Semantics and Domain theory

Exercises 13

1. Prove that, for M a closed λ-term, if M has a head-normal-form, then there is a sequence of terms $P_1, \ldots, P_n$ such that $M\, P_1 \ldots P_n =_\beta \mathbf{I}$.
(For closed terms, the reverse implication also holds, so this criterion is equivalent to *having a hnf*. This is where the terminology *solvable* comes from.)
Answer: ....
If M is a closed term that has a head-normal-form, then we know that $M =_\beta \lambda x_1, \ldots, x_n.y\, Q_1 \cdots Q_m$ for some $n \geq 0$, $m \geq 0$. As M is closed, we know that y is one of the x_i, so we have
$$M =_\beta \lambda x_1, \ldots, x_n.x_i\, Q_1 \cdots Q_m$$
with $n \geq 1$. We apply M to the sequence of terms $P_1, \ldots, P_n$, where it doesn't matter which terms we choose, except for P_i for which we take $P_i := \lambda z_1, \ldots, z_m.\mathbf{I}$. Then $M\, P_1 \ldots P_n =_\beta \mathbf{I}$.
End answer..

2. Define $T := \lambda x.x\, y\, (x\, x)$ and $M := T\, T$.
   (a) Draw the Böhm tree of M.
   Answer: ....
   We compute the head-normal form of M: $M = T\,T = T\, y\, (T\,T) = y\, y\, (y\, y)(T\,T)$. From there we construct the Böhm tree that is depicted on the next page.
   End answer..
   (b) Describe the set of approximations of M, $\mathcal{A}(M)$.
   Answer: ....
   We can give a grammar for $\mathcal{A}(M)$:
   $$A ::= \bot \mid y \bot \bot\, A \mid y\, y \bot A \mid y \bot (y \bot)\, A \mid y\, y\, (y \bot)\, A \mid y \bot (y\, y)\, A \mid y\, y\, (y\, y)\, A$$
   or alternatively we can give an inductive definition (which is actually just a different format for defining the same).
   - $\bot \in \mathcal{A}(M)$.
   - If $A \in \mathcal{A}(M)$, then $y \bot \bot\, A$, $y\, y \bot A$, $y \bot (y \bot)\, A$, $y\, y\, (y \bot)\, A$, $y \bot (y\, y)\, A$ and $y\, y\, (y\, y)\, A \in \mathcal{A}(M)$.

   End answer..

3. Remember that the $\mathbf{S}$ combinator is defined as $\lambda x\, y\, z.x\, z\, (y\, z)$.
   (a) Draw the Böhm tree of $\mathbf{S\,S\,S}$.
   Answer: ....
   We compute the normal form of $\mathbf{S\,S\,S}$: $\lambda z.\mathbf{S}\, z\, (\mathbf{S}\, z) = \lambda z.\lambda z'.z\, z'\, (\mathbf{S}\, z\, z') = \lambda z.\lambda z'.z\, z'\, (\lambda z''.\, z\, z''\, (z' z'')) = \lambda x\, y.x\, y\, (\lambda z.x\, z\, (y\, z))$. From there we construct the Böhm tree that is depicted on the next page.
   End answer..
   (b) Give the approximations of $\mathbf{S\,S\,S}$, that is, describe $\mathcal{A}(\mathbf{S\,S\,S})$.
   Answer: ....
   $\lambda x\, y.x\, y\, (\lambda z.x\, z\, (y\, z))$, $\lambda x\, y.x\, y\, (\lambda z.x\, z\, (y \bot))$, $\lambda x\, y.x \bot (\lambda z.x\, z\, (y\, z))$, $\lambda x\, y.x \bot (\lambda z.x\, z\, (y \bot))$, $\lambda x\, y.x\, y\, (\lambda z.x\, z \bot)$, $\lambda x\, y.x \bot (\lambda z.x\, z\, (y\, z))$, $\lambda x\, y.x\, y\, (\lambda z.x \bot (y\, z))$, etc.
   End answer..

4. Suppose that the term B satisfies $B = x\,B\,B$. Draw the Böhm tree of B.
Answer: ....
See the next page.
End answer..

5. (a) Give a term P that has the Böhm tree given below.
Answer: ....
P should satisy $P = x\,(x\,P\,y)$, so we take $P := \mathbf{Y}(\lambda p.x\,(x\,p\,y))$ where $\mathbf{Y}$ is the fixed-point combinator.
End answer..

(b) (Hard) Give a term Q that has the Böhm tree given below.
Answer: ....
We define a family of terms Q_n where

$$\begin{aligned} Q = Q_0 &= x\,(x\,Q_1\,y), \\ Q_1 &= x\,(x\,Q_2\,(y\,y)), \\ Q_2 &= x\,(x\,Q_3\,(y\,(y\,y))), \\ & \quad \text{etcetera} \end{aligned}$$

$$Q_n = x\,(x\,Q_{n+1}\,(c_n\,y\,y))$$

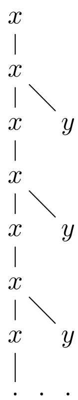
where $c_n = \lambda f\,x.f^n(x)$ is the n-th Church numeral. Note that $c_n\,y\,y$ is exactly the term that we need at that place, and $c_0\,y\,y = y$ is also correct. If we place the index n also as an argument, we are looking for a term Q satisfying

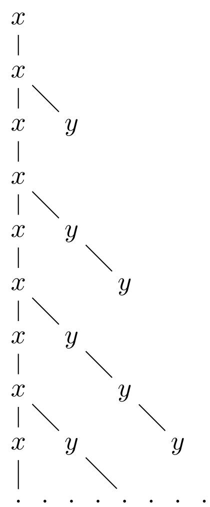
$$Q\,c_n = x\,(x\,(Q\,c_{n+1})\,(c_n\,y\,y)).$$

So we can take

$$Q := \mathbf{Y}(\lambda q.\lambda n.x\,(x\,(Q\,(\mathbf{succ}\,n))\,(n\,y\,y)))$$

where **succ** is the successor function on Church numerals. The term we need, that has the Böhm tree given in the exercise is then $Q\,c_0$.
End answer..

BT(P) = 

BT(Q) = 

6. Let M and N be λ-terms that satisfy the following equations

$$\begin{aligned} M &= \lambda xy.x\,(M\,x\,y)\,(M\,x\,y) \\ N &= \lambda xy.x\,(N\,x\,x)\,(N\,x\,x) \end{aligned}$$

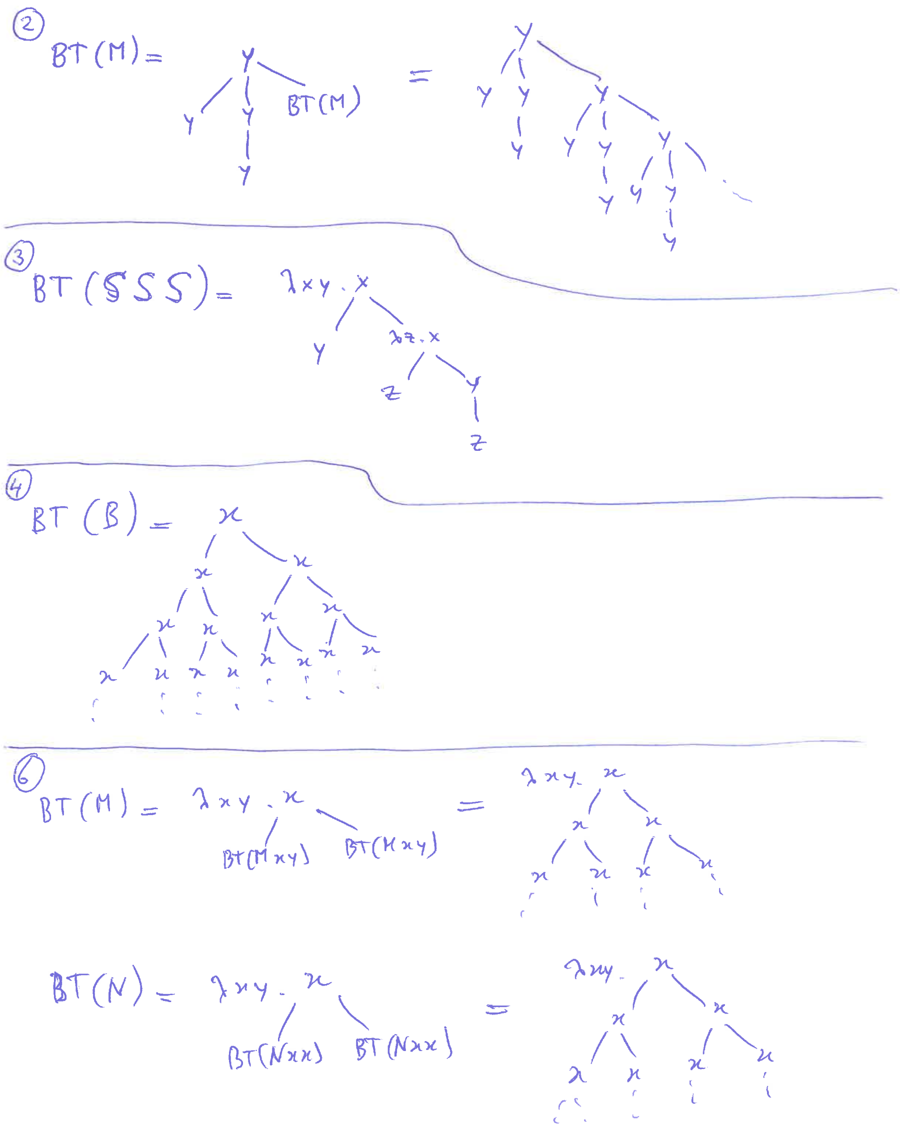
Prove that $M = N$ in D_A.

Answer: ....

We compute the Böhm trees of M and of N. This results in the Böhm trees that are depicted on the next page. We observe that $\mathrm{BT}(M) = \mathrm{BT}(N)$ and so $M = N$ in D_A.

End answer...

Answer: ....



End answer...