = 1$, we have $\mathbf{K}\, \mathbf{I} = \mathbf{K}_*$ and if $n - m > 1$ we have $\mathbf{K}\, \mathbf{I} \ldots \mathbf{I} = \mathbf{I}$. From both $\mathbf{K} = \mathbf{K}_*$ and $\mathbf{K} = \mathbf{I}$ we can prove that $M = N$ for all M, N, so $c_n = c_m$ is inconsistent if $n \neq m$.

   End answer..

2. The applicative structure $(M, \cdot)$, with $M = \mathbb{N}$ (the natural numbers) and $\cdot = *$ (multiplication) cannot be made into a (consistent) model of the untyped λ-calculus. We prove this in the following steps:

   Consider $(\mathbb{N}, *)$ and assume that there is an interpretation $[\![-]\!]$ satisfying the definition that we have given in the lecture (see Definition 59 of Berline). Now:

   (a) Show that $[\![\mathbf{K}\, \mathbf{I}]\!] = [\![\mathbf{I}\, \mathbf{K}]\!]$.

   Answer: ....

   $[\![\mathbf{K}\, \mathbf{I}]\!] = [\![\mathbf{K}]\!] * [\![\mathbf{I}]\!] = [\![\mathbf{I}]\!] * [\![\mathbf{K}]\!] = [\![\mathbf{I}\, \mathbf{K}]\!]$.

   End answer..

   (b) Conclude that $d = e$ for all $d, e \in \mathbb{N}$.

   Answer: ....

Let $d, e \in \mathbb{N}$. Take the environment $\rho : \mathbb{V} \to \mathbb{N}$ ($\mathbb{V}$ is the set of variables) with $\rho(x) = d$ and $\rho(y) = e$. We have

$$d = \rho(x) = [\![x]\!]_\rho = [\![\mathbf{I}\,\mathbf{K}\,x\,y]\!]_\rho = [\![\mathbf{K}\,\mathbf{I}\,x\,y]\!]_\rho = [\![y]\!]_\rho = \rho(y) = e.$$

End answer..

So: all elements are equal in the model.

3. Prove that the theory that equates all λ-terms that don't have a normal form is inconsistent by showing that the following equation is inconsistent in untyped λ calculus:

$$\lambda\,x\,y.x\,y\,\Omega = \lambda\,x\,y.y\,x\,\Omega.$$

Answer: ....
Assume $\lambda\,x\,y.x\,y\,\Omega = \lambda\,x\,y.y\,x\,\Omega$. There are various ways to show that $M = N$ for all M, N; here is just one. Take $\mathbf{U}^4_4 := \lambda x\,y\,z\,v.v$ and $\mathbf{U}^4_3 := \lambda x\,y\,z\,v.z$ and let M, N be arbitrary terms. Then $(\lambda\,x\,y.x\,y\,\Omega)\,\mathbf{U}^4_4\,\mathbf{U}^4_3\,M\,N = \mathbf{U}^4_4\,\mathbf{U}^4_3\,\Omega\,M\,N = N$, but also $(\lambda\,x\,y.y\,x\,\Omega)\,\mathbf{U}^4_4\,\mathbf{U}^4_3\,M\,N = \mathbf{U}^4_3\,\mathbf{U}^4_4\,\Omega\,M\,N = M$, so $M = N$.
End answer..