Semantics and Domain theory

Exercises 12

1. We consider the model definition as explained in the lecture. (See Definition 57 of Berline; so we assume that the interpretation $[\![-]\!]_\rho$ is well-defined.) Assume $G \circ A = \mathrm{id}_M$. Show that the η-rule holds in the model. (The η-rule says: $\lambda x.N\,x = N$ if $x \notin \mathrm{FV}(N)$.) NB. You may need to use the follwoing property (without proof): If $\rho(x) = \rho'(x)$ for all $x \in \mathrm{FV}(M)$, then $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.
   Answer: ....
   Assume we have a model where $G \circ A = \mathrm{id}_M$ and let N be a term with $x \notin \mathrm{FV}(N)$ and $\rho : \mathbb{V} \to M$ a valuation.

$$\begin{aligned}
[\![\lambda x.N\,x]\!]_\rho &= G(\boldsymbol{\lambda} d \in M.[\![N\,x]\!]_{\rho[x\mapsto d]}) \\
&= G(\boldsymbol{\lambda} d \in M.A([\![N]\!]_{\rho[x\mapsto d]})(d)) \\
&\overset{*}{=} G(\boldsymbol{\lambda} d \in M.A([\![N]\!]_\rho)(d) \\
&\overset{**}{=} G(A([\![N]\!]_\rho) \\
&= [\![N]\!]_\rho,
\end{aligned}$$

   where $\overset{*}{=}$ is because $x \notin \mathrm{FV}(N)$ and $\overset{**}{=}$ is because $\boldsymbol{\lambda} d \in M.A([\![N]\!]_\rho)(d) = A([\![N]\!]_\rho)$.
   End answer..

2. Which of the following sets are complete lattices.

   (a) The set of flat natural numbers $\mathbb{N}_\perp$.
   Answer: ....
   No, $\{0, 1\}$ has no upperbound.
   End answer..

   (b) The set $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ of *finite subsets* of $\mathbb{N}$.
   Answer: ....
   No, the collection of all sets $\{n,\}$ (for $n \in \mathbb{N}$) has no upperbound.
   End answer..

   (c) The set Ω ($= \mathbb{N} \cup \{\omega\}$, with the ordering we have seen before).
   Answer: ....
   Yes: each finite set has a maximal element, which is its lub and an infitie set has ω as lub.
   End answer..

   (d) The set of monotone functions from $\mathbb{B}_\perp^\top$ to $\mathbb{B}_\perp^\top$.
   (Remember that the set of flat booleans with a top element added, $\mathbb{B}_\perp^\top$, is a complete lattice.)

   Answer: ....
   Let X be a set of monotone functions from $\mathbb{B}_\perp^\top$ to $\mathbb{B}_\perp^\top$. (There are only finitely many functions from $\mathbb{B}_\perp^\top$ to $\mathbb{B}_\perp^\top$, so X is finite, but we don't need to use that.) Define $F := \boldsymbol{\lambda} b \in \mathbb{B}_\perp^\top. \bigsqcup\{f(b) \mid f \in X\}$. Then $F = \bigsqcup X$, because if $f \in X$, then $f \sqsubseteq F$ (because for all $b \in \mathbb{B}_\perp^\top$, $f(b) \sqsubseteq F(b)$) and if $\forall f \in X(f \sqsubseteq g)$, then $F \sqsubseteq g$ (because $\forall f \in X \forall b \in \mathbb{B}_\perp^\top(f(b) \sqsubseteq g(b))$ implies $\forall b \in \mathbb{B}_\perp^\top(F(b) \sqsubseteq g(b))$. Also, F is monotone, because if $b \sqsubseteq b'$, then $f(b) \sqsubseteq f(b')$ for all $f \in X$ and so
   $\bigsqcup\{f(b') \mid f \in X\} \sqsubseteq \bigsqcup\{f(b) \mid f \in X\}$.
   End answer..

3. Complete the proof of Proposition 3.1.7.
That is, show that in a complete lattice $(D, \sqsubseteq)$, if we define

$$\bigsqcap X := \bigsqcup \{y \in D \mid y \sqsubseteq X\},$$

then $\bigsqcap X$ is indeed the *greatest lower bound* (also called the *inf*) of X.

Answer: ....
[This is basically the same exercise as Exercise 10 of the dI-domains notes.]
Given a complete lattice $(D, \sqsubseteq)$, with $X \subseteq D$, define

$$d_0 := \bigsqcup \{d \in D \mid \forall x \in X (d \sqsubseteq x)\}.$$

We claim that $d_0 = \bigsqcap X$. We need to prove the following two properties.

(glb1) To prove: $\forall x \in X (d_0 \sqsubseteq x)$.
Suppose $x \in X$. Then $\forall d \in Y (d \sqsubseteq x)$, so x is an upperbound of Y, so $d_0 \sqsubseteq x$, because d_0 is the least upperbound of Y.

(glb2) To prove: $\forall d (\forall x \in X (d \sqsubseteq x)) \Rightarrow d \sqsubseteq d_0$.
Suppose $\forall x \in X (d \sqsubseteq x)$. Then $d \in Y$, so $d \sqsubseteq d_0$, because d_0 is an upperbound of Y.

End answer..

4. Prove the correctness of Definition 3.2.5. To prove this, you have to show that the function

$$\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)}$$

is continuous for every P and ρ. (You may assume that F and G are continuous and all the other results about continuity from the notes.)

Answer: ....
By induction on P

- $P = x$. Then $\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)} = \boldsymbol{\lambda} d.d$ and the identity is continuous.
- $P = y$. Then $\boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)} = \boldsymbol{\lambda} d.\rho(y)$, a constant function which is continuous.
- $P = MN$. Then

$$\begin{aligned} \boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)} &= \boldsymbol{\lambda} d.F([\![M]\!]_{\rho(x:=d)})([\![N]\!]_{\rho(x:=d)}) \\ &= \mathsf{App} \circ (\boldsymbol{\lambda} d.F([\![M]\!]_{\rho(x:=d)}), \boldsymbol{\lambda} d.[\![N]\!]_{\rho(x:=d)}), \end{aligned}$$

  which is continuous, because application, $\mathsf{App} : [D \to D] \times D \to D$ is continuous, composition preserves continuity, $\boldsymbol{\lambda} d.[\![N]\!]_{\rho(x:=d)})$ is continuous (IH) and $\boldsymbol{\lambda} d.F([\![M]\!]_{\rho(x:=d)}) = F \circ \boldsymbol{\lambda} d.[\![M]\!]_{\rho(x:=d)}$ is continuous because F is continuous and $\boldsymbol{\lambda} d.[\![M]\!]_{\rho(x:=d)}$ is continuous (IH).
- $P = \lambda y.M$. Then

$$\begin{aligned} \boldsymbol{\lambda} d.[\![P]\!]_{\rho(x:=d)} &= \boldsymbol{\lambda} d.G(\boldsymbol{\lambda} e.[\![M]\!]_{\rho(x:=d,y:=e)}) \\ &= G \circ (\boldsymbol{\lambda} d.\boldsymbol{\lambda} e.[\![M]\!]_{\rho(x:=d,y:=e)}). \end{aligned}$$

  Note that by (IH), both $\boldsymbol{\lambda} d.[\![M]\!]_{\rho(x:=d,y:=e)}$ (for any fixed e) and $\boldsymbol{\lambda} e.[\![M]\!]_{\rho(x:=d,y:=e)}$ (for any fixed d) are continuous, so $G(\boldsymbol{\lambda} e.[\![M]\!]_{\rho(x:=d,y:=e)})$ above is well-defined and $\boldsymbol{\lambda}(d,e).[\![M]\!]_{\rho(x:=d,y:=e)}$ is continuous. We can conclude that $G \circ (\boldsymbol{\lambda} d.\boldsymbol{\lambda} e.[\![M]\!]_{\rho(x:=d,y:=e)})$ is continuous, because it is a composition of continuous functions.

End answer ....

5. At the lecture, we have seen the interpretations in D_A of **I** ($= \lambda x.x$), **K** ($= \lambda x.\lambda y.x$) and **II**.

   (a) Compute the interpretation of $\lambda x.x\,x$.
   Answer: ....

$$\begin{aligned}
[\![\lambda x.x\,x]\!] &= G(\boldsymbol{\lambda} d \in D_A.F(d)(d)) \\
&= \{(\beta, b) \mid b \in F(\beta)(\beta)\} \\
&= \{(\beta, b) \mid b \in \{b \mid \exists \alpha \subseteq \beta((\alpha, b) \in \beta)\}\} \\
&= \{(\beta, b) \mid \exists \alpha \subseteq \beta((\alpha, b) \in \beta)\}
\end{aligned}$$

   Note that this set is not empty! For example if $A = \{0,1\}$, then B_2 contains element $(\{(\emptyset, 0), (\emptyset, 1)\}, 0)$, so this element is in $B := \cup_{i\in\mathbb{N}} B_i$. This element is also in $[\![\lambda x.x\,x]\!]$, because for $\alpha = \emptyset$ we have $(\alpha, 0) \in \{(\emptyset, 0), (\emptyset, 1)\}$.
   End answer ....

   (b) Show that $[\![\mathbf{KI}]\!] = \{(\beta, (\gamma, c)) \mid c \in \gamma\}$ (without doing a β-reduction first).
   Answer: ....
   Remember (or compute for yourself) that $[\![\mathbf{K}]\!]) = \{(\alpha, (\beta, b)) \mid b \in \alpha\}$, $[\![\mathbf{I}]\!] = \{(\gamma, c)) \mid c \in \gamma\}$.

$$\begin{aligned}
[\![\mathbf{KI}]\!] &= F([\![\mathbf{K}]\!])([\![\mathbf{I}]\!]) \\
&= F(\{(\alpha, (\beta, b)) \mid b \in \alpha\})(\{(\gamma, c)) \mid c \in \gamma\}) \\
&= \{d \mid \exists \delta \subseteq \{(\gamma, c)) \mid c \in \gamma\}((\delta, d) \in \{(\alpha, (\beta, b)) \mid b \in \alpha\})\} \\
&\stackrel{*}{=} \{(\beta, b) \mid \exists \delta \subseteq \{(\gamma, c)) \mid c \in \gamma\}(b \in \delta\})\} \\
&\stackrel{*}{=} \{(\beta, (\gamma, c)) \mid c \in \gamma\}
\end{aligned}$$

   For the $\stackrel{*}{=}$: if you don't see them, show $\subseteq$ and $\supseteq$ in both cases.

   End answer ....

6. Let Y be an element of D_A and let ρ be a valuation with $\rho(y) = Y$.

   (a) Compute in D_A the interpretation of $\lambda x.y\,x$ by expressing $[\![\lambda x.y\,x]\!]_\rho$ in terms of Y.
   Answer: ....

$$\begin{aligned}
[\![\lambda x.y\,x]\!]_\rho &= G(\boldsymbol{\lambda} d \in D_A.F(Y)(d)) \\
&= \{(\beta, b) \mid b \in F(Y)(b)\} \\
&= \{(\beta, b) \mid \exists \gamma \subseteq \beta((\gamma, b) \in Y\}
\end{aligned}$$

   End answer ....

   (b) Conclude that the η-rule does not hold in D_A. (The η-rule says that $\lambda x.M\,x = M$ if $x \notin \mathrm{FV}(M)$.)

Answer: ....
If the η-rule holds in D_A, then, for all M with $x \notin \mathrm{FV}(M)$, for all ρ: $[\![\lambda x.M\,x]\!]_\rho = [\![M]\!]_\rho$. For $M = y$ and ρ with $\rho(y) = Y$ this means

$$\{(\beta, b) \mid \exists\gamma \subseteq \beta((\gamma, b) \in Y\} = Y.$$

But the latter equation doesn't hold:

- If we let Y contain 'urelements, i.e. elements from A, then $Y \not\subseteq \{(\beta, b) \mid \exists\gamma \subseteq \beta((\gamma, b) \in Y\}$.
- If $(\gamma, b) \in Y$, then for any β with $\beta \supseteq \gamma$ we have $(\beta, b) \in \{(\beta, b) \mid \exists\gamma \subseteq \beta((\gamma, b) \in Y\}$. This means that, if we choose Y appropriately, $\{(\beta, b) \mid \exists\gamma \subseteq \beta((\gamma, b) \in Y\} \not\subseteq Y$.

End answer ....

7. Use the result of the following exercise ($[\![\Omega]\!] = \emptyset$) to:

   (a) Compute the interpretation of $\lambda y.\Omega$ in D_A.
   Answer: ....
   $[\![\lambda y.\Omega]\!] = G(\boldsymbol{\lambda} d.\emptyset) = \{(\beta, b) \mid b \in \emptyset\} = \emptyset$.
   End answer ....

   (b) Compute the interpretation of $\lambda y.y\,\Omega$ in D_A.
   Answer: ....

$$\begin{aligned} [\![\lambda y.y\Omega]\!] &= G(\boldsymbol{\lambda} d.F(d)(\emptyset)) \\ &= \{(\beta, b) \mid b \in F(\beta)(\emptyset)\} \\ &= \{(\beta, b) \mid \exists\gamma \subseteq \emptyset((\gamma, b) \in \beta)\} \\ &= \{(\beta, b) \mid (\emptyset, b) \in \beta\}. \end{aligned}$$

   NB. This is not the empty set!
   End answer ....

8. [Challenging] Show that the interpretation of Ω ($= (\lambda x.x\,x)(\lambda x.x\,x)$) in D_A is $\emptyset$.
   (Hint: From a $c \in [\![\Omega]\!]$ you can construct an infinite sequence $(\alpha_i)_{i \in \mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i$ for all i, which is impossible in D_A.)

   Answer: ....
   Remember that $[\![\lambda x.x\,x]\!] = \{(\beta, b) \mid \exists\alpha \subseteq \beta((\alpha, b) \in \beta)\}$. Let's call this set X, so

$$X = \{(\beta, b) \mid \exists\alpha \subseteq \beta((\alpha, b) \in \beta)\}$$

   and $[\![\Omega]\!] = F(X)(X) = \{c \mid \exists\gamma \subseteq X((\gamma, c) \in X)\} =$

$$\{c \mid \exists\gamma \subseteq X \exists\alpha \subseteq \gamma((\alpha, c) \in \gamma)\}.$$

   If $c \in [\![\Omega]\!]$, then $(\alpha_1, c) \in X$ for some $\alpha_1 \subseteq X$. We now define a sequence $\alpha_1, \alpha_2, \ldots$ as follows.

   $(\alpha_1, c) \in X$ so $(\alpha_2, c) \in \alpha_1$ for some $\alpha_2 \subseteq \alpha_1 \subseteq X$, so
   $(\alpha_2, c) \in X$ so $(\alpha_3, c) \in \alpha_2$ for some $\alpha_3 \subseteq \alpha_2 \subseteq X$, so
   $(\alpha_3, c) \in X$ so $(\alpha_4, c) \in \alpha_3$ for some $\alpha_4 \subseteq \alpha_3 \subseteq X$, so
   etcetera

We find $(\alpha_i)_{i\in\mathbb{N}}$ with $(\alpha_{i+1}, c) \in \alpha_i$ for all i. The α_i are all finite subsets of X that have a *norm* $|\alpha_i|$, which is an element in $\mathbb{N}$, see Definition 3.3.4. We have

$$\begin{aligned} |\alpha_i| &= \mathrm{max\{|a| \mid a \in \alpha_i\} \geq |(\alpha_i, c)|} \\ &= \mathrm{max\{|a| \mid a \in \alpha_{i+1}\} + |c| + 1} \\ &> |\alpha_{i+1} \end{aligned}$$

but this gives an infinte decreasing sequence in $\mathbb{N}$. Contradiction, so the ere is noo element in $[\![\Omega]\!]$.

End answer ....