Semantics and Domain theory

Exercises 2

NB. We write $\mathsf{State} \rightharpoonup \mathsf{State}$ for the set of partial functions from State to State.

1. Define the denotational semantics of **repeat** P **until** b as a fixed point of a function $g : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$.
   (NB this program executes statement P and then checks the boolean b; if b holds, execution stops, if b doesn't hold, it iterates.)

   Answer: ....

   We use the idea that **repeat** P **until** b *has the same meaning as*

   $$P;\ \ \mathbf{if}\ b\ \mathbf{then}\ \ \mathbf{skip}\ \ \mathbf{else}\ (\mathbf{repeat}\ P\ \mathbf{until}\ b).$$

   We define $[\![\mathbf{repeat}\ P\ \mathbf{until}\ b]\!]$ *as the fixed point of* $g : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$*, where*

   $$g := \boldsymbol{\lambda}\, w : \mathsf{State} \rightharpoonup \mathsf{State}.\ \boldsymbol{\lambda}\, s : \mathsf{State}.\ \mathrm{IF}([\![b]\!]([\![P]\!](s)), [\![P]\!](s), w([\![P]\!](s))),$$

   where $\mathrm{IF} : \mathbb{B} \times \mathsf{State} \times \mathsf{State} \to \mathsf{State}$ *is defined by* $\mathrm{IF}(tt, s_1, s_2) = s_1$ *and* $\mathrm{IF}(\mathit{ff}, s_1, s_2) = s_2$.

   Alternatively:

   $$g := \boldsymbol{\lambda}\, w : \mathsf{State} \rightharpoonup \mathsf{State}.\ \mathrm{If}([\![b]\!], \mathrm{Id}\,, w) \circ [\![P]\!],$$

   where $\mathrm{If} : (\mathsf{State} \to \mathbb{B}) \times (\mathsf{State} \to \mathsf{State}) \times (\mathsf{State} \to \mathsf{State}) \to (\mathsf{State} \to \mathsf{State})$ *is defined by* $\mathrm{If}(f, w_1, w_2) = \boldsymbol{\lambda}\, s : \mathsf{State}.\mathrm{IF}(f(s), w_1(s), w_2(s))$.

   End answer..

2. [Exercise 4.2 of Nielsen & Nielsen] Consider the statement

   $$S :=\ \mathbf{while}\ x \neq 0\ \mathbf{do}\ x := x - 1$$

   (a) Determine the functional $F : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$ associated with this statement. (The F we need to take the fixed point of to determine the semantics of S.)

   Answer: ....

   $$F(w)(s) := \begin{cases} s & \textit{if } s(x) = 0 \\ w(s[x \mapsto s(x) - 1]) & \textit{if } s(x) \neq 0 \end{cases}$$

   End answer..

   (b) Determine for each of the following partial functions $g : \mathsf{State} \rightharpoonup \mathsf{State}$ whether it is a fixed point of F.

   - $g_1(s) := \uparrow$ for all $s \in \mathsf{State}$
   - $g_2(s) := \begin{cases} s[x \mapsto 0] & \text{if} & s(x) \geq 0 \\ \uparrow & \text{if} & s(x) < 0 \end{cases}$
   - $g_3(s) := \begin{cases} s[x \mapsto 0] & \text{if} & s(x) \geq 0 \\ s & \text{if} & s(x) < 0 \end{cases}$
   - $g_4(s) := s[x \mapsto 0]$ for all $s \in \mathsf{State}$
   - $g_5(s) := s$ for all $s \in \mathsf{State}$

Answer: ....
g is a fixed point of F precisely when it satisfies

$$\begin{array}{lll} g(s) = s & \textit{if } s(x) = 0 & (a) \\ g(s) = g(s[x \mapsto s(x) - 1]) & \textit{if } s(x) \neq 0 & (b) \end{array}$$

So g_1 is not a fixed point, because it violates (a), g_2 and g_4 are fixed points because they satisfy both (a) and (b), g_3 and g_5 are not a fixed point, because they violate (b), e.g. on a state s with $s(x) = -1$.
End answer..

(c) Which of the above (if any) is the least fixed point of F?
Answer: ....
$g_2 \sqsubseteq g_4$ and g_2 is the least fixed point of F.
End answer..

3. Consider the function $f : (\mathsf{State} \rightharpoonup \mathsf{State}) \to (\mathsf{State} \rightharpoonup \mathsf{State})$ as defined by Pitts on slide 12, but now with $\mathsf{State} = \mathbb{L} \to \mathbb{Z}$:

$$f(w)(s) := \begin{cases} s & \text{if } s(x) \leq 0 \\ w(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \text{if } s(x) > 0 \end{cases}$$

(a) Prove $f(w_n) = w_{n+1}$ for $w_n : \mathsf{State} \rightharpoonup \mathsf{State}$ as defined in the lecture, for our notion of State.
Answer: ....
For $n \geq 1$, w_n has the following definition

$$w_n(s) := \begin{cases} s & \textit{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(x)! * s(y)]) & \textit{if } 0 < s(x) < n \\ \uparrow & \textit{if } s(x) \geq n \end{cases}$$

We have

$$\begin{aligned} f(w_n)(s) &= \begin{cases} s & \textit{if } s(x) \leq 0 \\ w_n(s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \textit{if } s(x) > 0 \end{cases} \\ &= \begin{cases} s & \textit{if } s(x) \leq 0 \\ s[x \mapsto s(x) - 1, y \mapsto s(x) * s(y)]) & \textit{if } s(x) = 1 \\ s[x \mapsto 0, y \mapsto (s(x) - 1)! * s(x) * s(y)] & \textit{if } 0 < s(x) - 1 < n \\ \uparrow & \textit{if } s(x) - 1 \geq n \end{cases} \\ &= \begin{cases} s & \textit{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(x)! * s(y)] & \textit{if } 0 < s(x) < n + 1 \\ \uparrow & \textit{if } s(x) \geq n + 1 \end{cases} \\ &= w_{n+1}(s) \end{aligned}$$

End answer..

(b) Prove $f(w_\infty) = w_\infty$ for $w_\infty : \mathsf{State} \rightharpoonup \mathsf{State}$ as defined in the lecture, for our notion of State.
Answer: ....
w_∞ has the following definition

$$w_\infty(s) := \begin{cases} s & \textit{if } s(x) \leq 0 \\ s[x \mapsto 0, y \mapsto s(x)! * s(y)]) & \textit{if } s(x) > 0 \end{cases}$$

We have

$$
\begin{array}{rcl}
f(w_\infty)(s) & = & \begin{cases} s & \textit{if } s(x) \le 0 \\ w_\infty(s[x \mapsto s(x)-1, y \mapsto s(x) * s(y)]) & \textit{if } s(x) > 0 \end{cases} \\
& = & \begin{cases} s & \textit{if } s(x) \le 0 \\ s[x \mapsto s(x)-1, y \mapsto s(x) * s(y)] & \textit{if } s(x) = 1 \\ s[x \mapsto 0, y \mapsto (s(x)-1)! * s(x) * s(y)] & \textit{if } s(x) > 1 \end{cases} \\
& = & \begin{cases} s & \textit{if } s(x) \le 0 \\ s[x \mapsto 0, y \mapsto (s(x)-1)! * s(x) * s(y)] & \textit{if } s(x) > 0 \end{cases} \\
& = & w_\infty
\end{array}
$$

End answer...

(c) Show implication (3) on page 19, that is: for all w,

$$w = f(w) \Rightarrow w_\infty \sqsubseteq w.$$

Answer: ....

Suppose $w = f(w)$. We show that for all $s \in \mathsf{State}$, $w_\infty(s) = w(s)$. We distinguish two cases.

- *$s(x) \le 0$. Then $w(s) = f(w)(s) = s = w_\infty(s)$, so done.*
- *$s(x) \ge 0$. We do induction on $n := s(x)$.*
  - *$n = 0$. Done.*
  - *IH: for states s with $s(x) = n$ we know $w(s) = w_\infty(s)$. Now suppose s is a state with $s(x) = n+1$. Then*

$$
\begin{array}{rcl}
w(s) = f(w)(s) & = & w(s[x \mapsto s(x)-1, y \mapsto s(x) * s(y)]) \\
& \overset{IH}{=} & w_\infty(s[x \mapsto s(x)-1, y \mapsto s(x) * s(y)]) \\
& = & s[x \mapsto 0, y \mapsto s(x-1)! * s(x) * s(y)]) \\
& = & s[x \mapsto 0, y \mapsto s(x)! * s(y)]) \\
& = & w_\infty(s).
\end{array}
$$

End answer...

(d) Prove that $\forall\, s \in \mathsf{State}\, \exists\, n[f^n(\bot)(s) = f^{n+1}(\bot)(s)]$.

Answer: ....

Let $s \in State$. As $w_n = f^n(\bot)$, we are done if we prove $\exists\, n[w_n(s) = w_{n+1}(s)]$, where w_n has been defined above. In case $s(x) \le 0$, take $n := 1$. In case $s(x) > 0$: take $n := s(x)+1$. Then $s(x) < n$, so $w_n(s) = s[x \mapsto 0, y \mapsto s(x)! * s(y)]) = w_{n+1}(s)$.

End answer...

4. [Extra exercise to possibly think about] Define a denotational semantics for the statement **for** $x := e_1$ **to** e_2 **do** P:

   (a) First with e_1, e_2 fixed numbers in $\mathbb{Z}$, say n and m.

   (b) Discuss some of the choices and problems with giving the general semantics, where e_1 and e_2 are arbitrary expressions.
   What semantics would you give to **for** $x := 1$ **to** $x+1$ **do skip** ? And to **for** $x := 1$ **to** 3 **do** $x := x-1$?