Semantics and Domain theory

Exercises 4

1. Let $(D, \sqsubseteq)$ be the domain of finite and infinite sequences over $\Sigma := \{a, b\}$ with $\sqsubseteq$ the prefix ordering. (So $D = \Sigma^* \cup \Sigma^\omega$.)

   (a) Which of the following functions $f : D \to D$ is monotonic / continuous?

   i. $f(s) = s$ with all a's removed.

   Answer: ....

   We use the following observations: (1) for v a finite word, $v \sqsubseteq vw$ for all $w \in D$; (2) for v an infinite word, we only have $v \sqsubseteq v$. Furthermore, if $(v)_{i\in\mathbb{N}}$ is a chain we have, for all $i \in \mathbb{N}$, $v_{i+1} = v_i w_{i+1}$ for some w_{i+1}, and we can write v_{i+1} as $v_0 w_1 w_2 \ldots w_{i+1}$ and $\sqcup_{i\in\mathbb{N}} v_i = v_0 w_1 w_2 w_3 \ldots$.

   For this f, we have $f(vw) = f(v)f(w)$. It is monotonic (using (1) and (2) above): if $v \sqsubseteq vw$, then $f(v) \sqsubseteq f(v)f(w) = f(vw)$. It is also continuous: if $(v)_{i\in\mathbb{N}}$ is a chain, (using (3) above) $f(\sqcup_{i\in\mathbb{N}} v_i) = f(v_0 w_1 w_2 w_3 \ldots) = f(v_0)f(w_1)f(w_2)f(w_3)\ldots = \sqcup_{i\in\mathbb{N}} f(v_i)$.

   The lfp of f is λ, the empty word, as this is $\sqcup_{n\in\mathbb{N}} f^n(\lambda)$.

   End answer ....

   ii. $f(s) = abba$ if s is finite; $f(s) = s$ if s is infinite.

   Answer: ....

   This f is not monotonic: $f(a) = abba$ and $a \sqsubseteq a^\omega$, whereas $f(a^\omega) = a^\omega$ and $abba \not\sqsubseteq a^\omega$.

   NB. f doesn't preserve lubs either: $f(\sqcup_{n\in\mathbb{N}} a^n) = f(a^\omega) = a^\omega$, whereas $\sqcup_{n\in\mathbb{N}} f(a^n) = abba$.

   End answer ....

   iii. $f(s) = abbas$.

   Answer: ....

   This f is monotonic: if $v \sqsubseteq w$, then obviously $abbav \sqsubseteq abbaw$. It is also continuous: for $(v_i)_{i\in\mathbb{N}}$ a chain, $f(\sqcup_{i\in\mathbb{N}} v_i) = abba(\sqcup_{i\in\mathbb{N}} v_i) \stackrel{*}{=} \sqcup(abbav_i)_{i\in\mathbb{N}} = \sqcup_{i\in\mathbb{N}} f(v_i)$, where the equality $\stackrel{*}{=}$ can be seen by using (3) above.

   The lfp of of f is $(abba)^\omega$, infinitely often $abba$, as this is $\sqcup_{n\in\mathbb{N}} f^n(\lambda)$.

   End answer ....

   iv. $f(s) = a$ if s contains finitely many b's; $f(s) = b$ if s contains infinitely many b's

   Answer: ....

   This f is not monotonic: $f(b) = a$ and $b \sqsubseteq b^\omega$, whereas $f(b^\omega) = b$ and $a \not\sqsubseteq b$.

   NB. f doesn't preserve lubs either: $f(\sqcup_{n\in\mathbb{N}} b^n) = f(b^\omega) = b$, whereas $\sqcup_{n\in\mathbb{N}} f(b^n) = a$.

   End answer ....

   (b) For each of the functions f in (a) that is continuous, compute the least fixed point of f.

   Answer: ....

   See above

   End answer ....

2. Let $(D, \sqsubseteq)$ be a domain with some element d_0 and let $f : D \to D$ be continuous. Suppose $d_0 \sqsubseteq f(d_0)$. Prove that $\sqcup_{i\in\mathbb{N}} f^i(d_0)$ is a fixed point of f.

Answer: ....
Basically, we redo the proof of Tarski's theorem. First note that, due to the fact that $d_0 \sqsubseteq f(d_0)$, we know that $(f^i(d_0))_{i\in\mathbb{N}}$ is a chain and we can talk about $\sqcup_{i\in\mathbb{N}} f^i(d_0)$ at all!!

Now, $f(\sqcup_{i\in\mathbb{N}} f^i(d_0)) = \sqcup_{i\in\mathbb{N}} f^{i+1}(d_0) \stackrel{\pm}{=} \sqcup_{i\in\mathbb{N}} f^i(d_0)$, where $\stackrel{\pm}{=}$ follows from the fact that the second chain differs from the first only by adding an extra first element.
End answer ....

3. Let $f, g : (D, \sqsubseteq) \to (D, \sqsubseteq)$ be continuous functions on domain $(D, \sqsubseteq)$. Prove

$$\text{fix}(f \circ g) = f(\text{fix}(g \circ f))$$

   (a) by unfolding the *definition* of fix (slide 29)
   Answer: ....

   $\text{fix}(f\circ g) = \sqcup_{n\in\mathbb{N}}(f\circ g)^n(\bot) = \sqcup_{n\in\mathbb{N}} f(g\circ f)^n(g(\bot)) \sqsupseteq \sqcup_{n\in\mathbb{N}} f(g\circ f)^n(\bot) = f(\sqcup_{n\in\mathbb{N}}(g\circ f)^n(\bot)) = f(\text{fix}(g\circ f))$.
   The other way round: $f(\text{fix}(g \circ f) = f(\sqcup_{n\in\mathbb{N}}(g\circ f)^n(\bot)) = \sqcup_{n\in\mathbb{N}} f(g \circ f)^n(\bot) = \sqcup_{n\in\mathbb{N}}(f\circ g)^n((f(\bot)) \sqsupseteq \sqcup_{n\in\mathbb{N}}(f\circ g)^n(\bot) = \text{fix}(f\circ g)$.
   So $\text{fix}(f \circ g) = f(\text{fix}(g \circ f))$.

   End answer ....

   (b) by using the *properties* of pre-fixed point (slide 20) and fixed point (slide 29) and proving

      i. $\text{fix}(f \circ g) \sqsubseteq f(\text{fix}(g \circ f))$

      Answer: ....
      We use the pre-fixed point properties of fix: for $h : D \to D$ a continuous function we have (fix1): $\text{fix}(h)$ is a pre-fixed point of h, and (fix2): if d is a pre-fixed point of h, then $\text{fix}(h) \sqsubseteq d$.

$$\frac{}{h(\text{fix}(h)) \sqsubseteq \text{fix}(h)}\ (\text{fix1}) \qquad\qquad \frac{h(d) \sqsubseteq d}{\text{fix}(h) \sqsubseteq d}\ (\text{fix2})$$

      We prove $\text{fix}(f \circ g) \sqsubseteq f(\text{fix}(g \circ f))$ by proving that $f(\text{fix}(g \circ f))$ is a pre-fixed point of $f \circ g$:

$$(f \circ g)(f(\text{fix}(g \circ f))) = f(g \circ f)(\text{fix}(g \circ f)) \sqsubseteq f(\text{fix}(g \circ f))$$

      where the $\sqsubseteq$ is by monotonicity and because $\text{fix}(g \circ f)$ is a pre-fixed point of $g \circ f$. Done.
      End answer ....

      ii. $f(\text{fix}(g \circ f)) \sqsubseteq \text{fix}(f \circ g)$

      Answer: ....
      From (i) we reuse the result $\text{fix}(g \circ f) \sqsubseteq g(\text{fix}(f \circ g))$, which follows by replacing f with g in the statement of (i). We also borrow the pre-fixed point reasoning rules (fix1) and (fix2). We derive

$$\frac{\dfrac{\text{fix}(g \circ f) \sqsubseteq g(\text{fix}(f \circ g))}{f(\text{fix}(g \circ f)) \sqsubseteq f(g(\text{fix}(f \circ g)))}\ (mon) \qquad \dfrac{}{f(g(\text{fix}(f \circ g))) \sqsubseteq \text{fix}(f \circ g))}\ (\text{fix1 for } f \circ g)}{f(\text{fix}(g \circ f)) \sqsubseteq \text{fix}(f \circ g)}$$

End answer ....

4. For the *disjoint union* of two domains (also called the *binary sum* of domains), there are two choices: the *coalesced sum* (or *smashed sum*) $D +_c E$, or the *separated sum* $D +_s E$.

   For the coalesced sum, the set $D +_c E$ is defined as

$$\{\perp\} \cup \{(0, d) \mid d \in D, d \neq \perp_D\} \cup \{(1, e) \mid e \in E, e \neq \perp_E\}$$

   For the separated sum, the set $D +_s E$ is defined as

$$\{\perp\} \cup \{(0, d) \mid d \in D\} \cup \{(1, e) \mid e \in E\}$$

   So, the separated sum introduces a new $\perp$ element, whereas the coalesced sum "coalesces (or smashes) them together".

   (NB. The 0 and 1 in the pairs have no special significance, apart from being able to distinguish the "elements coming from D" from the "elements coming from E"; we want to define the *disjoint* union, which should also work, for example, for $\mathbb{N}_\perp + \mathbb{N}_\perp$.)

   Let two domains $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be given.

   (a) Define the partial ordering $\sqsubseteq$ on $D +_s E$ and give the $\perp$-element.
   Answer: ....
   On $D +_s E$, $x \sqsubseteq y$ if $x = \perp$ or ($x = (0, d)$ and $y = (0, d')$ and $d \sqsubseteq_D d'$) or ($x = (1, e)$ and $y = (1, e')$ and $e \sqsubseteq_E e'$). The bottom element is $\perp$ (the newly added $\perp$-element).
   End answer ....

   (b) Define the partial ordering $\sqsubseteq$ on $D +_c E$ and give the $\perp$-element.
   Answer: ....
   On $D +_c E$, $x \sqsubseteq y$ if $x = \perp$ or ($x = (0, d)$ and $y = (0, d')$ and $d \sqsubseteq_D d'$) or ($x = (1, e)$ and $y = (1, e')$ and $e \sqsubseteq_E e'$). The bottom element is $\perp$ (the newly added $\perp$-element).
   End answer ....

   (c) For $(f_i)_{i\in\mathbb{N}}$ a chain in $D +_s E$ define $\sqcup_{i\in\mathbb{N}} f_i$ and prove that it is the least upperbound.
   Answer: ....
   Let $(f_i)_{i\in\mathbb{N}}$ be a chain in $D +_s E$. There are three possibilities:

      i. $\forall\, i \in \mathbb{N}.\, f_i = \perp$. Then $\sqcup_{i\in\mathbb{N}} f_i = \perp$, which is obviously the lub.

      ii. $\exists\, j \in \mathbb{N}\, \exists\, d \in D.\, f_j = (0, d)$. Then $\forall\, n \in \mathbb{N}\, \exists\, d_n \in D.\, f_{j+n} = d_n$, that is: the chain $(f_i)_{i\in\mathbb{N}}$ "remains in D" from j onwards. Then define $\sqcup_{i\in\mathbb{N}} f_i := (0, \sqcup_{n\in\mathbb{N}} d_n)$, which is obviously the lub.

      iii. $\exists\, j \in \mathbb{N}\, \exists\, e \in E.\, f_j = (1, e)$. The case is the same as the previous, with D replaced by E. Now $\forall\, n \in \mathbb{N}\, \exists\, e_n \in E.\, f_{j+n} = e_n$ and define $\sqcup_{i\in\mathbb{N}} f_i := (1, \sqcup_{n\in\mathbb{N}} e_n)$, which is obviously the lub.

   End answer ....

   (d) For $(f_i)_{i\in\mathbb{N}}$ a chain in $D +_c E$ define $\sqcup_{i\in\mathbb{N}} f_i$ and prove that it is the least upperbound.
   Answer: ....
   Let $(f_i)_{i\in\mathbb{N}}$ be a chain in $D +_c E$. We can make the same three case

distinctions as for $D +_s E$, and the definitions and proofs are exactly the same.

End answer..

(e) Define injections $\mathsf{inl} : D \to D +_s E$ and $\mathsf{inr} : E \to D +_s E$ that are continuous. (You don't have to prove that they are continuous.)

Answer: ....

For the separated sum, there are actually two obvious choices for inl and inr (and they are both continuous): we can map $\perp_D$ to $(0, \perp_D)$ in $D +_s E$ or to $\perp$ (the "new" $\perp$-element) in $D +_s E$.

- Option 1. Define, for $d \in D$, $\mathsf{inl}(d) := (0, d)$ (so now $\mathsf{inl}(\perp_D) := (0, \perp_D)$). Similarly, for $e \in E$, $\mathsf{inr}(e) := (1, e)$.
- Option 2. Define, for $d \in D$, $\mathsf{inl}'(d) := (0, d)$ if $d \neq \perp_D$ and $\mathsf{inl}'(\perp_D) := \perp$. Similarly, for $e \in E$, $\mathsf{inr}'(e) := (1, e)$ if $e \neq \perp_E$ and $\mathsf{inr}'(\perp_E) := \perp$.

End answer..

(f) Define injections $\mathsf{inl} : D \to D +_c E$ and $\mathsf{inr} : E \to D +_c E$ that are continuous. (You don't have to prove that they are continuous.)

Answer: ....

Define, for $d \in D$, $\mathsf{inl}(d) := (0, d)$ if $d \neq \perp_D$ and $\mathsf{inl}(\perp_D) := \perp$. Similarly, for $e \in E$, $\mathsf{inr}(e) := (1, e)$ if $e \neq \perp_E$ and $\mathsf{inr}(\perp_E) := \perp$.

End answer..

(g) (*) For F a domain and $f : D \to F$, $g : E \to F$ we want to define a continuous function $[f, g] : D + E \to F$ such that $[f, g](\mathsf{inl}(x)) = f(x)$ and $[f, g](\mathsf{inr}(x)) = g(x)$.

Show how to define $[f, g]$ for the case of $D +_c E$ and for the case of $D +_s E$. For one of these cases, we can only define $[f, g]$ if we place additional requirements on f and g. Which?

Answer: ....

For the coalesced sum, $D +_c E$, given (continuous) $f : D \to F$, $g : E \to F$, we define $[f, g] : D +_c E \to F$ by

$$\begin{array}{rcl} [f,g](\perp) & := & \perp_F \\ [f,g](0,d) & := & f(d) \\ [f,g](1,e) & := & g(e) \end{array}$$

Then, for $d \neq \perp_D$, we have $[f, g](\mathsf{inl}(d) = [f, g](0, d) = f(d)$, and similar for $e \neq \perp_E$ and $\mathsf{inr}(e)$. For $\perp_D$ we have $[f, g](\mathsf{inl}(\perp_d) = [f, g](\perp) = \perp_F$, and this is equal to $f(\perp_D)$ only if f is strict. We have a similar situation for $\perp_E$, so the required equations are only (fully) satisfied if we limit to strict functions.

For the separated sum, $D +_s E$, given (continuous) $f : D \to F$, $g : E \to F$, we define $[f, g] : D +_s E \to F$ by

$$\begin{array}{rcl} [f,g](\perp) & := & \perp_F \\ [f,g](0,d) & := & f(d) \\ [f,g](1,e) & := & g(e) \end{array}$$

NB. For $\perp$, one could propose to put $[f, g](\perp) := f(\perp_D)$ (or $g(\perp_E)$). However, that doesn't work in general, as it violates monotonicity for non-strict f and g: $\perp \sqsubseteq (1, \perp_E)$, so we need $[f, g](\perp) \sqsubseteq g(\perp_E)$.

Above, we have two choices for the injections. We analyze them both.

First inl and inr:

We have $[f, g](\mathsf{inl}(d) = [f, g](0, d) = f(d)$, and similarly $[f, g](\mathsf{inr}(e) = [f, g](1, e) = g(e)$, so the equations are satisfied.

Now for inl' and inr':

For $d \neq \bot_D$, the situation for inl' is the same as for inl: $[f, g](\mathsf{inl}'(d) = [f, g](0, d) = f(d)$. Similarly for $e \neq \bot_E$. For $\bot_D$ we have $[f, g](\mathsf{inl}'(\bot_d) = [f, g](\bot) = \bot_F$, and this is equal to $f(\bot_D)$ only if f is strict. We have a similar situation for $\bot_E$, so the required equations are only (fully) satisfied if we limit to strict functions.

End answer..